Two-Grid *hp*-Version DDGFEM for Second-Order Quasilinear Elliptic PDEs using Agglomerated Coarse Meshes

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Standard Formulation

Nonlinear Problem

Given a semilinear form $\mathcal{N}(\cdot;\cdot,\cdot)$, find $u \in V$ such that

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Define V_h be the FE space on the mesh, then:

(Standard) Discretization Method

Find $u_h \in V_h$ such that

$$\mathcal{N}_h(u_h;u_h,v_h) = 0 \qquad \forall v_h \in V_h.$$

Two-Grid Methods

Create a mesh which is 'coarser' than the original mesh and define V_H as the FE space on this mesh, then:

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Find $u_H \in V_H$ such that

$$\mathcal{N}_H(u_H; u_H, v_H) = 0 \qquad \forall v_H \in V_H,$$

find $u_{2G} \in V_h$ such that

$$\mathcal{N}_h(u_H; u_{2G}, v_h) = 0 \qquad \forall v_h \in V_h.$$

Xu 1992, 1994, 1996, Xu & Zhou 1999, Axelsson & Layton 1996, Dawson, Wheeler & Woodward 1998, Utnes 1997, Marion & Xu 1995, Wu & Allen 1999, Bi & Ginting 2007, 2011

Second-Order Quasilinear PDEs

Quasilinear Problem

Given $\Omega \subset \mathbb{R}^d, d = 2, 3$ and $f \in L^2(\Omega)$, find u such that

$$\begin{aligned} -\nabla \cdot \{\mu(\boldsymbol{x}, |\nabla u|)\nabla u\} &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

Assumption

1. *$\mu \in C(\bar{\Omega} \times [0, \infty))$ and*
2. *there exists positive constants m_μ and M_μ such that*

$$M_\mu(t - s) \leq \mu(\boldsymbol{x}, t)t - \mu(\boldsymbol{x}, s)s \leq M_\mu(t - s), \quad t \geq s \geq 0, \quad \boldsymbol{x} \in \bar{\Omega}.$$

- $\mathcal{T}_h$ is a mesh consisting of triangles/tetrahedrons elements κ of granularity h, which are an affine map of a reference element $\hat{\kappa}$; i.e., there exists an affine mapping $T_\kappa : \hat{\kappa} \to \kappa$ such that $\kappa = T_\kappa(\hat{\kappa})$.
- Define polynomial degree p_κ for all $\kappa \in \mathcal{T}_h$
- (Fine) *hp*-DG finite element space:

$$V_{hp}(\mathcal{T}_h, \boldsymbol{p}) = \{v \in L^2(\Omega) : v|_\kappa \circ T_\kappa \in \mathcal{P}_{p_\kappa}(\hat{\kappa}), \kappa \in \mathcal{T}_h\}.$$

- $\mathcal{F}_h = \mathcal{F}_h^B \cup \mathcal{F}_h^I$ denotes the set of all faces in the mesh $\mathcal{T}_h$.
- Trace operators

$$\{\!\{\cdot\}\!\} : \text{Average Operator} \qquad [\![\cdot]\!] : \text{Jump Operator}.$$

(Standard) Incomplete Interior Penalty Method

Find $u_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ such that

$$A_{hp}(u_{hp}; u_{hp}, v_{hp}) = F_{hp}(v_{hp})$$

for all $v_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.

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$$\begin{aligned} A_{hp}(\psi; u, v) &= \int_\Omega \mu(|\nabla_h \psi|)\nabla_h u \cdot \nabla_h v \, d\boldsymbol{x} + \int_{\mathcal{F}_h} \sigma_{hp} [\![u]\!] \cdot [\![v]\!] \, ds \\ &\quad - \int_{\mathcal{F}_h} \{\!\{\mu(|\nabla_h \psi|)\nabla_h u\}\!\} \cdot [\![v]\!] \, ds, \\ F_{hp}(v) &= \int_\Omega f v \, d\boldsymbol{x}. \end{aligned}$$

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Interior penalty parameter:

$$\sigma_{hp} = \gamma_{hp} \frac{p_F^2}{h_F},$$

where $p_F = \max(p_{\kappa_1}, p_{\kappa_2})$ and h_F is the diameter of the face.

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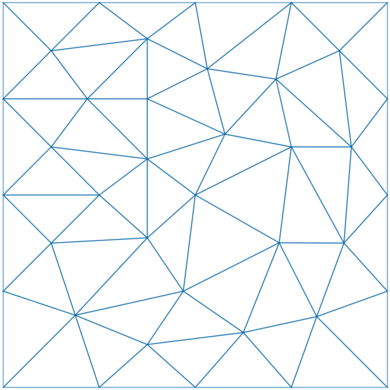
References:

Bustinza & Gatica 2004, Gatica, Gonzáles & Meddahi 2004, Houston, Robson & Suli 2005, Bustinza, Cockburn & Gatica 2005, Houston, Süli & Wihler 2007, Gudi, Nataraj & Pani 2008

Two-Grid *hp*-DGFEM

We construct a coarse mesh $\mathcal{T}_H$, consisting of general polygons/polyhedra κ_H by agglomerating elements in the fine mesh $\mathcal{T}_h$.



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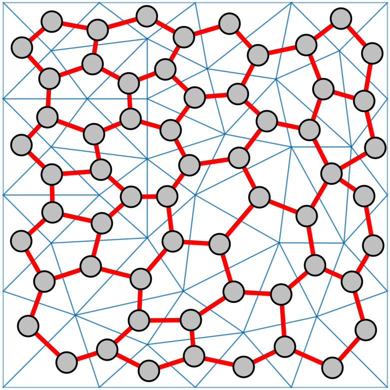

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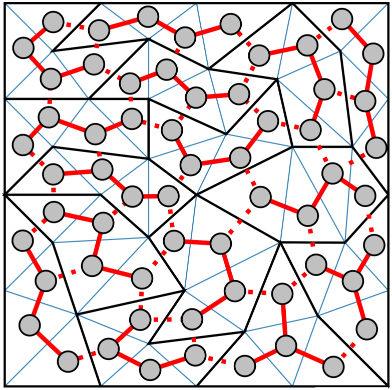

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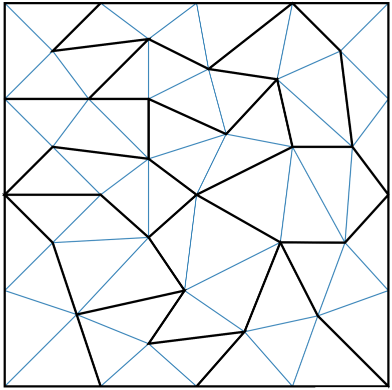

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- Define $\mathcal{T}_h(\kappa_H) = \{\kappa \in \mathcal{T}_h : \kappa \subseteq \kappa_H\}$ for all $\kappa_H \in \mathcal{T}_H$.
- Define polynomial degree P_{κ_H}, for all $\kappa_H \in \mathcal{T}_H$, such that

$$P_{\kappa_H} \leq p_\kappa \text{ for all } \kappa \in \mathcal{T}_h(\kappa_H).$$

- (Coarse) *hp*-DG finite element space:

$$V_{HP}(\mathcal{T}_H, \boldsymbol{P}) = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{P}_{P_\kappa}(\kappa), \kappa \in \mathcal{T}_H\}.$$

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- $V_{HP}(\mathcal{T}_H, \boldsymbol{P}) \subseteq V_{hp}(\mathcal{T}_h, \boldsymbol{p})$
- We use a *slightly* different *interior penalty parameter*:

$$\sigma_{HP} = \gamma_{HP} \max_{\kappa \in \{\kappa^+, \kappa^-\}} \left(C_{\mathrm{INV}} \frac{P_\kappa^2}{H_\kappa} \right),$$

for an interior face $F = \partial\kappa \cap \partial\kappa^-$, where C_{INV} is a constant from an inverse inequality for agglomerated elements.

[Cangiani, Dong, Georgoulis, & Houston 2017]

Two-Grid Approximation

1. Construct coarse and fine FE spaces $V_{HP}(\mathcal{T}_H, \boldsymbol{P})$ and $V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.
2. Compute the coarse grid approximation $u_{HP} \in V_{HP}(\mathcal{T}_H, \boldsymbol{P})$ such that

$$A_{HP}(u_{HP}; u_{HP}, v_{HP}) = F_{HP}(v_{HP})$$

   for all $v_{HP} \in V_{HP}(\mathcal{T}_H, \boldsymbol{P})$.
3. Determine the fine grid approximation $u_{2G} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ such that

$$A_{hp}(u_{HP}; u_{2G}, v_{hp}) = F_{hp}(v_{hp})$$

   for all $v_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.

[C., Houston, & Wihler 2013]

We define the following extension of the form $A_{HP}(\cdot;\cdot,\cdot)$, cf. to $\mathcal{V}\times\mathcal{V}$, where $\mathcal{V} = H^1(\Omega) + V_{HP}(\mathcal{T}_H, \boldsymbol{P})$.

$$\widetilde{A}_{HP}(u,v) = \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \mu(|\nabla u|)\nabla u\cdot\nabla v\,d\boldsymbol{x}$$
$$-\sum_{F\in\mathcal{F}_h}\int_F \{\!\{\mu(|\boldsymbol{\Pi}_{L^2}(\nabla u)|)\boldsymbol{\Pi}_{L^2}(\nabla u)\}\!\}\cdot[\![v]\!]\,ds$$
$$+\sum_{F\in\mathcal{F}_h}\int_F \sigma_{HP}[\![u]\!]\cdot[\![v]\!]\,ds,$$

Here, $\boldsymbol{\Pi}_{L^2} : [L^2(\Omega)]^d \to [V_{HP}(\mathcal{T}_H, \boldsymbol{P})]^d$ denotes the orthogonal L^2-projection onto the finite element space $[V_{HP}(\mathcal{T}_H, \boldsymbol{P})]^d$.

Existence & Uniqueness

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$$\begin{aligned}\widetilde{A}_{HP}(u,v) = &\sum_{\kappa\in\mathcal{T}_h}\int_\kappa \mu(|\nabla u|)\nabla u\cdot\nabla v\,d\boldsymbol{x}\\ &-\sum_{F\in\mathcal{F}_h}\int_F \{\!\{\mu(|\boldsymbol{\Pi}_{L^2}(\nabla u)|)\boldsymbol{\Pi}_{L^2}(\nabla u)\}\!\}\cdot[\![v]\!]\,ds\\ &+\sum_{F\in\mathcal{F}_h}\int_F \sigma_{HP}[\![u]\!]\cdot[\![v]\!]\,ds,\end{aligned}$$

Here, $\boldsymbol{\Pi}_{L^2} : [L^2(\Omega)]^d \to [V_{HP}(\mathcal{T}_H, \boldsymbol{P})]^d$ denotes the orthogonal L^2-projection onto the finite element space $[V_{HP}(\mathcal{T}_H, \boldsymbol{P})]^d$.
We note, that

$$\widetilde{A}_{HP}(u,v) = A_{HP}(u;u,v), \qquad \text{for all } u,v\in V_{HP}(\mathcal{T}_H,\boldsymbol{P}).$$

Existence & Uniqueness

Lemma

Let $\gamma_{HP} > \gamma_{\min}\epsilon$, *where* $\epsilon > 1/4$ *and* $\gamma_{\min}$ *is a positive constant; then, given a regularity assumption on the element (cf., Cangiani, Dong, Georgoulis, Houston 2017) holds, we have that the semi-linear form* $\widetilde{A}_{HP}(\cdot,\cdot)$ *is strongly monotone in the sense that*

$$\widetilde{A}_{HP}(v_1, v_1 - v_2) - \widetilde{A}_{HP}(v_2, v_1 - v_2) \geq C_{\mathrm{mono}} \|v_1 - v_2\|_{HP}^2,$$

and Lipschitz continuous in the sense that

$$|\widetilde{A}_{HP}(v_1, w) - \widetilde{A}_{HP}(v_2, w)| \leq C_{\mathrm{cont}} \|v_1 - v_2\|_{HP} \|w\|_{HP}$$

for all $v_1, v_2, w \in \mathcal{V}$, *where* C_{mono} *and* C_{cont} *are positive constants independent of the discretization parameters.*

Proof.

Application of the bounds of the non-linearity, along with standard arguments, prove these bounds. [C., Houston (In Prep.)] $\square$

Existence & Uniqueness

Theorem

Suppose that γ_{hp} and γ_{HP} are sufficiently large. Then, there exists a unique solution $u_{2G} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ to the two-grid IIP DGFEM.

Proof.

As from the previous lemma we have Lipschitz continuity and strong monotonicity of the semi-linear form $\widetilde{A}_{HP}(\cdot,\cdot)$ and

$$\widetilde{A}_{HP}(u_{HP}, v_{HP}) = A_{HP}(u_{HP}; u_{HP}, v_{HP}) = F_{HP}(v_{HP}),$$

for all $v_{HP} \in V_{HP}(\mathcal{T}_H, \boldsymbol{P})$, we can follow the proof of Houston, Robson, Süli 2005 (Theorem 2.5) to show that u_{HP} is a unique solution of the coarse approximation. Furthermore, as the fine grid formulation is an interior penalty discretization of a linear elliptic PDE, where the coefficient $\mu(|\nabla_h u_{HP}|)$ is a known function, the existence and uniqueness of the solution u_{2G} to this problem follows immediately. □

A Priori Error Estimation

Lemma (Standard Qualilinear DGFEM)

Assuming that $u \in C^1(\Omega)$ *and* $u|_\kappa \in H^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$, *for* $\kappa \in \mathcal{T}_h$ *then the solution* $u_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ *of the standard DGFEM satisfies the error bound*

$$\|u - u_{hp}\|_{hp}^2 \leq C_1 \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{H^{k_\kappa}(\kappa)}^2$$

with $s_\kappa = \min(p_\kappa + 1, k_\kappa)$.

Proof.

See Houston, Robson, & Süli 2005. □

A Priori Error Estimation

Theorem (Coarse Mesh Approximation)

Let $\mathcal{T}_H^\sharp = \{\mathcal{K}\}$ be a covering of $\mathcal{T}_H$ consisting of d-simplices and $u_{HP} \in V_{HP}(\mathcal{T}_H, \boldsymbol{P})$ be the coarse mesh approximation. If $u|_\kappa \in H^{K_\kappa}(\kappa)$, $K_\kappa \geq 3/2$, for $\kappa \in \mathcal{T}_H$, such that $\mathfrak{E}u|_\mathcal{K} \in H^{K_\kappa}(\mathcal{K})$, where $\mathfrak{E}$ is an extension operator and $\mathcal{K} \in \mathcal{T}_H^\sharp$ with $\kappa \subset \mathcal{K}$; then,

$$\|u - u_{HP}\|_{HP}^2 \leq C_2 \sum_{K \in \mathcal{T}_H} \frac{H_\kappa^{2S_\kappa - 2}}{P_\kappa^{2K_\kappa - 2}} (1 + \mathcal{G}_\kappa(H_\kappa, P_\kappa)) \|\mathfrak{E}u\|_{H^{K_\kappa}(\mathcal{K})}^2$$

where $S_\kappa = \min(P_\kappa + 1, K_\kappa)$ and

$$\mathcal{G}_\kappa(H_\kappa, P_\kappa) := (P_\kappa + P_\kappa^2) H_\kappa^{-1} \max_{F \subset \partial\kappa} \sigma_{HP}^{-1}|_F + H_\kappa P_\kappa^{-1} \max_{F \subset \partial\kappa} \sigma_{HP}|_F.$$

Proof.

Due to Lipschitz continuity and monotonicity the prove follows almost identically to Cangiani, Dong, Georgoulis, & Houston 2017. □

Theorem (Two-Grid Quasilinear Approximation)

Let $\mathcal{T}_H^\sharp = \{\mathcal{K}\}$ be a covering of $\mathcal{T}_H$ consisting of d-simplices. If $u|_\kappa \in H^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$ and $u|_\kappa \in H^{K_\kappa}(\kappa)$, $K_\kappa \geq 3/2$, for $\kappa \in \mathcal{T}_H$, such that $\mathfrak{E}u|_\mathcal{K} \in H^{K_\kappa}(\mathcal{K})$, where $\mathcal{K} \in \mathcal{T}_H^\sharp$ with $\kappa \subset \mathcal{K}$; then, the solution $u_{2G} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ of the two-grid DGFEM satisfies the error bounds

$$\|u_{hp} - u_{2G}\|_{hp}^2 \leq C_3 \left(C_1 \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 + C_2 \sum_{K \in \mathcal{T}_H} \frac{H_\kappa^{2S_\kappa - 2}}{P_\kappa^{2K_\kappa - 2}} (1 + \mathcal{G}_\kappa(H_\kappa, P_\kappa)) \|\mathfrak{E}u\|_{H^{K_\kappa}(\mathcal{K})}^2 \right)$$

$$\|u - u_{2G}\|_{hp}^2 \leq (1 + C_3) C_1 \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 + C_2 C_3 \sum_{K \in \mathcal{T}_H} \frac{H_\kappa^{2S_\kappa - 2}}{P_\kappa^{2K_\kappa - 2}} (1 + \mathcal{G}_\kappa(H_\kappa, P_\kappa)) \|\mathfrak{E}u\|_{H^{K_\kappa}(\mathcal{K})}^2.$$

Proof.

Defining $\phi = u_{2G} - u_{hp}$; then,

$$
\begin{aligned}
C_c\|\phi\|_{hp}^2 &\le A_{hp}(u_{HP}; u_{2G}, \phi) - A_{hp}(u_{HP}; u_{hp}, \phi) \\
&= A_{hp}(u_{hp}; u_{hp}, \phi) - A_{hp}(u_{HP}; u_{hp}, \phi) \\
&\le \sum_{\kappa\in\mathcal{T}_h} \int_\kappa |(\mu(|\nabla u_{hp}|) - \mu(|\nabla u_{HP}|))\nabla u_{hp}||\nabla\phi| \, d\boldsymbol{x} \\
&\qquad + \sum_{F\in\mathcal{F}_h} \int_F \{\!\!\{|(\mu(|\nabla u_{hp}|) - \mu(|\nabla u_{HP}|))\nabla u_{hp}|\}\!\!\}|[\![\phi]\!]| \, ds \\
&\le C\Bigg(\sum_{\kappa\in\mathcal{T}_h} \int_\kappa |\nabla(u_{hp} - u_{HP})||\nabla\phi| \, d\boldsymbol{x} \\
&\qquad\qquad + \sum_{F\in\mathcal{F}_h} \int_F \{\!\!\{|\nabla(u_{hp} - u_{HP})|\}\!\!\}|[\![\phi]\!]| \, ds \Bigg) \\
&\le C\left(\|\nabla_h(u - u_{hp})\|_{L^2(\Omega)} + \|\nabla_h(u - u_{HP})\|_{L^2(\Omega)} \right) \|\phi\|_{hp}.
\end{aligned}
$$

Lemma (Standard Quasilinear DGFEM)

The following bound holds:

$$\|u - u_{hp}\|_{hp}^2 \leq C_1 \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 \qquad .$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\begin{aligned}\eta_\kappa^2 &= h_\kappa^2 p_\kappa^{-2} \|f + \nabla \cdot \{\mu(|\nabla u_{hp}|)\nabla u_{hp}\}\|_{L^2(\kappa)}^2 \\ &\quad + h_\kappa p_\kappa^{-1} \|[\![\mu(|\nabla u_{hp}|)\nabla u_{hp}]\!]\|_{L^2(\partial\kappa \setminus \Gamma)}^2 + \gamma_{hp}^2 p_\kappa^3 h_\kappa^{-1} \|[\![u_{hp}]\!]\|_{L^2(\partial\kappa)}^2\end{aligned}$$

Proof.

See Houston, Süli & Wihler 2008. □

Lemma (Two-Grid Quasilinear Approximation)

The following bound holds:

$$\|u - u_{2G}\|_{hp}^2 \leq C_2 \sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \xi_\kappa^2 \right).$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\begin{aligned} \eta_\kappa^2 &= h_\kappa^2 p_\kappa^{-2} \|f + \nabla \cdot \{\mu(|\nabla u_{HP}|) \nabla u_{2G}\}\|_{L^2(\kappa)}^2 \\ &\quad + h_\kappa p_\kappa^{-1} \|[\![\mu(|\nabla u_{HP}|) \nabla u_{2G}]\!]\|_{L^2(\partial\kappa \setminus \Gamma)}^2 + \gamma_{hp}^2 p_\kappa^3 h_\kappa^{-1} \|[\![u_{2G}]\!]\|_{L^2(\partial\kappa)}^2 \end{aligned}$$

and the local two-grid error indicators are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\xi_\kappa^2 = \|(\mu(|\nabla u_{HP}|) - \mu(|\nabla u_{2G}|)) \nabla u_{2G}\|_{L^2(\kappa)}^2.$$

Proof.

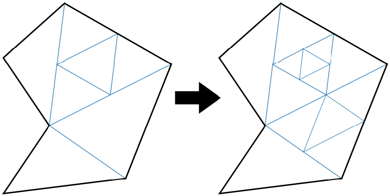
See C., Houston, & Wihler 2013 for the case of a *normal* coarse mesh. This analysis is performed on the fine mesh and the only requirement on the coarse mesh is that $V_{HP}(\mathcal{T}_H, \boldsymbol{P}) \subseteq V_{hp}(\mathcal{T}_h, \boldsymbol{p})$, which still holds. □

Two-Grid Adaptivity

1. Construct initial coarse and fine FE spaces, with coarse mesh created by agglomerating the fine mesh.
2. Compute the coarse grid approximation and two-grid solution.
3. Select elements for refinement based on η_κ and ξ_κ:
   3.1 Use $\sqrt{\eta_K^2 + \xi_K^2}$ to determine set $\mathfrak{R}(\mathcal{T}_h) \subseteq \mathcal{T}_h$ of elements to refine.
   3.2 Choose fine or coarse mesh refinement. For all $\kappa \in \mathfrak{R}(\mathcal{T}_h)$
      - if $\lambda_F \xi_\kappa \leq \eta_\kappa$ refine the fine element κ, and
      - if $\lambda_C \eta_\kappa \leq \xi_\kappa$ refine the coarse element $\kappa_H \in \mathcal{T}_H$, where $\kappa \in \mathcal{T}_h(\kappa_H)$.
4. Perform *h*-/*hp*-mesh refinement of the fine space.
5. Select *h*- or *p*-refinement for each coarse element to refine.
6. Perform *h*-/*hp*-refinement of the coarse space.
7. Goto 2.

The constants λ_F and λ_C are steering parameters.

Fine Element Refine:



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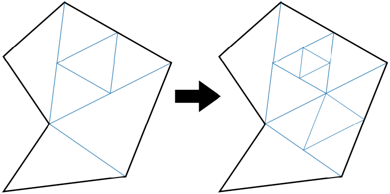
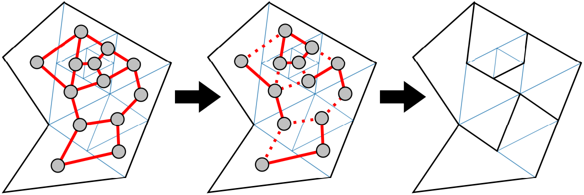


Coarse Element Refine — Partition patch of fine elements into 2^d elements



[Collis & Houston, 2016]

Coarse Element *h*-Refinement

Using a standard graph partition algorithm will attempt to create agglomerated elements with the same number of *child* fine elements, minimising the number of edge cuts.

However, we have information about the error for each fine element — can we distribute the agglomeration using this information?

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Possible to assign *weights* to each vertex and use a graph partitioning algorithm that balances these weights, rather than the number of elements. [Karypis & Kumar 1998]

We set the weight to the total local error indicator: $\eta_\kappa^2 + \xi_\kappa^2$

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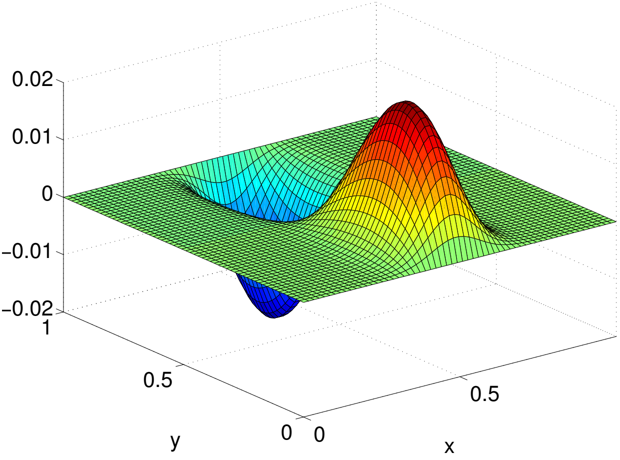
The coarse element refinement uses the fine elements *after* refinement; therefore, we divide the (square) of each error indicator equally between the new fine elements; i.e., $\eta_{\kappa_s} = {}^{\eta_\kappa}/\sqrt{N}$ and $\xi_{\kappa_s} = {}^{\xi_\kappa}/\sqrt{N}$, for $s = 1, \dots, N$, if κ is divided into N children $\kappa_1, \dots, \kappa_N$.

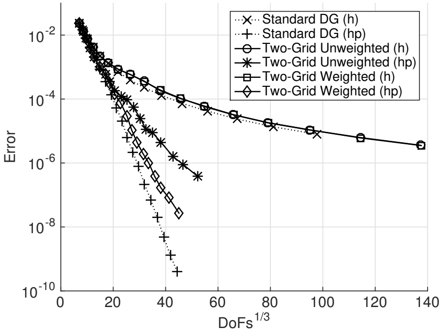
Quasilinear PDE: Smooth Solution

We let $\Omega = (0,1)^2, \mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select f so that

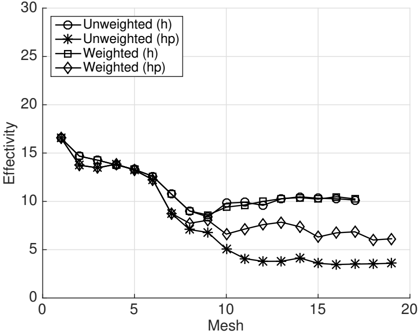
$$u(x,y) = x(1-x)y(1-y)(1-2y)e^{-20(2x-1)^2}.$$



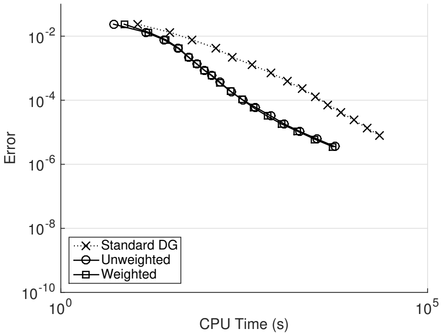
Quasilinear PDE: Smooth Solution



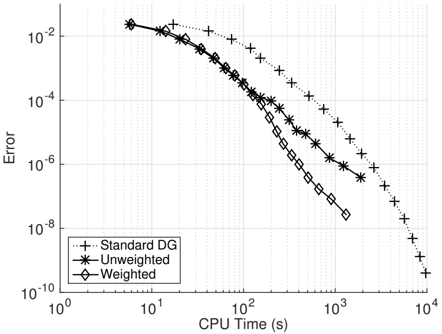
Error vs. #DoFs



Effectivity Indices

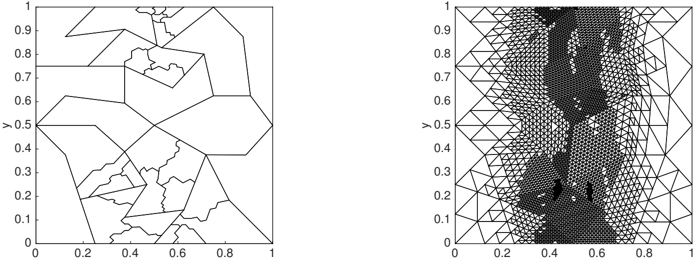


Error vs. CPU Time
h-refinement

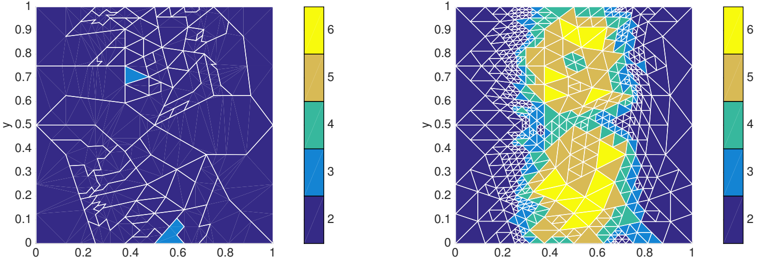


Error vs. CPU Time
hp-refinement

Quasilinear PDE: Smooth Solution



8 *h*-refinement (Weighted Coarse Refinement)

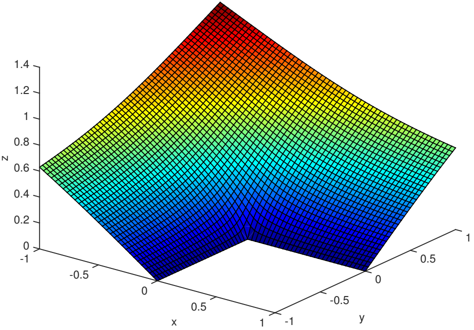


8 *hp*-refinement (Weighted Coarse Refinement)

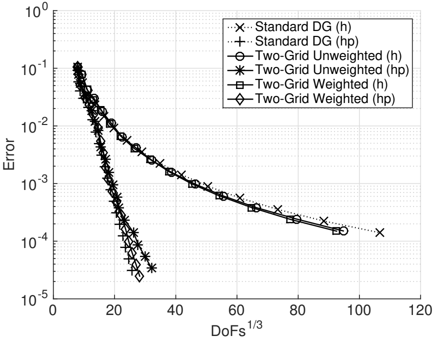
We let $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$, $\mu(\boldsymbol{x}, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2}$ and select f so that

$$u(r,\phi) = r^{2/3} \sin\left(\frac{2}{3}\varphi\right).$$

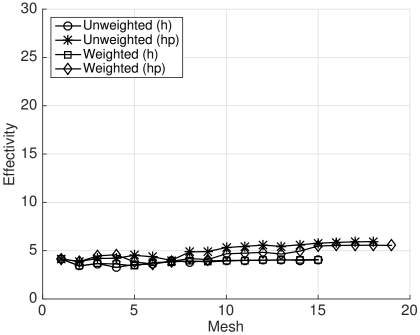
Note that u in analytic in $\overline{\Omega} \setminus \{\mathbf{0}\}$, but ∇u is singular at the origin.



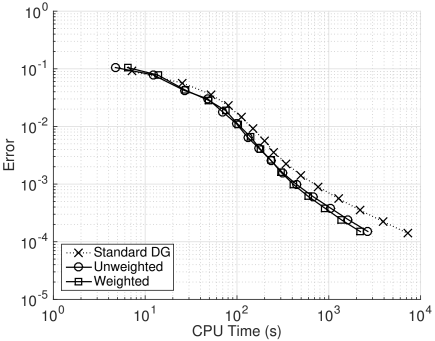
Quasilinear PDE: Singular Solution



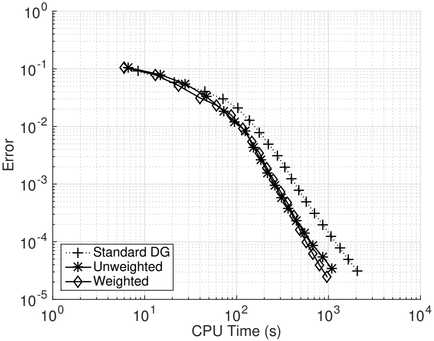
Error vs. #DoFs



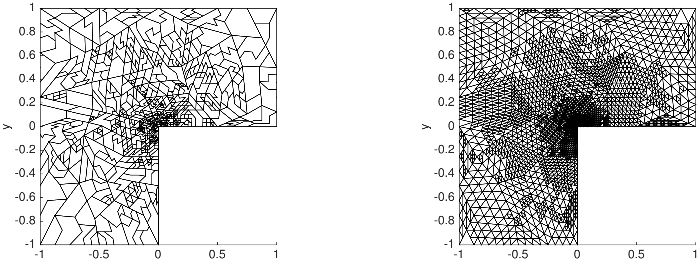
Effectivity Indices



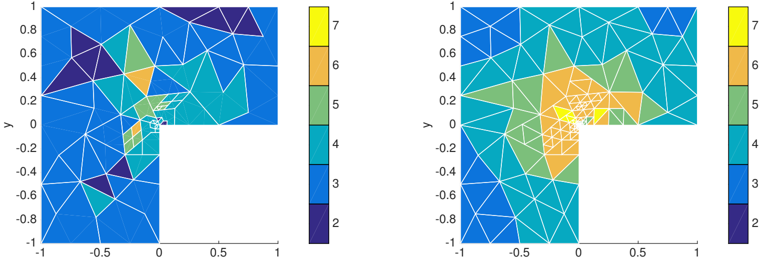
Error vs. CPU Time
h-refinement



Error vs. CPU Time
hp-refinement



8 *h*-refinement (Weighted Coarse Refinement)



8 *hp*-refinement (Weighted Coarse Refinement)

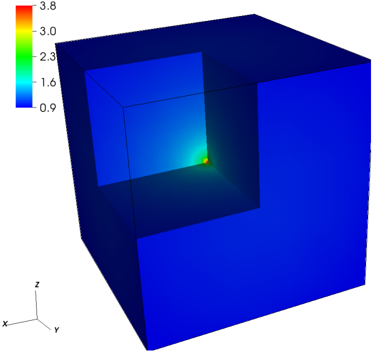
Quasilinear PDE: Singular Solution

We let Ω be the Fichera corner $(-1,1)^3 \setminus [0,1)^3$, $\mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select f so that

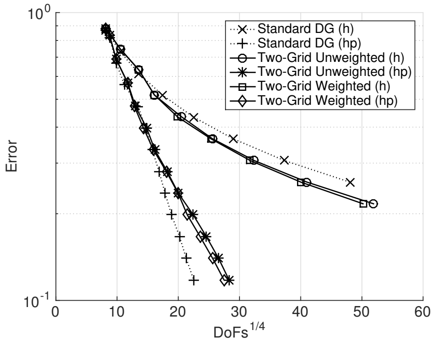
$$u(\boldsymbol{x}) = (x^2 + y^2 + z^2)^{q/2}, \quad q \in \mathbb{R};$$

for $q > -1/2, u \in H^1(\Omega)$. Here, we select $q = -1/4$.

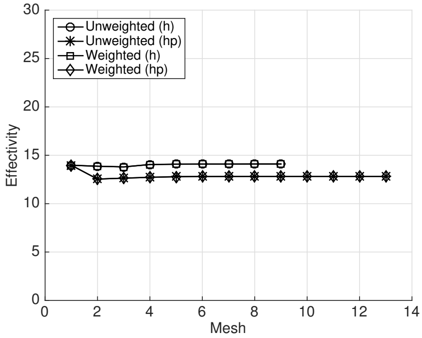
Beilina, Korotov & Křížek 2005



Quasilinear PDE: Singular Solution

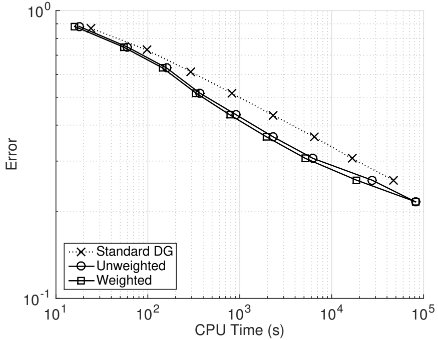


Error vs. #DoFs

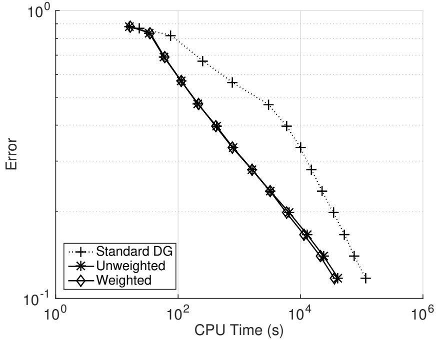


Effectivity Indices

Quasilinear PDE: Singular Solution



Error vs. CPU Time
h-refinement



Error vs. CPU Time
hp-refinement

Conclusion

Summary:

- Two-Grid DG *a posteriori* error estimates still hold for agglomerated coarse mesh of polygons and fine mesh of simplices.
- We can adaptively refine the coarse mesh based on the error estimates.

Future Aims:

- Extend to general nonlinearities.