Homework 2

Finite Element Methods 1

Due date: 3rd December 2024

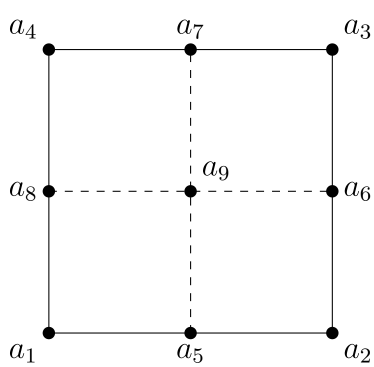
Submit a PDF/scan of the answers to the following questions before the deadline via the *Study Group Roster* (*Záznamník učitele*) in SIS, or hand-in directly at the practical class on 3rd December 2024.

1. (2 points) Consider finite elements (T, P_T, Σ_T), where

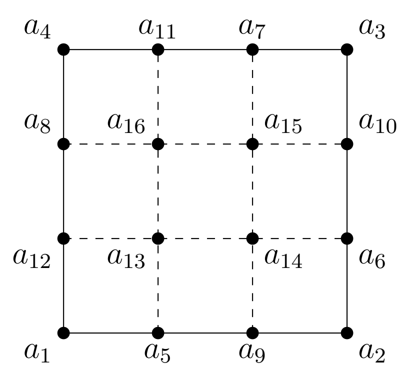
$$\begin{aligned} &T \text{ is a rectangle}, \\ &P_T = Q_3(T), \\ &\Sigma_T = \{p(z) : z \in M_3(T)\}. \end{aligned}$$

   For $T = [0,1]^2$, and the points from the principal lattice $M_3(T)$ numbered as per Figure 1b, write basis functions of the finite element (T, P_T, Σ_T). It is sufficient to derive functions for only four basis functions, as the remaining twelve can be obtained by circular permutations of the indices. Let $\mathcal{T}_h$ be a triangulation of a bounded domain $\Omega \subset \mathbb{R}^2$ consisting of rectangles and assign the above finite element to each $T \in \mathcal{T}_h$. Write the definition of the corresponding finite element space X_h and verify that $X_h \subset C(\overline{\Omega})$.

2. (2 points) Let the points $a_1, \dots a_9$ be the points of the principal lattice $M_2(T)$, see Fig-



(a) $M_2(T)$



(b) $M_3(T)$

Figure 1: Principal lattices for rectangles

ure 1a, and define the space

$$Q_2'(T) = \left\{ p \in Q_2(T) : 4\, p(a_9) + \sum_{i=1}^{4} p(a_i) - 2 \sum_{i=5}^{8} p(a_i) = 0 \right\}.$$

Show that any polynomial $p \in Q_2'(T)$ is uniquely determined by the values at the points $a_1, \ldots, a_8$ and derive basis functions $p_1', \ldots, p_8'$ of $Q_2'(T)$ satisfying $p_i'(a_j) = \delta_{ij}$, $i, j = 1, \ldots, 8$. Prove that $P_2(T) \subset Q_2'(T)$.

Hint. We can proceed similarly as for the reduced Lagrange cubic n-simplex. It is sufficient to derive functions for only two basis functions, as the remaining six can be obtained by circular permutations of the indices.

3. (2 points) Let T be a pentahedral prism, see Figure 2, with vertices $a_1, \ldots, a_6$. The triangular faces are orthogonal to the x_3 axis, and the quadrilateral faces are parallel to the x_3 axis. Let

$$\begin{aligned} P_T = \{p(x_1, x_2, x_3) = \gamma_1 + \gamma_2 x_1 + \gamma_3 x_2 + \gamma_4 x_3 \\ + \gamma_5 x_1 x_3 + \gamma_6 x_2 x_3 \\ : \gamma_1, \ldots, \gamma_6 \in \mathbb{R}\}. \end{aligned}$$

Show that any function $p \in P_T$ is uniquely determined by its values at the vertices $a_1, \ldots, a_6$ and that, for any $p \in P_T$ and face $F \subset \partial T$, the restriction $p|_F$ is uniquely determined by its values at the vertices of the face F.

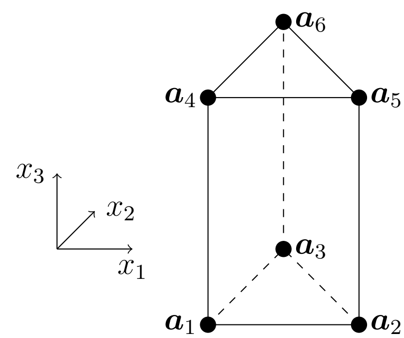


Figure 2: Pentahedral prism