Two-Grid *hp*-Version DGFEM for Second-Order Quasilinear Elliptic PDEs using Agglomerated Coarse Meshes

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Joint work with
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Overview

Section 1

Overview of Two-Grid Methods

Standard Formulation

Nonlinear Problem

Given a semilinear form $\mathcal{N}(\cdot;\cdot,\cdot)$, find $u \in V$ such that

$$\mathcal{N}(u;u,v) = 0 \qquad \forall v \in V.$$

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Define V_h be the FE space on the mesh, then:

(Standard) Discretization Method

Find $u_h \in V_h$ such that

$$\mathcal{N}_h(u_h;u_h,v_h) = 0 \qquad \forall v_h \in V_h.$$

Two-Grid Methods

Create a mesh which is 'coarser' than the original mesh and define V_H as the FE space on this mesh, then:

Two-Grid Discretization Method

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Two-Grid Discretization Method

Find $u_H \in V_H$ such that

$$\mathcal{N}_H(u_H; u_H, v_H) = 0 \qquad \forall v_H \in V_H,$$

find $u_{2G} \in V_h$ such that

$$\mathcal{N}_h(u_H; u_{2G}, v_h) = 0 \qquad \forall v_h \in V_h.$$

Xu 1992, 1994, 1996, Xu & Zhou 1999, Axelsson & Layton 1996, Dawson, Wheeler & Woodward 1998, Utnes 1997, Marion & Xu 1995, Wu & Allen 1999, Bi & Ginting 2007, 2011

Section 2

Second-Order Quasilinear PDE

Quasilinear Problem

Given $\Omega \subset \mathbb{R}^d, d = 2, 3$ and $f \in L^2(\Omega)$, find u such that

$$-\nabla \cdot \{\mu(\boldsymbol{x}, |\nabla u|)\nabla u\} = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega.$$

Assumption

1. $\mu \in C(\bar{\Omega} \times [0, \infty))$ *and*
2. *there exists positive constants* m_μ *and* M_μ *such that*

$$m_\mu(t - s) \leq \mu(\boldsymbol{x}, t)t - \mu(\boldsymbol{x}, s)s \leq M_\mu(t - s), \quad t \geq s \geq 0, \quad \boldsymbol{x} \in \bar{\Omega}.$$

By multiplication by a test function and integrating by parts we get the weak formulation:

Weak Formulation

Find $u \in H_0^1(\Omega) := \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$ such that

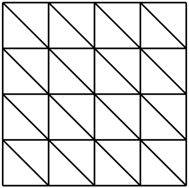
$$\int_\Omega \mu(|\nabla u|)\nabla u \cdot \nabla v \, ds = \int_\Omega fv \, ds,$$

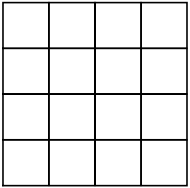
for all $v \in H_0^1(\Omega)$.

CGFEM (Linear Basis, $p = 1$)

With continuous Galerkin finite element methods we want to search for a solution in a finite dimensional subspace of $H_0^1(\Omega)$.

- Subdivide the domain Ω into a mesh $\mathcal{T}_h$ of non-overlapping triangular, tetrahedral, quadrilateral, or hexehedral elements K, with size h_K, which are an affine map of a reference element $\hat{K}$; i.e., there exists an affine mapping $T_K : \hat{K} \to K$ such that $K = T_K(\hat{K})$.





- We'll consider linear basis functions on each element for now.
- Define the CG finite element space (continuous over Ω):

$$V_{CG}(\mathcal{T}_h) = \{v \in H_0^1(\Omega) : v|_K \circ T_K \in \mathcal{Q}_1(\hat{K}), K \in \mathcal{T}_h\} \subset H_0^1(\Omega).$$

CGFEM (Linear Basis, $p = 1$)

By using the finite dimensional subspace we get a CGFEM approximation:

CGFEM

Find $u_{CG} \in V_{CG}(\mathcal{T}_h)$ such that

$$\int_\Omega \mu(|\nabla u_{CG}|)\nabla u_{CG} \cdot \nabla v_{CG}\, ds = \int_\Omega f v_{CG}\, ds,$$

for all $v_{CG} \in V_{CG}(\mathcal{T}_h)$.

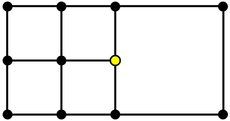
We can define u_{CG} and v_{CG} in terms of nodal hat basis functions (one per interior vertex of the mesh); i.e.,

$$u_{CG} = \sum_{i \in \mathcal{N}_h^{\mathcal{I}}} \alpha_i \varphi_i, \qquad \text{where } \alpha_i \in \mathbb{R},\ \text{for all } i \in \mathcal{N}_h^{\mathcal{I}}.$$

From this we get a nonlinear system of equations of $\#\mathcal{N}_h^{\mathcal{I}}$ unknowns, which can be solved using Newton's method, solving a linear system at each iteration.

In this talk we interested in discontinuous Galerkin finite element methods, where we don't enforce continuity of the basis functions across faces.

- This is results in more degrees of freedom (as no sharing between neighbouring elements).
- Allows us to handle so-called hanging nodes in the mesh easily:



- Allows us to easily use different order polynomials on each element — to that end we define a polynomial degree p_K for all $K \in \mathcal{T}_h$.

Now we can define the (fine) *hp*-DG finite element space:

$$V_{hp}(\mathcal{T}_h, \boldsymbol{p}) = \{v \in L^2(\Omega) : v|_K \circ T_K \in \mathcal{P}_{p_K}(\hat{K}), K \in \mathcal{T}_h\} \not\subset H_0^1(\Omega).$$

By elementwise integration by parts, and selection of suitable fluxes on edges/faces we can derive a discontinuous Galerkin finite element method.

(Standard) Incomplete Interior Penalty Method

Find $u_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ such that

$$A_{hp}(u_{hp}; u_{hp}, v_{hp}) = F_{hp}(v_{hp})$$

for all $v_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.

$$A_{hp}(\psi; u, v) = \sum_{K \in \mathcal{T}_h} \int_\Omega \mu(|\nabla_h \psi|) \nabla_h u \cdot \nabla_h v \, d\boldsymbol{x} + \sum_{F \in \mathcal{F}_h} \int_F \sigma_{hp} [\![u]\!] \cdot [\![v]\!] \, ds$$
$$- \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\mu(|\nabla_h \psi|) \nabla_h u\}\!\} \cdot [\![v]\!] \, ds,$$
$$F_{hp}(v) = \int_\Omega f v \, d\boldsymbol{x}.$$

where $\mathcal{F}_h = \mathcal{F}_h^B \cup \mathcal{F}_h^I$ denotes the set of all faces in the mesh $\mathcal{T}_h$.

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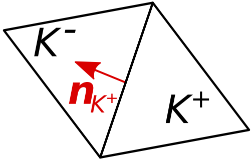
for all $v_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.

Penalty parameter: $\sigma_{hp} = \gamma_{hp} \dfrac{p_F^2}{h_F}$,

Average: $\{\!\{u\}\!\} = \dfrac{1}{2}(u|_{K^+} + u|_{K^-})$,

Jump: $[\![u]\!] = (u|_{K^+} - u|_{K^-})\boldsymbol{n}_{K^+}$,



where $p_F = \max(p_{K^+}, p_{K^-})$, h_F is the diameter of the face, and γ_{hp} is a (sufficiently large) constant.

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for all $v_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.

References:

Bustinza & Gatica 2004, Gatica, Gonzáles & Meddahi 2004, Houston, Robson & Suli 2005, Bustinza, Cockburn & Gatica 2005, Houston, Süli & Wihler 2007, Gudi, Nataraj & Pani 2008

Polygonal Elements

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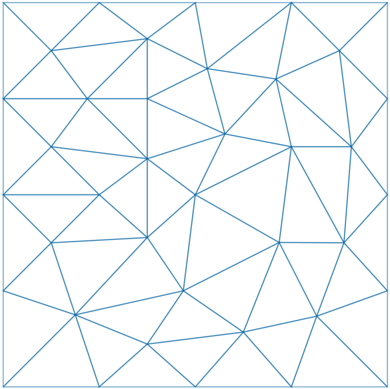
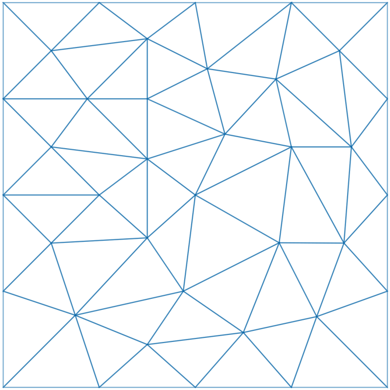
- This is fine for structured meshes, but what about unstructured?



- For two-grid, we would like to be able to construct a coarse mesh, where the mesh skeleton of the coarse mesh is contained within the fine mesh skeleton.
- This is fine for structured meshes, but what about unstructured?
- Recent work (Cangiani, Dong, Georgoulis, & Houston 2017) has extended DG methods to general polygonal elements (notable deriving trace/inverse inequalities we require) — providing one of two conditions are met:
    1. A bound exists on the number of edges/faces in the elements.
    2. A shape regularity type condition holds — essentially the element can be divided into simplices, with each face of the element sharing a complete face with one of these simplices, and a bound exists on the ratio between this simplex and the element size.

We construct a coarse mesh $\mathcal{T}_H$, consisting of general polygons/polyhedra K_H by agglomerating elements in the fine mesh $\mathcal{T}_h$; using, for example, METIS — Karypis & Kumar 1999.



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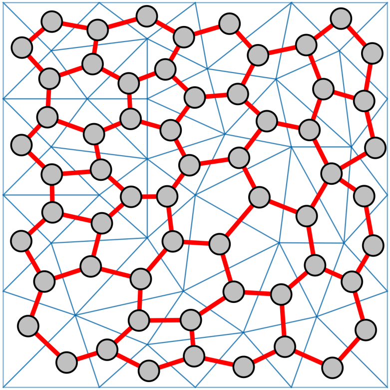

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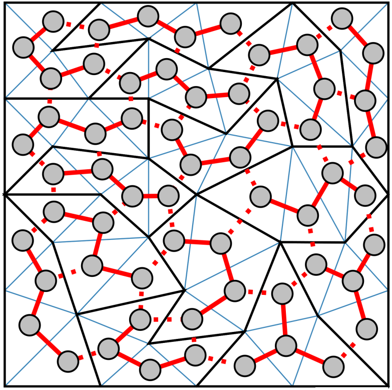

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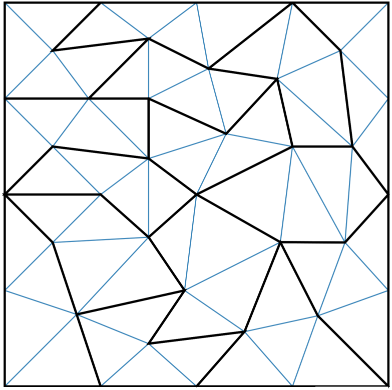

Due to this agglomeration and adaptive refinement (see later), we cannot guarantee any bound on the number of faces.

Two-Grid *hp*-DGFEM

- Define $\mathcal{T}_h(K_H) = \{K \in \mathcal{T}_h : K \subseteq K_H\}$ for all $K_H \in \mathcal{T}_H$.
- Define polynomial degree P_{K_H}, for all $K_H \in \mathcal{T}_H$, such that

$$P_{K_H} \leq p_K \text{ for all } K \in \mathcal{T}_h(K_H).$$

- (Coarse) *hp*-DG finite element space:

$$V_{HP}(\mathcal{T}_H, \boldsymbol{P}) = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_{P_K}(K), K \in \mathcal{T}_H\}.$$

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- (Coarse) *hp*-DG finite element space:

$$V_{HP}(\mathcal{T}_H, \boldsymbol{P}) = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_{P_K}(K), K \in \mathcal{T}_H\}.$$

- $V_{HP}(\mathcal{T}_H, \boldsymbol{P}) \subseteq V_{hp}(\mathcal{T}_h, \boldsymbol{p})$
- We use a *slightly* different *interior penalty parameter*:

$$\sigma_{HP} = \gamma_{HP} \max_{K \in \{K^+, K^-\}} \left(C_{\mathrm{INV}} \frac{P_K^2}{H_K} \right),$$

for an interior face $F = \partial K^+ \cap \partial K^-$, where C_{INV} is a constant from an inverse inequality for agglomerated elements.

[Cangiani, Dong, Georgoulis, & Houston 2017] universität wien

Two-Grid Approximation

1. Construct coarse and fine FE spaces $V_{HP}(\mathcal{T}_H, \boldsymbol{P})$ and $V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.
2. Compute the coarse grid approximation $u_{HP} \in V_{HP}(\mathcal{T}_H, \boldsymbol{P})$ such that

$$A_{HP}(u_{HP}; u_{HP}, v_{HP}) = F_{HP}(v_{HP})$$

   for all $v_{HP} \in V_{HP}(\mathcal{T}_H, \boldsymbol{P})$.

3. Determine the fine grid approximation $u_{2G} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ such that

$$A_{hp}(u_{HP}; u_{2G}, v_{hp}) = F_{hp}(v_{hp})$$

   for all $v_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.

[C., Houston, & Wihler 2013]

We define the following extension of the form $A_{HP}(\cdot;\cdot,\cdot)$, to $\mathcal{V} \times \mathcal{V}$, where $\mathcal{V} = H^1(\Omega) + V_{HP}(\mathcal{T}_H, \boldsymbol{P})$.

$$\begin{aligned}\widetilde{A}_{HP}(u,v) = &\sum_{K\in\mathcal{T}_h} \int_K \mu(|\nabla_h u|)\nabla_h u \cdot \nabla_h v \, d\boldsymbol{x} \\ &- \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\mu(|\boldsymbol{\Pi}_{L^2}(\nabla_h u)|)\boldsymbol{\Pi}_{L^2}(\nabla_h u)\}\!\} \cdot [\![v]\!] \, ds \\ &+ \sum_{F\in\mathcal{F}_h} \int_F \sigma_{HP} [\![u]\!] \cdot [\![v]\!] \, ds,\end{aligned}$$

Here, $\boldsymbol{\Pi}_{L^2} : [L^2(\Omega)]^d \to [V_{HP}(\mathcal{T}_H, \boldsymbol{P})]^d$ denotes the orthogonal L^2-projection onto the finite element space $[V_{HP}(\mathcal{T}_H, \boldsymbol{P})]^d$.

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Here, $\boldsymbol{\Pi}_{L^2}:[L^2(\Omega)]^d\rightarrow[V_{HP}(\mathcal{T}_H,\boldsymbol{P})]^d$ denotes the orthogonal L^2-projection onto the finite element space $[V_{HP}(\mathcal{T}_H,\boldsymbol{P})]^d$.
We note, that

$$\widetilde{A}_{HP}(u,v)=A_{HP}(u;u,v),\qquad \text{for all } u,v\in V_{HP}(\mathcal{T}_H,\boldsymbol{P}).$$

Existence & Uniqueness

Lemma

Let $\gamma_{HP} > \gamma_{\min}\epsilon$, where $\epsilon > 1/4$ and $\gamma_{\min}$ is a positive constant; then, given the regularity assumption on the element (cf., Cangiani, Dong, Georgoulis, Houston 2017) holds, we have that the semi-linear form $\widetilde{A}_{HP}(\cdot,\cdot)$ is strongly monotone in the sense that

$$\widetilde{A}_{HP}(v_1, v_1 - v_2) - \widetilde{A}_{HP}(v_2, v_1 - v_2) \geq C_{\mathrm{mono}} \|v_1 - v_2\|_{HP}^2,$$

and Lipschitz continuous in the sense that

$$|\widetilde{A}_{HP}(v_1, w) - \widetilde{A}_{HP}(v_2, w)| \leq C_{\mathrm{cont}} \|v_1 - v_2\|_{HP} \|w\|_{HP}$$

for all $v_1, v_2, w \in \mathcal{V}$, where C_{mono} and C_{cont} are positive constants independent of the discretization parameters.

Proof.

Application of the bounds of the non-linearity, along with standard arguments, prove these bounds. [C., Houston (In Prep.)] □

Existence & Uniqueness

Theorem

Suppose that γ_{hp} and γ_{HP} are sufficiently large. Then, there exists a unique solution $u_{2G} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ to the two-grid IIP DGFEM.

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Proof.

As from the previous lemma we have Lipschitz continuity and strong monotonicity of the semi-linear form $\widetilde{A}_{HP}(\cdot, \cdot)$ and

$$\widetilde{A}_{HP}(u_{HP}, v_{HP}) = A_{HP}(u_{HP}; u_{HP}, v_{HP}) = F_{HP}(v_{HP}),$$

for all $v_{HP} \in V_{HP}(\mathcal{T}_H, \boldsymbol{P})$, we can follow the proof of Houston, Robson, Süli 2005 (Theorem 2.5) to show that u_{HP} is a unique solution of the coarse approximation. Furthermore, as the fine grid formulation is an interior penalty discretization of a linear elliptic PDE, where the coefficient $\mu(|\nabla_h u_{HP}|)$ is a known function, the existence and uniqueness of the solution u_{2G} to this problem follows immediately. □

We would like to show that the method converges as the coarse/fine meshes are refined (or polynomial degrees are increased).

To that end we first introduce the DG-norm

$$\|v\|_{hp}^2 = \sum_{K\in\mathcal{T}_h} \|\nabla_h v\|_{L^2(\Omega)}^2 + \sum_{F\in\mathcal{F}_h} \int_F \sigma_{hp} |[\![v]\!]|^2 \, ds.$$

Lemma (Standard Qualilinear DGFEM)

Assuming that $u \in C^1(\Omega)$ *and* $u|_K \in H^{k_K}(K)$, $k_K \geq 2$, *for* $K \in \mathcal{T}_h$ *then the solution* $u_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ *of the standard DGFEM satisfies the error bound*

$$\|u - u_{hp}\|_{hp}^2 \leq C_1 \sum_{K \in \mathcal{T}_h} \frac{h_K^{2s_K - 2}}{p_K^{2k_K - 3}} \|u\|_{H^{k_K}(K)}^2$$

with $s_K = \min(p_K + 1, k_K)$.

Proof.

See Houston, Robson, & Süli 2005. □

A Priori Error Estimation

Theorem (Coarse Mesh Approximation)

Let $\mathcal{T}_H^\sharp = \{\mathcal{K}\}$ be a covering of $\mathcal{T}_H$ consisting of d-simplices and $u_{HP} \in V_{HP}(\mathcal{T}_H, \boldsymbol{P})$ be the coarse mesh approximation. If $u|_K \in H^{K_K}(K)$, $K_K \geq 3/2$, for $K \in \mathcal{T}_H$, such that $\mathfrak{E}u|_{\mathcal{K}} \in H^{K_K}(\mathcal{K})$, where $\mathfrak{E}$ is an extension operator and $\mathcal{K} \in \mathcal{T}_H^\sharp$ with $K \subset \mathcal{K}$; then,

$$\|u - u_{HP}\|_{HP}^2 \leq C_2 \sum_{K \in \mathcal{T}_H} \frac{H_K^{2S_K-2}}{P_K^{2K_K-2}} (1 + \mathcal{G}_K(H_K, P_K)) \|\mathfrak{E}u\|_{H^{K_K}(\mathcal{K})}^2$$

where $S_K = \min(P_K + 1, K_K)$ and

$$\mathcal{G}_K(H_K, P_K) := (P_K + P_K^2) H_K^{-1} \max_{F \subset \partial K} \sigma_{HP}^{-1}|_F + H_K P_K^{-1} \max_{F \subset \partial K} \sigma_{HP}|_F.$$

Proof.

Due to Lipschitz continuity and monotonicity the prove follows almost identically to Cangiani, Dong, Georgoulis, & Houston 2017. □

Theorem (Two-Grid Quasilinear Approximation)

Let $\mathcal{T}_H^\sharp = \{\mathcal{K}\}$ be a covering of $\mathcal{T}_H$ consisting of d-simplices. If $u|_K \in H^{k_K}(K)$, $k_K \geq 2$ and $u|_K \in H^{K_K}(K)$, $K_K \geq 3/2$, for $K \in \mathcal{T}_H$, such that $\mathfrak{E}u|_\mathcal{K} \in H^{K_K}(\mathcal{K})$, where $\mathcal{K} \in \mathcal{T}_H^\sharp$ with $K \subset \mathcal{K}$; then, the solution $u_{2G} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ of the two-grid DGFEM satisfies the error bounds

$$\|u_{hp} - u_{2G}\|_{hp}^2 \leq C_3 \left(C_1 \sum_{K \in \mathcal{T}_h} \frac{h_K^{2s_K - 2}}{p_K^{2k_K - 3}} \|u\|_{H^{k_K}(K)}^2 + C_2 \sum_{K \in \mathcal{T}_H} \frac{H_K^{2S_K - 2}}{P_K^{2K_K - 2}} (1 + \mathcal{G}_K(H_K, P_K)) \|\mathfrak{E}u\|_{H^{K_K}(\mathcal{K})}^2 \right)$$

$$\|u - u_{2G}\|_{hp}^2 \leq (1 + C_3) C_1 \sum_{K \in \mathcal{T}_h} \frac{h_K^{2s_K - 2}}{p_K^{2k_K - 3}} \|u\|_{H^{k_K}(K)}^2 + C_2 C_3 \sum_{K \in \mathcal{T}_H} \frac{H_K^{2S_K - 2}}{P_K^{2K_K - 2}} (1 + \mathcal{G}_K(H_K, P_K)) \|\mathfrak{E}u\|_{H^{K_K}(\mathcal{K})}^2.$$

Proof.

Defining $\phi = u_{2G} - u_{hp}$; then,

$$\begin{aligned}
C_c\|\phi\|_{hp}^2 &\le A_{hp}(u_{HP}; u_{2G}, \phi) - A_{hp}(u_{HP}; u_{hp}, \phi) \\
&= A_{hp}(u_{hp}; u_{hp}, \phi) - A_{hp}(u_{HP}; u_{hp}, \phi) \\
&\le \sum_{K\in\mathcal{T}_h} \int_K |(\mu(|\nabla u_{hp}|) - \mu(|\nabla u_{HP}|))\nabla u_{hp}||\nabla\phi| \, d\boldsymbol{x} \\
&\qquad + \sum_{F\in\mathcal{F}_h} \int_F \{\!\{|(\mu(|\nabla u_{hp}|) - \mu(|\nabla u_{HP}|))\nabla u_{hp}|\}\!\}|[\![\phi]\!]| \, ds \\
&\le C\Bigg(\sum_{K\in\mathcal{T}_h} \int_K |\nabla(u_{hp} - u_{HP})||\nabla\phi| \, d\boldsymbol{x} \\
&\qquad\qquad + \sum_{F\in\mathcal{F}_h} \int_F \{\!\{|\nabla(u_{hp} - u_{HP})|\}\!\}|[\![\phi]\!]| \, ds \Bigg) \\
&\le C\left(\|\nabla_h(u - u_{hp})\|_{L^2(\Omega)} + \|\nabla_h(u - u_{HP})\|_{L^2(\Omega)}\right)\|\phi\|_{hp}.
\end{aligned}$$

Section 3

Adaptive Mesh Refinement

It would be useful to be able to automatically adjust the coarse and fine meshes in a way that allows us to reduce the error, ideally to point where we can estimate that the error is below a desired tolerance.
This can be done if we have several things:

1. an error bound we can compute *a posteriori* based on the numerical solution,
2. a way to estimate the elements contributing the most to the error,
3. a way to select which elements to refine based on this contribution,
4. a method for deciding whether to refine the coarse or fine element, and
5. a method for deciding on whether to perform h– or p–refinement.

Multiple methods already exist for steps 3 and 5 (and are unimportant for this talk).
For steps 1 and 2 we consider residual-based *a posteriori* error estimation, modified for the two-grid method, and also develop an algorithm for step 4.

Lemma (Standard Quasilinear DGFEM)

The following bound holds:

$$\|u - u_{hp}\|_{hp}^2 \leq C_1 \sum_{K \in \mathcal{T}_h} \eta_K^2 .$$

Here the local error indicators η_K are defined, for all $K \in \mathcal{T}_h$, as

$$\eta_K^2 = h_K^2 p_K^{-2} \|f + \nabla \cdot \{\mu(|\nabla u_{hp}|)\nabla u_{hp}\}\|_{L^2(K)}^2 + h_K p_K^{-1} \|[\![\mu(|\nabla u_{hp}|)\nabla u_{hp}]\!]\|_{L^2(\partial K \setminus \Gamma)}^2 + \gamma_{hp}^2 p_K^3 h_K^{-1} \|[\![u_{hp}]\!]\|_{L^2(\partial K)}^2$$

Proof.

See Houston, Süli & Wihler 2008. □

Lemma (Two-Grid Quasilinear Approximation)

The following bound holds:

$$\|u - u_{2G}\|_{hp}^2 \leq C_2 \sum_{K \in \mathcal{T}_h} \left(\eta_K^2 + \xi_K^2\right).$$

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$$\eta_K^2 = h_K^2 p_K^{-2} \|f + \nabla \cdot \{\mu(|\nabla u_{HP}|)\nabla u_{2G}\}\|_{L^2(K)}^2 + h_K p_K^{-1} \|[\![\mu(|\nabla u_{HP}|)\nabla u_{2G}]\!]\|_{L^2(\partial K \setminus \Gamma)}^2 + \gamma_{hp}^2 p_K^3 h_K^{-1} \|[\![u_{2G}]\!]\|_{L^2(\partial K)}^2$$

and the local two-grid error indicators are defined, for all $K \in \mathcal{T}_h$*, as*

$$\xi_K^2 = \|(\mu(|\nabla u_{HP}|) - \mu(|\nabla u_{2G}|))\nabla u_{2G}\|_{L^2(K)}^2.$$

Proof.

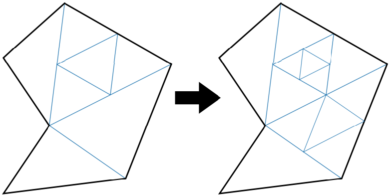
See C., Houston, & Wihler 2013 for the case of a *normal* coarse mesh. This analysis is performed on the fine mesh and the only requirement on the coarse mesh is that $V_{HP}(\mathcal{T}_H, \boldsymbol{P}) \subseteq V_{hp}(\mathcal{T}_h, \boldsymbol{p})$, which still holds. □

Two-Grid Adaptivity

1. Construct initial coarse and fine FE spaces, with coarse mesh created by agglomerating the fine mesh.
2. Compute the coarse grid approximation and two-grid solution.
3. Select elements for refinement based on η_K and ξ_K:
   3.1 Use $\sqrt{\eta_K^2 + \xi_K^2}$ to determine set $\mathfrak{R}(\mathcal{T}_h) \subseteq \mathcal{T}_h$ of elements to refine.
   3.2 Choose fine or coarse mesh refinement. For all $K \in \mathfrak{R}(\mathcal{T}_h)$
      - if $\lambda_F \xi_K \leq \eta_K$ refine the fine element K, and
      - if $\lambda_C \eta_K \leq \xi_K$ refine the coarse element $K_H \in \mathcal{T}_H$, where $K \in \mathcal{T}_h(K_H)$.
4. Perform *h*-/*hp*-mesh refinement of the fine space.
5. Select *h*- or *p*-refinement for each coarse element to refine.
6. Perform mesh smoothing to ensure any coarse element marked for refinement has at least 2^d child fine elements.
7. Perform *h*-/*hp*-refinement of the coarse space.
8. Goto 2.

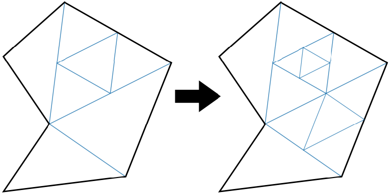
Coarse Element *h*-Refinement

Fine Element Refine:

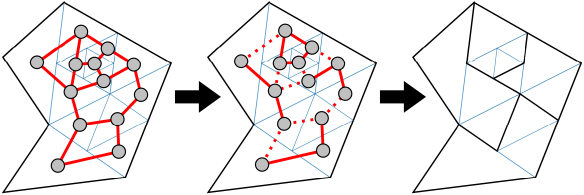


Coarse Element *h*-Refinement

Fine Element Refine:



Coarse Element Refine — Partition patch of fine elements into 2^d elements



[Collis & Houston, 2016]

Using a standard graph partition algorithm will attempt to create agglomerated elements with the same number of *child* fine elements, minimising the number of edge cuts.

However, we have information about the error for each fine element — can we distribute the agglomeration using this information?

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Possible to assign *weights* to each vertex and use a graph partitioning algorithm that balances these weights, rather than the number of elements. [Karypis & Kumar 1998]

We set the weight to the total local error indicator: $\eta_K^2 + \xi_K^2$

Coarse Element h-Refinement

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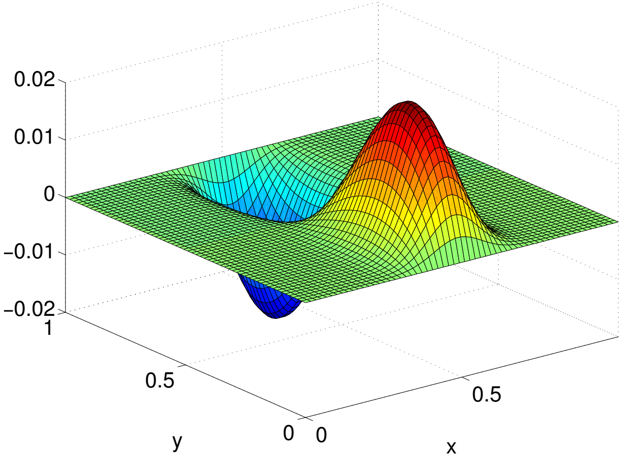
The coarse element refinement uses the fine elements *after* refinement; therefore, we divide the (square) of each error indicator equally between the new fine elements; i.e., $\eta_{K_s} = {}^{\eta_K}/\sqrt{N}$ and $\xi_{K_s} = {}^{\xi_K}/\sqrt{N}$, for $s = 1, \ldots, N$, if K is divided into N children $K_1, \ldots, K_N$.

Section 4

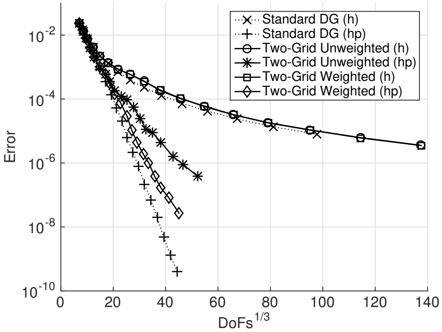
Numerical Experiments

We let $\Omega = (0,1)^2, \mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select f so that

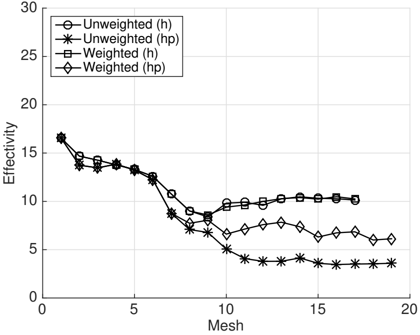
$$u(x,y) = x(1-x)y(1-y)(1-2y)\mathrm{e}^{-20(2x-1)^2}.$$



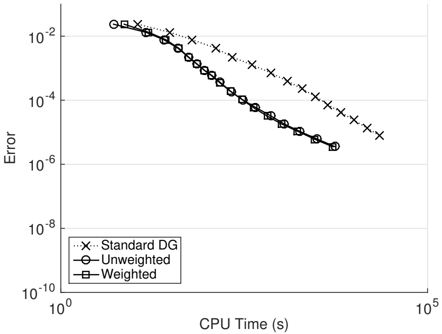
Quasilinear PDE: Smooth Solution



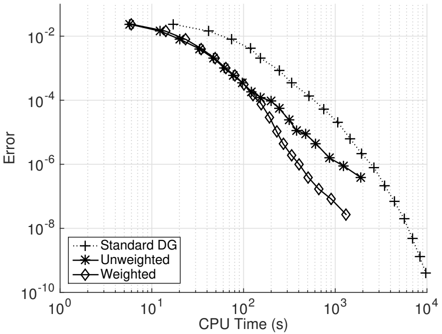
Error vs. #DoFs



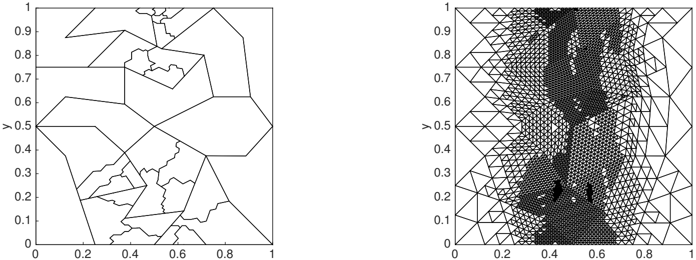
Effectivity Indices



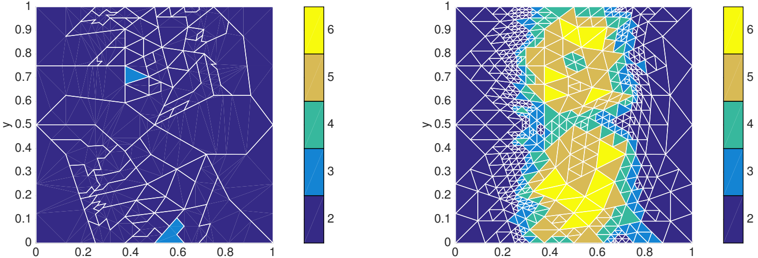
Error vs. CPU Time
h-refinement



Error vs. CPU Time
hp-refinement



8 *h*-refinement (Weighted Coarse Refinement)

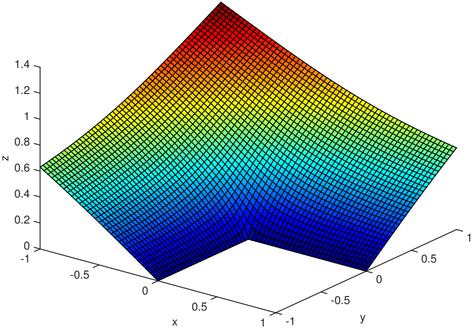


8 *hp*-refinement (Weighted Coarse Refinement)

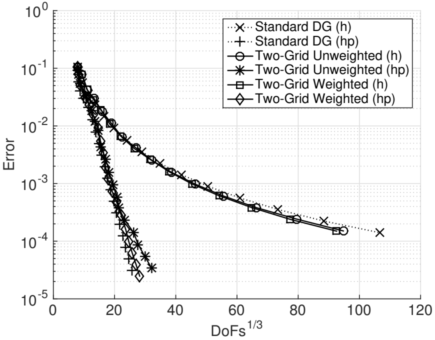
We let $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$, $\mu(\boldsymbol{x}, |\nabla u|) = 1 + e^{-|\nabla u|^2}$ and select f so that

$$u(r,\phi) = r^{2/3} \sin\left(\frac{2}{3}\varphi\right).$$

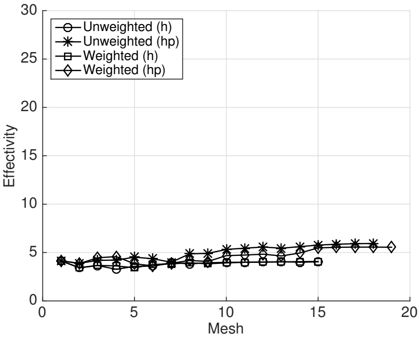
Note that u in analytic in $\overline{\Omega} \setminus \{\mathbf{0}\}$, but ∇u is singular at the origin.



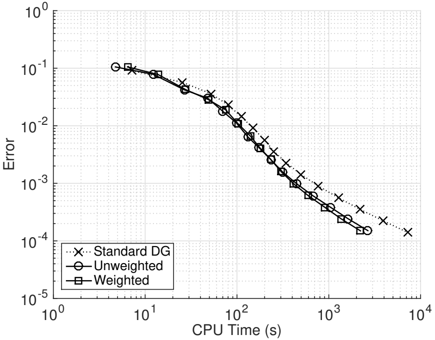
Quasilinear PDE: Singular Solution



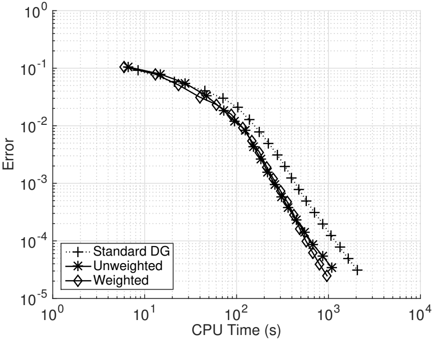
Error vs. #DoFs



Effectivity Indices

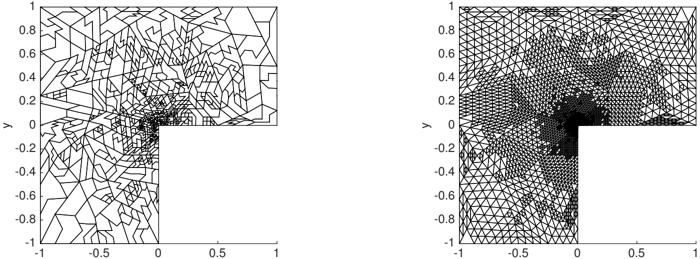


Error vs. CPU Time
h-refinement



Error vs. CPU Time
hp-refinement

Quasilinear PDE: Singular Solution



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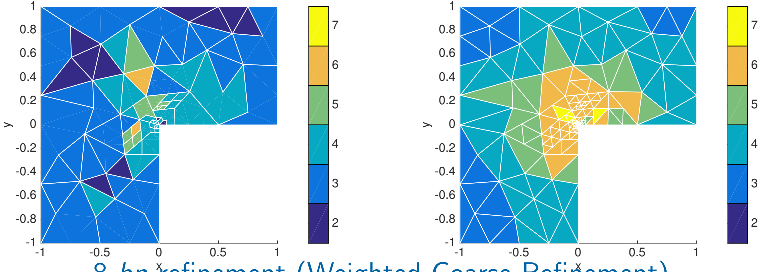


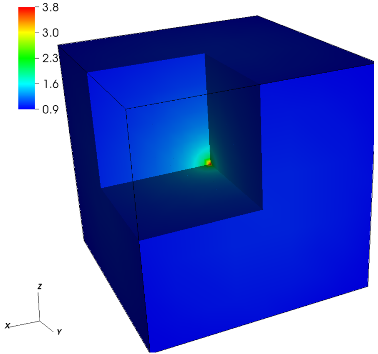
8 *hp*-refinement (Weighted Coarse Refinement)

We let Ω be the Fichera corner $(-1,1)^3 \setminus [0,1)^3$, $\mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select f so that

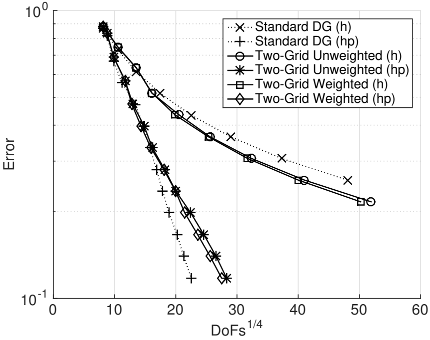
$$u(\boldsymbol{x}) = (x^2 + y^2 + z^2)^{q/2}, \quad q \in \mathbb{R};$$

for $q > -1/2, u \in H^1(\Omega)$. Here, we select $q = -1/4$.

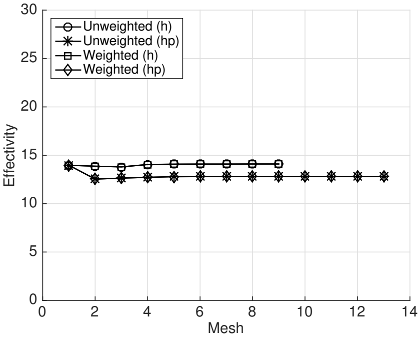
Beilina, Korotov & Křížek 2005



Quasilinear PDE: Singular Solution

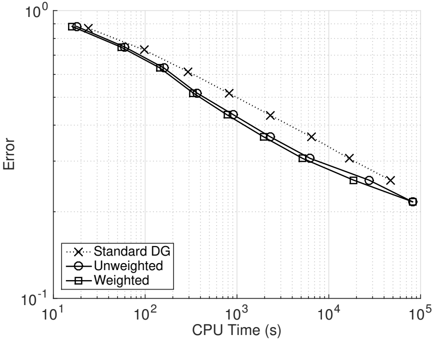


Error vs. #DoFs

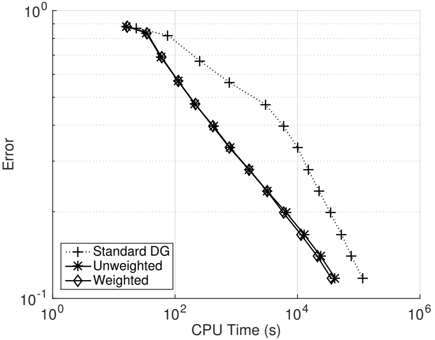


Effectivity Indices

Quasilinear PDE: Singular Solution



Error vs. CPU Time
h-refinement



Error vs. CPU Time
hp-refinement

Conclusion

Summary:

- Derived *a priori* error estimates for agglomerated coarse meshes.
- Two-Grid DG *a posteriori* error estimates still hold for agglomerated coarse mesh of polygons and fine mesh of simplices.
- We can adaptively refine the coarse mesh based on the error estimates.

Future Aims:

- Extend to general nonlinearities.