Numerical Solution of ODEs

Exercise Class

19th December 2023

Consider the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} && \text{in } \Omega := \{(t,x), 0 \le x \le \pi, t_0 \le t \le T\}, \\ u(0,t) = u(\pi,t) &= 0, && \text{for } t \in [t_0, T], \\ u(x,0) &= u^0(x), && \text{for } x \in [0,\pi]. \end{aligned}$$

Via the method of lines and a *central difference* approximation at discrete spatial points $x_j = hj$, $j = 1, \dots, N$, $h = \frac{\pi}{N+1}$ of

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{h^2}, \qquad j = 1, \dots, N,$$

we derive the following initial value:

$$\boldsymbol{u}'(t) = A\boldsymbol{u}(t) \qquad \text{for } t \in [t_0, T],$$

with initial conditions $\boldsymbol{u}^0 \in \mathbb{R}^N$ at time $t_0 = 0$, given by $u_j^0 = u^0(x_j)$, $j = 1, \dots, N$, where

$$A = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix} \in \mathbb{R}^{N \times N}.$$

This is the Jacobian of the initial value problem at any point.

1. Solve this system for $N = 100, 1000$.
2. Numerically compute (using MATLAB) the spectrum $\sigma(A)$ and stiffness ratio

$$L = \frac{\max\limits_{i=1,\dots,N} |\mathrm{Re}(\lambda_i)|}{\min\limits_{i=1,\dots,N} |\mathrm{Re}(\lambda_i)|}$$

for $N = 10, 100, 1000$.