20.12.2024 — Homework 4

Finite Element Methods 1

Due date: 7th January 2025

Submit a PDF/scan of the answers to the following questions before the deadline via the *Study Group Roster* (*Záznamník učitele*) in SIS, or hand-in directly at the practical class on the 7th January 2025.

1. (2 points) Let T be an n-simplex in $\mathbb{R}^n$ and let $\lambda_1, \dots, \lambda_{n+1}$ be the barycentric coordinates with respect to the vertices of T. Prove the formula

$$\int_T \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_{n+1}^{\alpha_{n+1}} \, \mathrm{d}\boldsymbol{x} = \frac{\alpha_1! \alpha_2! \cdots \alpha_{n+1}! n!}{(\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} + n)!} |T|, \quad \forall \alpha_1, \dots, \alpha_{n+1} \in \mathbb{N}_0.$$

Hint. Transform the integral over T to an integral over the reference simplex $\widehat{T}$.

Solution:

Let F_T be an invertible mapping which maps the unit n-simplex $\widehat{T}$ onto the n-simplex T. Then,

$$\begin{aligned} I &:= \int_T \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_{n+1}^{\alpha_{n+1}} \, \mathrm{d}\boldsymbol{x} \\ &= \frac{|T|}{|\widehat{T}|} \int_{\widehat{T}} \widehat{\lambda}_1^{\alpha_1} \widehat{\lambda}_2^{\alpha_2} \cdots \widehat{\lambda}_{n+1}^{\alpha_{n+1}} \, \mathrm{d}\widehat{\boldsymbol{x}} \\ &= \frac{|T|}{|\widehat{T}|} \int_{\widehat{T}} \widehat{x}_1^{\alpha_1} \widehat{x}_2^{\alpha_2} \cdots \widehat{x}_n^{\alpha_n} \left(1 - \sum_{i=1}^{n} \widehat{x}_i\right)^{\alpha_{n+1}} \, \mathrm{d}\widehat{\boldsymbol{x}}. \end{aligned}$$

Since

$$\begin{aligned} \widehat{T} &= \left\{ \widehat{\boldsymbol{x}} \in \mathbb{R}^n : \widehat{x}_i \in [0,1], i = 1, \dots, n, \sum_{i=1}^{n} \widehat{x}_i \le 1 \right\} \\ &= \Bigg\{ \widehat{\boldsymbol{x}} \in \mathbb{R}^n : \widehat{x}_1 \in [0,1], \widehat{x}_2 \in [0, 1 - \widehat{x}_1], \\ &\qquad \widehat{x}_3 \in [0, 1 - (\widehat{x}_1 + \widehat{x}_2)], \dots, \widehat{x}_n \in \left[0, 1 - \sum_{i=1}^{n-1} \widehat{x}_i\right] \Bigg\} \end{aligned}$$

the integral becomes

$$I = \frac{|T|}{|\widehat{T}|} \int_0^1 \widehat{x}_1^{\alpha_1} \cdots \int_0^{1-\sum\limits_{i=1}^{n-2} \widehat{x}_i} \widehat{x}_{n-1}^{\alpha_{n-1}} \int_0^{1-\sum\limits_{i=1}^{n-1} \widehat{x}_i} \widehat{x}_n^{\alpha_n} \left(1 - \sum_{i=1}^{n} \widehat{x}_i\right)^{\alpha_{n+1}} \mathrm{d}\widehat{x}_n \, \mathrm{d}\widehat{x}_{n-1} \ldots \mathrm{d}\widehat{x}_1.$$

We consider a more generic integral

$$\int_0^c \xi^\alpha (c-\xi)^\beta \, \mathrm{d}\xi,$$

for $\alpha, \beta \in \mathbb{N}_0$ and $c \in [0,1]$. If $\beta = 0$ then

$$\int_0^c \xi^\alpha (c-\xi)^\beta \, \mathrm{d}\xi = \frac{c^{\alpha+1}}{\alpha+1};$$

and when $\beta > 0$

$$\begin{aligned}
\int_0^c \xi^\alpha (c-\xi)^\beta \, \mathrm{d}\xi = \int_0^c \left(\frac{\xi^{\alpha+1}}{\alpha+1}\right)' (c-\xi)^\beta \, \mathrm{d}\xi &= \int_0^c \frac{\xi^{\alpha+1}}{\alpha+1} \beta (c-\xi)^{\beta-1} \, \mathrm{d}\xi = \ldots \\
&= \int_0^c \frac{\xi^{\alpha+\beta}}{(\alpha+1)\ldots(\alpha+\beta)} \beta! \, \mathrm{d}\xi \\
&= \frac{\alpha!\beta!}{(\alpha+\beta+1)!} c^{\alpha+\beta+1}.
\end{aligned}$$

Thus, we have be selecting $\alpha = \alpha_n$, $\beta = \alpha_{n+1}$, $\xi = \widehat{x}_n$ and $c = 1 - \sum_{i=1}^{n-1} \widehat{x}_i$ that

$$\int_0^{1-\sum\limits_{i=1}^{n-1} \widehat{x}_i} \widehat{x}_n^{\alpha_n} \left(1 - \sum_{i=1}^{n} \widehat{x}_i\right)^{\alpha_{n+1}} \mathrm{d}\widehat{x}_n = \frac{\alpha_n! \alpha_{n+1}!}{(\alpha_n + \alpha_{n+1} + 1)!} \left(1 - \sum_{i=1}^{n-1} \widehat{x}_i\right)^{\alpha_n + \alpha_{n+1} + 1}.$$

Similarly, we have that

$$\begin{aligned}
&\int_0^{1-\sum\limits_{i=1}^{n-2} \widehat{x}_i} \widehat{x}_{n-1}^{\alpha_{n-1}} \int_0^{1-\sum\limits_{i=1}^{n-1} \widehat{x}_i} \widehat{x}_n^{\alpha_n} \left(1 - \sum_{i=1}^{n} \widehat{x}_i\right)^{\alpha_{n+1}} \mathrm{d}\widehat{x}_n \, \mathrm{d}\widehat{x}_{n-1} \\
&\qquad = \frac{\alpha_n! \alpha_{n+1}!}{(\alpha_n + \alpha_{n+1} + 1)!} \int_0^{1-\sum\limits_{i=1}^{n-2} \widehat{x}_i} \widehat{x}_{n-1}^{\alpha_{n-1}} \left(1 - \sum_{i=1}^{n-1} \widehat{x}_i\right)^{\alpha_n + \alpha_{n+1} + 1} \mathrm{d}\widehat{x}_{n-1} \\
&\qquad = \frac{\alpha_n! \alpha_{n+1}!}{(\alpha_n + \alpha_{n+1} + 1)!} \frac{\alpha_{n-1}! (\alpha_n + \alpha_{n+1} + 1)!}{(\alpha_{n-1} + \alpha_n + \alpha_{n+1} + 2)!} \left(1 - \sum_{i=1}^{n-2} \widehat{x}_i\right)^{\alpha_{n-1} + \alpha_n + \alpha_{n+1} + 2} \\
&\qquad = \frac{\alpha_{n-1}! \alpha_n! \alpha_{n+1}!}{(\alpha_{n-1} + \alpha_n + \alpha_{n+1} + 2)!} \left(1 - \sum_{i=1}^{n-2} \widehat{x}_i\right)^{\alpha_{n-1} + \alpha_n + \alpha_{n+1} + 2}.
\end{aligned}$$

Recursively, we get that for $k = 1, \dots, n$,

$$\int_0^{1-\sum\limits_{i=1}^{n-k}\widehat{x}_i} \widehat{x}_{n-k+1}^{\alpha_1} \cdots \int_0^{1-\sum\limits_{i=1}^{n-1}\widehat{x}_i} \widehat{x}_n^{\alpha_n} \left(1 - \sum_{i=1}^{n} \widehat{x}_i\right)^{\alpha_{n+1}} \mathrm{d}\widehat{x}_n \, \mathrm{d}\widehat{x}_{n-1} \dots \mathrm{d}\widehat{x}_{n-k+1}$$
$$= \frac{\alpha_{n-k+1}! \dots \alpha_{n+1}!}{(\alpha_{n-k+1} + \cdots + \alpha_{n+1} + k)!} \left(1 - \sum_{i=1}^{n-k} \widehat{x}_i\right)^{\alpha_{n-k+1}+\cdots+\alpha_{n+1}+k}.$$

Therefore, we have that

$$\int_{\widehat{T}} \widehat{x}_1^{\alpha_1} \widehat{x}_2^{\alpha_2} \cdots \widehat{x}_n^{\alpha_n} \left(1 - \sum_{i=1}^{n} \widehat{x}_i\right)^{\alpha_{n+1}} \mathrm{d}\widehat{\boldsymbol{x}} = \frac{\alpha_1! \dots \alpha_{n+1}!}{(\alpha_1 + \cdots + \alpha_{n+1} + n)!}.$$

By setting $\alpha_1 = \cdots = \alpha_{n+1} = 0$ we have that

$$|\widehat{T}| = \int_{\widehat{T}} 1 \, \mathrm{d}\widehat{\boldsymbol{x}} = \frac{1}{n!};$$

hence,

$$I = \frac{\alpha_1! \dots \alpha_{n+1}! n!}{(\alpha_1 + \cdots + \alpha_{n+1} + n)!} |T|.$$

2. (2 points) Let $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$ be a finite element, and let $l, m \in \mathbb{N}_0$ and $r, q \in [1, \infty]$ be such that $l \leq m$ and $\widehat{P} \subset W^{l,r}(\widehat{T}) \cap W^{m,q}(\widehat{T})$. For any $T \in \mathcal{T}_h$, let (T, P_T, Σ_T) be a finite element which is affine-equivalent to $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$. Then, there exists a positive constant C, depending only on $\widehat{T}$, $\widehat{P}$, l, m, r, q, and n such that

$$|v|_{m,q,T} \leq C \frac{h_T^l}{\varrho_T^m} |T|^{\frac{1}{q} - \frac{1}{r}} |v|_{l,r,T}, \qquad \text{for all } v \in P_T, T \in \mathcal{T}_h. \tag{2.1}$$

Let X_h be the finite element space corresponding to $\mathcal{T}_h$ and the finite elements (T, P_T, Σ_T). Introduce the seminorms

$$|v|_{m,q,h} = \left(\sum_{T \in \mathcal{T}_h} |v|_{m,q,T}^q\right)^{1/q} \quad \text{if } q < \infty, \qquad |v|_{m,\infty,h} = \max_{T \in \mathcal{T}_h} |v|_{m,\infty,T}.$$

Let $\mathcal{T}_h$ satisfy

$$\frac{h_T}{\varrho_T} \leq \sigma, \qquad \text{for all } T \in \mathcal{T}_h,$$

and the *inverse assumption*

$$\exists \kappa > 0 : \qquad \frac{h}{h_T} \leq \kappa \quad \text{for all } T \in \mathcal{T}_h,$$

where $h = \max_{T\in\mathcal{T}_h} h_T$. Prove that the *inverse inequality*

$$|v_h|_{m,q,h} \le C h^{l-m+\min(0,n/q-n/r)} |v_h|_{l,r,h}, \qquad \text{for all } v_h \in X_h,$$

where C is a positive constant depending only on $\widehat{T}$, $\widehat{P}$, l, m, r, q, n, σ, κ, and Ω.

Hint. The following inequalities may be useful.

Hölder Inequality For any non-negative numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$$

for any $p, q \in (1, \infty)$ satisfying $1/p + 1/q = 1$.

Jensen Inequality For any non-negative numbers $a_1, \dots, a_n$

$$\left(\sum_{i=1}^{n} a_i^q\right)^{1/q} \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}$$

for any $p, q \in (0, \infty)$ satisfying $p \le q$.

Solution:

From (2.1), the fact that there exists constants σ and C_2 such that for all $T \in \mathcal{T}_h$

$$\frac{h_T}{\varrho_T} \le \sigma \qquad \text{and} \qquad |T| \le C_2 h_T^n \quad \text{(from Homework 3, q.2(a))},$$

and the *inverse assumption* we have that

$$|v_h|_{m,q,T} \le \overline{C} h^{l-m} h^{n/q-n/r} |v_h|_{l,r,T}, \qquad \text{for all } v_h \in X_h, T \in \mathcal{T}_h,$$

where $\overline{C}$ depends only on $\widehat{T}$, $\widehat{P}$, l, m, r, q, n, σ, and κ. We also note that from Homework 3, q.2(a) there exists a constant C_1 such that $C_1 h_T^n \le |T|$ and, hence,

$$\operatorname{card} \mathcal{T}_h \le \frac{|\Omega|}{\min_{T\in\mathcal{T}_h} |T|} \le \frac{|\Omega|}{C_1 \min_{T\in\mathcal{T}_h} h_T^n} \le \frac{|\Omega|\kappa^n}{C_1} h^{-n} = \widetilde{C} h^{-n},$$

where $\widetilde{C}$ depends only on n, σ, κ, and Ω.

We now consider four separate cases:

- $q = \infty$: There exists a $T_0 \in \mathcal{T}_h$ such that $|v_h|_{m,\infty,h} = |v_h|_{m,\infty,T_0}$; therefore,

$$|v_h|_{m,\infty,h} \le \overline{C} h^{l-m} h^{-n/r} |v_h|_{l,r,T_0} \le \overline{C} h^{l-m-n/r} |v_h|_{l,r,h}$$

- $r = \infty$:

$$|v_h|_{m,q,h} \le \overline{C} h^{l-m} h^{n/q} \left(\sum_{T \in \mathcal{T}_h} |v|^q_{l,\infty,T} \right)^{1/q}$$
$$\le \overline{C} h^{l-m} h^{n/q} \left(\operatorname{card} \mathcal{T}_h\right)^{1/q} |v_h|_{l,\infty,h} \le \overline{C}\widetilde{C}^{1/q} h^{l-m} |v_h|_{l,\infty,h}$$

- $r \le q < \infty$: By Jensen inequality

$$|v_h|_{m,q,h} \le \overline{C} h^{l-m+n/q-n/r} \left(\sum_{T \in \mathcal{T}_h} |v|^q_{l,r,T} \right)^{1/q} \le \overline{C} h^{l-m+n/q-n/r} |v_h|_{l,r,h}$$

- $q < r < \infty$: By Hölder inequality

$$|v_h|_{m,q,h} \le \overline{C} h^{l-m+n/q-n/r} \left(\sum_{T \in \mathcal{T}_h} 1 \cdot |v|^q_{l,r,T} \right)^{1/q}$$
$$\le \overline{C} h^{l-m+n/q-n/r} \left(\left(\sum_{T \in \mathcal{T}_h} 1 \right)^{1-q/r} \left(\sum_{T \in \mathcal{T}_h} |v|^{q \cdot r/q}_{l,r,T} \right)^{q/r} \right)^{1/q}$$
$$\le \overline{C} h^{l-m+n/q-n/r} \left(\operatorname{card} \mathcal{T}_h\right)^{1/q-1/r} |v_h|_{l,r,h}$$
$$\le \overline{C}\widetilde{C}^{1/q-1/r} h^{l-m} |v_h|_{l,r,h}$$

3. (2 points) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-continuous boundary $\partial\Omega$ and consider the weak formulation of a boundary value problem for a second order elliptic partial differential equation with Dirichlet boundary conditions u_b on $\partial\Omega$: find $u_h \in H^1(\Omega)$ such that

$$u - \widetilde{u}_b \in H^1_0(\Omega), \qquad a(u, v) = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega),$$

where $\widetilde{u}_b$ is a function satisfying the Dirichlet boundary conditions; i.e. $\widetilde{u}_b|_{\partial\Omega} = u_b$. Assume that the bilinear form $a(\cdot, \cdot)$ satisfies the condition

$$a(v, v) \ge \alpha \|v\|^2_{1,\Omega} \qquad \forall v \in H^1_0(\Omega),$$

for some constant $\alpha > 0$.

Let $X_h \subset H^1(\Omega)$ be a finite element space and let

$$V_h = \{v_h \in X_h : \Phi(v_h) = 0 \; \forall \Phi \in \Sigma^{\partial\Omega}_h\},$$

where $\Sigma^{\partial\Omega}_h$ is the set of degrees of freedom of X_h corresponding to the nodes on the boundary of Ω; i.e., the boundary degrees of freedom. Assume that $V_h \subset H^1_0(\Omega)$, let

$\widetilde{u}_{bh} \in X_h$ be a function approximating $\widetilde{u}_b$ and consider the discrete problem: Find $u_h \in X_h$ such that

$$u_h - \widetilde{u}_{bh} \in V_h, \qquad a(u_h, v_h) = \langle f, v_h \rangle \quad \forall v \in V_h.$$

Show that u_h does not depend on how $\widetilde{u}_{bh}$ is defined for interior degrees of freedom; i.e., show that u_h does not depend on the values $\Phi(\widetilde{u}_{bh})$ for $\Phi \in \Sigma_h \setminus \Sigma_h^{\partial\Omega}$.

Solution:
From the statement of the problem we have a function $\widetilde{u}_{bh} \in X_h$ approximating $\widetilde{u}_b$ with corresponding approximate solution $u_h \in X_h$ which satisfies

$$u_h - \widetilde{u}_{bh} \in V_h, \qquad a(u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_h. \tag{3.1}$$

We now consider a different approximation $\overline{u}_{bh} \in X_h$ to $\widetilde{u}_b$ with different values for interior degrees of freedom to $\widetilde{u}_{bh}$ but the same of boundary; i.e., $\Phi(\widetilde{u}_{bh}) = \Phi(\overline{u}_{bh})$ for $\Phi \in \Sigma_h^{\partial\Omega}$. With this approximation, we obtain a corresponding approximate solution $\overline{u}_h \in X_h$ which satisfies

$$\overline{u}_h - \overline{u}_{bh} \in V_h, \qquad a(\overline{u}_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_h. \tag{3.2}$$

We note that $\widetilde{u}_{bh} - \overline{u}_{bh} \in X_h$ and $\Phi(\widetilde{u}_{bh} - \overline{u}_{bh}) = 0 \implies \widetilde{u}_{bh} - \overline{u}_{bh} \in V_h$; additionally,

$$u_h - \overline{u}_h = (u_h - \widetilde{u}_{bh}) - (\overline{u}_h - \overline{u}_{bh}) + (\widetilde{u}_{bh} - \overline{u}_{bh}) \in V_h \subset H_0^1(\Omega).$$

From (3.1)–(3.2) we have that

$$a(u_h - \overline{u}_h, v_h) = 0 \quad \forall v_h \in V_h \qquad \Longrightarrow \qquad a(u_h - \overline{u}_h, u_h - \overline{u}_h) = 0.$$

Additionally, we have that

$$a(v, v) \geq \alpha \|v\|_{1,\Omega}^2 \qquad \forall v \in H_0^1(\Omega);$$

therefore

$$0 = a(u_h - \overline{u}_h, u_h - \overline{u}_h) \geq \alpha \|u_h - \overline{u}_h\|_{1,\Omega}^2 \geq 0.$$

Hence we have that

$$\|u_h - \overline{u}_h\|_{1,\Omega} = 0 \qquad \Longrightarrow \qquad u_h = \overline{u}_h.$$

So, regardless of the values selected for the interior degrees of freedom we get the same solution; i.e., u_h does not depend on how $\widetilde{u}_{bh}$ is defined for interior degrees of freedom.