20.12.2024 — Homework 4

Finite Element Methods 1

Due date: 7th January 2025

Submit a PDF/scan of the answers to the following questions before the deadline via the *Study Group Roster* (*Záznamník učitele*) in SIS, or hand-in directly at the practical class on the 7th January 2025.

1. (2 points) Let T be an n-simplex in $\mathbb{R}^n$ and let $\lambda_1, \dots, \lambda_{n+1}$ be the barycentric coordinates with respect to the vertices of T. Prove the formula

$$\int_T \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_{n+1}^{\alpha_{n+1}} \, \mathrm{d}\boldsymbol{x} = \frac{\alpha_1! \alpha_2! \cdots \alpha_{n+1}! n!}{(\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} + n)!} |T|, \quad \forall \alpha_1, \dots, \alpha_{n+1} \in \mathbb{N}_0.$$

Hint. Transform the integral over T to an integral over the reference simplex $\widehat{T}$.

2. (2 points) Let $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$ be a finite element, and let $l, m \in \mathbb{N}_0$ and $r, q \in [1, \infty]$ be such that $l \le m$ and $\widehat{P} \subset W^{l,r}(\widehat{T}) \cap W^{m,q}(\widehat{T})$. For any $T \in \mathcal{T}_h$, let (T, P_T, Σ_T) be a finite element which is affine-equivalent to $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$. Then, there exists a positive constant C, depending only on $\widehat{T}$, $\widehat{P}$, l, m, r, q, and n such that

$$|v|_{m,q,T} \le C \frac{h_T^l}{\varrho_T^m} |T|^{\frac{1}{q} - \frac{1}{r}} |v|_{l,r,T}, \qquad \text{for all } v \in P_T, T \in \mathcal{T}_h. \tag{2.1}$$

Let X_h be the finite element space corresponding to $\mathcal{T}_h$ and the finite elements (T, P_T, Σ_T). Introduce the seminorms

$$|v|_{m,q,h} = \left(\sum_{T \in \mathcal{T}_h} |v|_{m,q,T}^q \right)^{1/q} \quad \text{if } q < \infty, \qquad |v|_{m,\infty,h} = \max_{T \in \mathcal{T}_h} |v|_{m,\infty,T}.$$

Let $\mathcal{T}_h$ satisfy

$$\frac{h_T}{\varrho_T} \le \sigma, \qquad \text{for all } T \in \mathcal{T}_h,$$

and the *inverse assumption*

$$\exists \kappa > 0 : \qquad \frac{h}{h_T} \le \kappa \quad \text{for all } T \in \mathcal{T}_h,$$

where $h = \max_{T \in \mathcal{T}_h} h_T$. Prove that the *inverse inequality*

$$|v_h|_{m,q,h} \le C h^{l - m + \min(0, n/q - n/r)} |v_h|_{l,r,h}, \qquad \text{for all } v_h \in X_h,$$

where C is a positive constant depending only on $\widehat{T}$, $\widehat{P}$, l, m, r, q, n, σ, κ, and Ω.

Hint. The following inequalities may be useful.

Hölder Inequality For any non-negative numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$$

for any $p, q \in (1, \infty)$ satisfying $1/p + 1/q = 1$.

Jensen Inequality For any non-negative numbers $a_1, \dots, a_n$

$$\left(\sum_{i=1}^{n} a_i^q\right)^{1/q} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}$$

for any $p, q \in (0, \infty)$ satisfying $p \leq q$.

3. (2 points) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-continuous boundary $\partial\Omega$ and consider the weak formulation of a boundary value problem for a second order elliptic partial differential equation with Dirichlet boundary conditions u_b on $\partial\Omega$: find $u_h \in H^1(\Omega)$ such that

$$u - \widetilde{u}_b \in H_0^1(\Omega), \qquad a(u, v) = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega),$$

where $\widetilde{u}_b$ is a function satisfying the Dirichlet boundary conditions; i.e. $\widetilde{u}_b|_{\partial\Omega} = u_b$. Assume that the bilinear form $a(\cdot, \cdot)$ satisfies the condition

$$a(v, v) \geq \alpha \|v\|_{1,\Omega}^2 \qquad \forall v \in H_0^1(\Omega),$$

for some constant $\alpha > 0$.

Let $X_h \subset H^1(\Omega)$ be a finite element space and let

$$V_h = \{v_h \in X_h : \Phi(v_h) = 0 \ \forall \Phi \in \Sigma_h^{\partial\Omega}\},$$

where $\Sigma_h^{\partial\Omega}$ is the set of degrees of freedom of X_h corresponding to the nodes on the boundary of Ω; i.e., the boundary degrees of freedom. Assume that $V_h \subset H_0^1(\Omega)$, let $\widetilde{u}_{bh} \in X_h$ be a function approximating $\widetilde{u}_b$ and consider the discrete problem: Find $u_h \in X_h$ such that

$$u_h - \widetilde{u}_{bh} \in V_h, \qquad a(u_h, v_h) = \langle f, v_h \rangle \quad \forall v \in V_h.$$

Show that u_h does not depend on how $\widetilde{u}_{bh}$ is defined for interior degrees of freedom; i.e., show that u_h does not depend on the values $\Phi(\widetilde{u}_{bh})$ for $\Phi \in \Sigma_h \setminus \Sigma_h^{\partial\Omega}$.