Two-Grid *hp*-Version DGFEM for Second-Order Quasilinear Elliptic PDEs using Agglomerated Coarse Meshes

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Standard Formulation

Nonlinear Problem

Given a semilinear form $\mathcal{N}(\cdot;\cdot,\cdot)$, find $u \in V$ such that

$$\mathcal{N}(u;u,v) = 0 \qquad \forall v \in V.$$

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(Standard) Discretization Method

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Create a mesh which is 'coarser' than the original mesh and define V_H as the FE space on this mesh, then:

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Find $u_H \in V_H$ such that

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find $u_{2G} \in V_h$ such that

$$\mathcal{N}_h(u_H; u_{2G}, v_h) = 0 \qquad \forall v_h \in V_h.$$

Xu 1992, 1994, 1996, Xu & Zhou 1999, Axelsson & Layton 1996, Dawson, Wheeler & Woodward 1998, Utnes 1997, Marion & Xu 1995, Wu & Allen 1999, Bi & Ginting 2007, 2011

Quasilinear Problem

Given $\Omega \subset \mathbb{R}^d, d = 2, 3$ and $f \in L^2(\Omega)$, find u such that

$$\begin{aligned} -\nabla \cdot \{\mu(\boldsymbol{x}, |\nabla u|)\nabla u\} &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Assumption

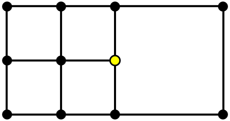
1. $\mu \in C(\bar{\Omega} \times [0, \infty))$ *and*
2. *there exists positive constants* m_μ *and* M_μ *such that*

$$m_\mu(t - s) \leq \mu(\boldsymbol{x}, t)t - \mu(\boldsymbol{x}, s)s \leq M_\mu(t - s), \quad t \geq s \geq 0, \quad \boldsymbol{x} \in \bar{\Omega}.$$

hp-DGFEM

In this talk we interested in discontinuous Galerkin finite element methods, where we don't enforce continuity of the basis functions across faces.

- This is results in more degrees of freedom (as no sharing between neighbouring elements).
- Allows us to handle so-called hanging nodes in the mesh easily:



- Allows us to easily use different order polynomials on each element — to that end we define a polynomial degree p_κ for all $\kappa \in \mathcal{T}_h$.

Now we can define the (fine) *hp*-DG finite element space:

$$V_{hp}(\mathcal{T}_h, \boldsymbol{p}) = \{v \in L^2(\Omega) : v|_\kappa \circ T_\kappa \in \mathcal{P}_{p_\kappa}(\hat{\kappa}), \kappa \in \mathcal{T}_h\} \not\subset H_0^1(\Omega).$$

By elementwise integration by parts, and selection of suitable fluxes on edges/faces we can derive a discontinuous Galerkin finite element method.

(Standard) Incomplete Interior Penalty Method

Find $u_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ such that

$$A_{hp}(u_{hp}; u_{hp}, v_{hp}) = F_{hp}(v_{hp})$$

for all $v_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.

$$\begin{aligned} A_{hp}(\psi; u, v) = \sum_{K\in\mathcal{T}_h} \int_\Omega \mu(|\nabla_h \psi|)\nabla_h u \cdot \nabla_h v \, d\boldsymbol{x} + \sum_{F\in\mathcal{F}_h} \int_F \sigma_{hp} [\![u]\!] \cdot [\![v]\!] \, ds \\ - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\mu(|\nabla_h \psi|)\nabla_h u\}\!\} \cdot [\![v]\!] \, ds, \end{aligned}$$

$$F_{hp}(v) = \int_\Omega fv \, d\boldsymbol{x}.$$

where $\mathcal{F}_h = \mathcal{F}_h^B \cup \mathcal{F}_h^I$ denotes the set of all faces in the mesh $\mathcal{T}_h$.

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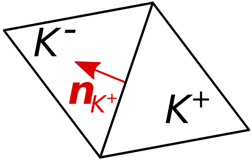
for all $v_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.

Penalty parameter: $\sigma_{hp} = \gamma_{hp} \frac{p_F^2}{h_F}$,

Average: $\{\!\{u\}\!\} = \frac{1}{2}(u|_{K^+} + u|_{K^-})$,

Jump: $[\![u]\!] = (u|_{K^+} - u|_{K^-})\boldsymbol{n}_{K^+}$,



where $p_F = \max(p_{\kappa^+}, p_{\kappa^-})$, h_F is the diameter of the face, and γ_{hp} is a (sufficiently large) constant.

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for all $v_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.

References:

Bustinza & Gatica 2004, Gatica, Gonzáles & Meddahi 2004, Houston, Robson & Suli 2005, Bustinza, Cockburn & Gatica 2005, Houston, Süli & Wihler 2007, Gudi, Nataraj & Pani 2008

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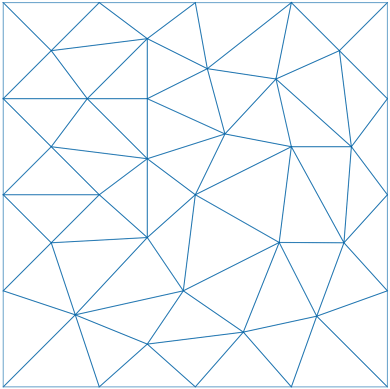
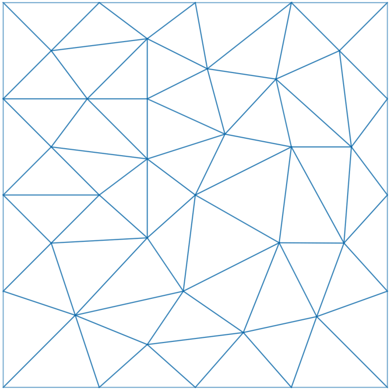
- This is fine for structured meshes, but what about unstructured?



- For two-grid, we would like to be able to construct a coarse mesh, where the mesh skeleton of the coarse mesh is contained within the fine mesh skeleton.
- This is fine for structured meshes, but what about unstructured?
- Recent work (Cangiani, Dong, Georgoulis, & Houston 2017) has extended DG methods to general polygonal elements (notably deriving trace/inverse inequalities we require) — providing one of two conditions are met:
  1. A bound exists on the number of edges/faces in the elements.
  2. A shape regularity type condition holds — essentially the element can be divided into simplices, with each face of the element sharing a complete face with one of these simplices, and a bound exists on the ratio between this simplex and the element size.

Two-Grid *hp*-DGFEM

We construct a coarse mesh $\mathcal{T}_H$, consisting of general polygons/polyhedra κ_H by agglomerating elements in the fine mesh $\mathcal{T}_h$; using, for example, METIS — Karypis & Kumar 1999.



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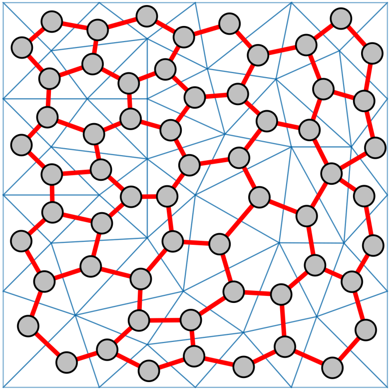

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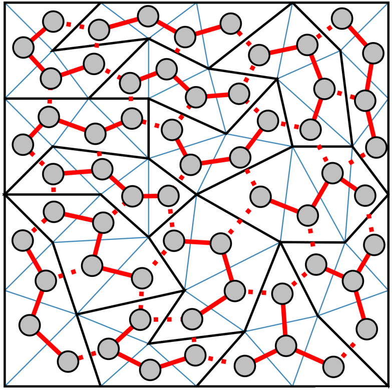

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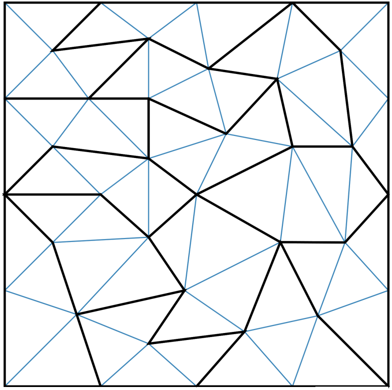

Due to this agglomeration and adaptive refinement (see later), we cannot guarantee any bound on the number of faces.

- Define $\mathcal{T}_h(\kappa_H) = \{\kappa \in \mathcal{T}_h : \kappa \subseteq \kappa_H\}$ for all $\kappa_H \in \mathcal{T}_H$.
- Define polynomial degree P_{κ_H}, for all $\kappa_H \in \mathcal{T}_H$, such that

$$P_{\kappa_H} \leq p_\kappa \text{ for all } \kappa \in \mathcal{T}_h(\kappa_H).$$

- (Coarse) *hp*-DG finite element space:

$$V_{HP}(\mathcal{T}_H, \boldsymbol{P}) = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{P}_{P_\kappa}(\kappa), \kappa \in \mathcal{T}_H\}.$$

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- $V_{HP}(\mathcal{T}_H, \boldsymbol{P}) \subseteq V_{hp}(\mathcal{T}_h, \boldsymbol{p})$
- We use a *slightly* different *interior penalty parameter*:

$$\sigma_{HP} = \gamma_{HP} \max_{\kappa \in \{\kappa^+, \kappa^-\}} \left(C_{\mathrm{INV}} \frac{P_\kappa^2}{H_\kappa} \right),$$

for an interior face $F = \partial\kappa^+ \cap \partial\kappa^-$, where C_{INV} is a constant from an inverse inequality for agglomerated elements.

[Cangiani, Dong, Georgoulis, & Houston 2017]

Two-Grid Approximation

1. Construct coarse and fine FE spaces $V_{HP}(\mathcal{T}_H, \boldsymbol{P})$ and $V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.
2. Compute the coarse grid approximation $u_{HP} \in V_{HP}(\mathcal{T}_H, \boldsymbol{P})$ such that

$$A_{HP}(u_{HP}; u_{HP}, v_{HP}) = F_{HP}(v_{HP})$$

   for all $v_{HP} \in V_{HP}(\mathcal{T}_H, \boldsymbol{P})$.
3. Determine the fine grid approximation $u_{2G} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ such that

$$A_{hp}(u_{HP}; u_{2G}, v_{hp}) = F_{hp}(v_{hp})$$

   for all $v_{hp} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$.

[C., Houston, & Wihler 2013]

Theorem

Suppose that γ_{hp} and γ_{HP} are sufficiently large. Then, there exists a unique solution $u_{2G} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ to the two-grid IIP DGFEM.

Proof.

For sufficiently large γ_{HP}, given a regularity assumption on the element (cf., Cangiani, Dong, Georgoulis, Houston 2017) holds, we can show Lipschitz continuity and strong monotonicity of the semi-linear form $A_{HP}(\cdot;\cdot,\cdot)$, we can follow the proof of Houston, Robson, Süli 2005 (Theorem 2.5) to show that u_{HP} is a unique solution of the coarse approximation.
Furthermore, as the fine grid formulation is an interior penalty discretization of a linear elliptic PDE, where the coefficient $\mu(|\nabla_h u_{HP}|)$ is a known function, the existence and uniqueness of the solution u_{2G} to this problem follows immediately if γ_{hp} is sufficiently large. $\square$

A Priori Error Estimation

We would like to show that the method converges as the coarse/fine meshes are refined (or polynomial degrees are increased).

To that end we first introduce the DG-norm

$$\|v\|_{hp}^2 = \sum_{K\in\mathcal{T}_h} \|\nabla_h v\|_{L^2(\Omega)}^2 + \sum_{F\in\mathcal{F}_h} \int_F \sigma_{hp} |[\![v]\!]|^2 \, ds.$$

Theorem (Two-Grid Quasilinear Approximation)

Let $\mathcal{T}_H^\sharp = \{\mathcal{K}\}$ be a covering of $\mathcal{T}_H$ consisting of d-simplices. If $u|_\kappa \in H^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$ and $u|_\kappa \in H^{K_\kappa}(\kappa)$, $K_\kappa \geq 3/2$, for $\kappa \in \mathcal{T}_H$, such that $\mathfrak{E}u|_\mathcal{K} \in H^{K_\kappa}(\mathcal{K})$, where $\mathcal{K} \in \mathcal{T}_H^\sharp$ with $\kappa \subset \mathcal{K}$; then, the solution $u_{2G} \in V_{hp}(\mathcal{T}_h, \boldsymbol{p})$ satisfies

$$\|u_{hp} - u_{2G}\|_{hp}^2 \leq C_3 \left(C_1 \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 \right.$$
$$\left. + C_2 \sum_{K \in \mathcal{T}_H} \frac{H_\kappa^{2S_\kappa - 2}}{P_\kappa^{2K_\kappa - 2}} (1 + \mathcal{G}_\kappa(H_\kappa, P_\kappa)) \|\mathfrak{E}u\|_{H^{K_\kappa}(\mathcal{K})}^2 \right)$$

$$\|u - u_{2G}\|_{hp}^2 \leq (1 + C_3) C_1 \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{H^{k_\kappa}(\kappa)}^2$$
$$+ C_2 C_3 \sum_{K \in \mathcal{T}_H} \frac{H_\kappa^{2S_\kappa - 2}}{P_\kappa^{2K_\kappa - 2}} (1 + \mathcal{G}_\kappa(H_\kappa, P_\kappa)) \|\mathfrak{E}u\|_{H^{K_\kappa}(\mathcal{K})}^2,$$

where $\mathcal{G}_{\kappa_H}(H_\kappa, P_\kappa) := (P_{\kappa_H} + P_{\kappa_H}^2) H_{\kappa_H}^{-1} \max_{F \subset \partial\kappa_H} \sigma_{HP}^{-1}|_F + \frac{H_{\kappa_H}}{P_{\kappa_H}} \max_{F \subset \partial\kappa_H} \sigma_{HP}|_F$

It would be useful to be able to automatically adjust the coarse and fine meshes in a way that allows us to reduce the error, ideally to point where we can estimate that the error is below a desired tolerance.

This can be done if we have several things:

1. an error bound we can compute *a posteriori* based on the numerical solution,
2. a way to estimate the elements contributing the most to the error,
3. a way to select which elements to refine based on this contribution,
4. a method for deciding whether to refine the coarse or fine element, and
5. a method for deciding on whether to perform h– or p–refinement.

Multiple methods already exist for steps 3 and 5 (and are unimportant for this talk).

For steps 1 and 2 we consider residual-based *a posteriori* error estimation, modified for the two-grid method, and also develop an algorithm for step 4.

Lemma (Standard Quasilinear DGFEM)

The following bound holds:

$$\|u - u_{hp}\|_{hp}^2 \leq C_1 \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 \quad .$$

Here the local error indicators η_κ *are defined, for all* $\kappa \in \mathcal{T}_h$*, as*

$$\eta_\kappa^2 = h_\kappa^2 p_\kappa^{-2} \|f + \nabla \cdot \{\mu(|\nabla u_{hp}|)\nabla u_{hp}\}\|_{L^2(\kappa)}^2 + h_\kappa p_\kappa^{-1} \|[\![\mu(|\nabla u_{hp}|)\nabla u_{hp}]\!]\|_{L^2(\partial\kappa\setminus\Gamma)}^2 + \gamma_{hp}^2 p_\kappa^3 h_\kappa^{-1} \|[\![u_{hp}]\!]\|_{L^2(\partial\kappa)}^2$$

Proof.

See Houston, Süli & Wihler 2008. □

Lemma (Two-Grid Quasilinear Approximation)

The following bound holds:

$$\|u - u_{2G}\|_{hp}^2 \leq C_2 \sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \xi_\kappa^2 \right).$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

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and the local two-grid error indicators are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\xi_\kappa^2 = \|(\mu(|\nabla u_{HP}|) - \mu(|\nabla u_{2G}|))\nabla u_{2G}\|_{L^2(\kappa)}^2.$$

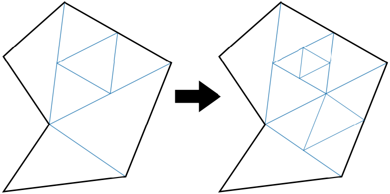
Proof.

See C., Houston, & Wihler 2013 for the case of a *normal* coarse mesh. This analysis is performed on the fine mesh and the only requirement on the coarse mesh is that $V_{HP}(\mathcal{T}_H, \boldsymbol{P}) \subseteq V_{hp}(\mathcal{T}_h, \boldsymbol{p})$, which still holds. □

Two-Grid Adaptivity

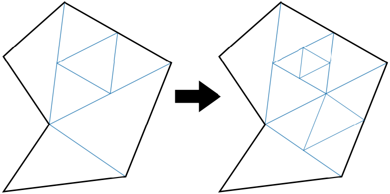
1. Construct initial coarse and fine FE spaces, with coarse mesh created by agglomerating the fine mesh.
2. Compute the coarse grid approximation and two-grid solution.
3. Select elements for refinement based on η_κ and ξ_κ:
   3.1 Use $\sqrt{\eta_K^2 + \xi_K^2}$ to determine set $\mathfrak{R}(\mathcal{T}_h) \subseteq \mathcal{T}_h$ of elements to refine.
   3.2 Choose fine or coarse mesh refinement. For all $\kappa \in \mathfrak{R}(\mathcal{T}_h)$
      - if $\lambda_F \xi_\kappa \leq \eta_\kappa$ refine the fine element κ, and
      - if $\lambda_C \eta_\kappa \leq \xi_\kappa$ refine the coarse element $\kappa_H \in \mathcal{T}_H$, where $\kappa \in \mathcal{T}_h(\kappa_H)$.
4. Perform *h*-/*hp*-mesh refinement of the fine space.
5. Select *h*- or *p*-refinement for each coarse element to refine.
6. Perform mesh smoothing to ensure any coarse element marked for refinement has at least 2^d child fine elements.
7. Perform *h*-/*hp*-refinement of the coarse space.
8. Goto 2.

Fine Element Refine:

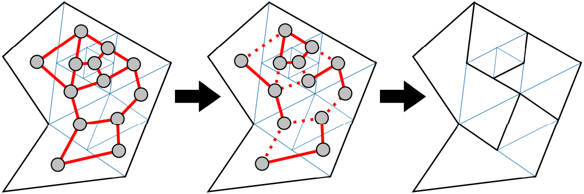


Coarse Element h-Refinement

Fine Element Refine:



Coarse Element Refine — Partition patch of fine elements into 2^d elements



Coarse Element *h*-Refinement

Using a standard graph partition algorithm will attempt to create agglomerated elements with the same number of *child* fine elements, minimising the number of edge cuts.

However, we have information about the error for each fine element — can we distribute the agglomeration using this information?

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Possible to assign *weights* to each vertex and use a graph partitioning algorithm that balances these weights, rather than the number of elements. [Karypis & Kumar 1998]

We set the weight to the total local error indicator: $\eta_\kappa^2 + \xi_\kappa^2$

Coarse Element *h*-Refinement

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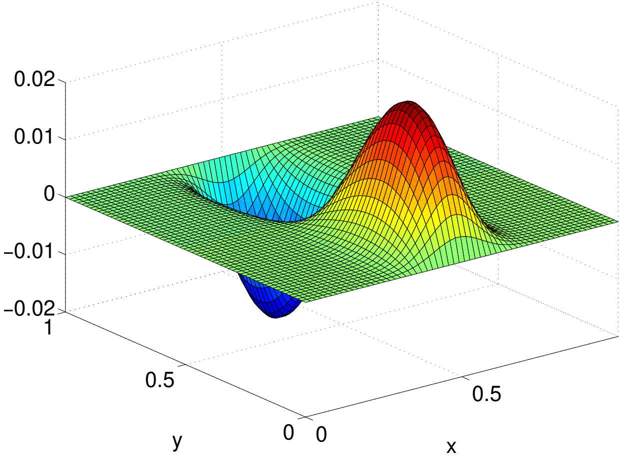
The coarse element refinement uses the fine elements *after* refinement; therefore, we divide the (square) of each error indicator equally between the new fine elements; i.e., $\eta_{\kappa_s} = {}^{\eta_\kappa}/\sqrt{N}$ and $\xi_{\kappa_s} = {}^{\xi_K}/\sqrt{N}$, for $s = 1, \ldots, N$, if κ is divided into N children $\kappa_1, \ldots, \kappa_N$.

Quasilinear PDE: Smooth Solution

We let $\Omega = (0,1)^2, \mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select f so that

$$u(x,y) = x(1-x)y(1-y)(1-2y)\mathrm{e}^{-20(2x-1)^2}.$$



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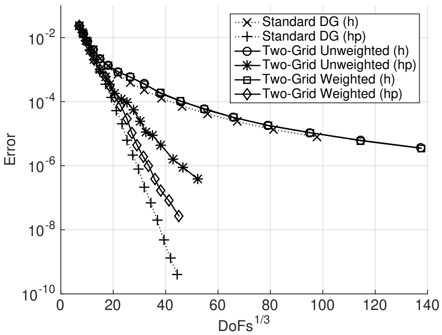
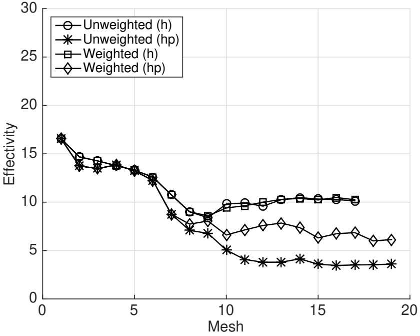
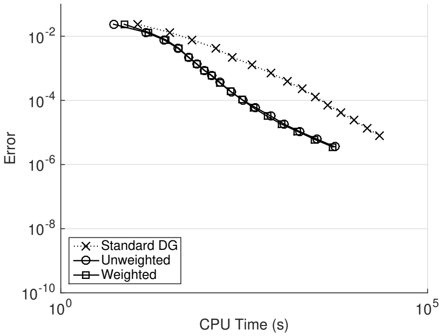


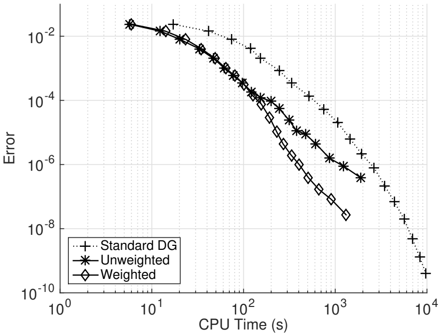
Error vs. #DoFs



Effectivity Indices



Error vs. CPU Time
h-refinement



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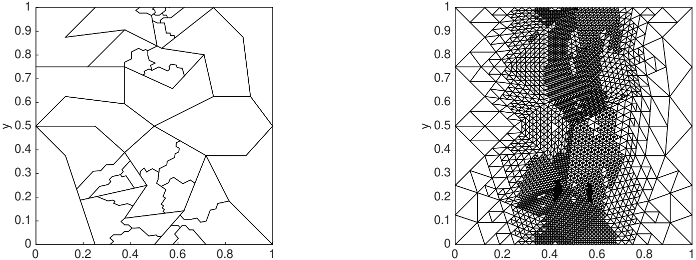
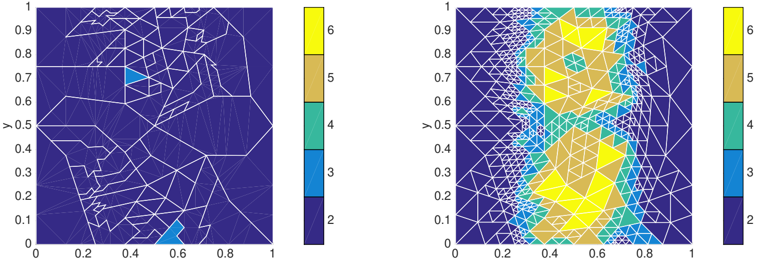


8 h-refinement (Weighted Coarse Refinement)



8 hp-refinement (Weighted Coarse Refinement)

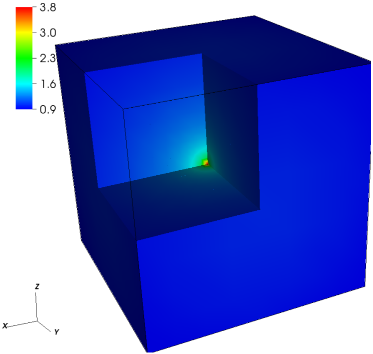
Quasilinear PDE: Singular Solution

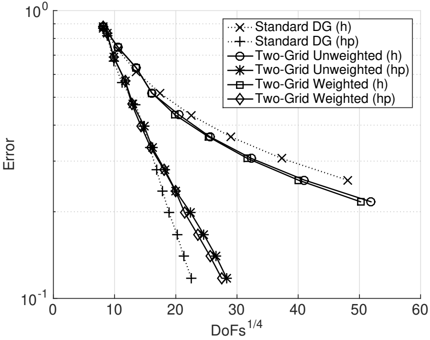
We let Ω be the Fichera corner $(-1,1)^3 \setminus [0,1)^3$, $\mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select f so that

$$u(\boldsymbol{x}) = (x^2 + y^2 + z^2)^{q/2}, \quad q \in \mathbb{R};$$

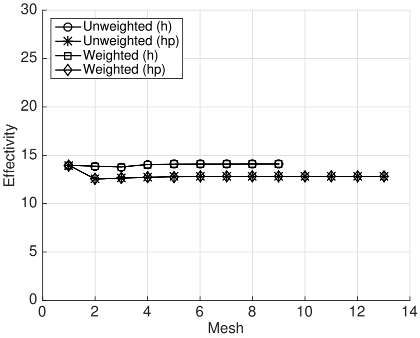
for $q > -1/2, u \in H^1(\Omega)$. Here, we select $q = -1/4$.

Beilina, Korotov & Křížek 2005



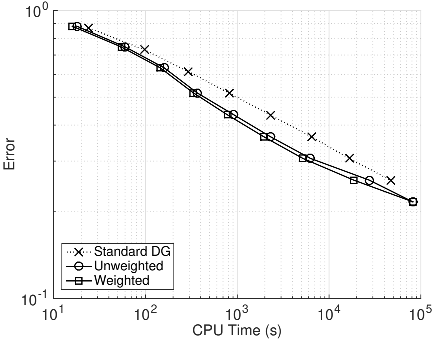


Error vs. #DoFs

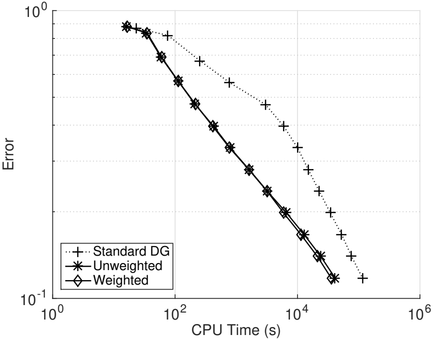


Effectivity Indices

Quasilinear PDE: Singular Solution



Error vs. CPU Time
h-refinement



Error vs. CPU Time
hp-refinement

Conclusion

Summary:

- Derived *a priori* error estimates for agglomerated coarse meshes.
- Two-Grid DG *a posteriori* error estimates still hold for agglomerated coarse mesh of polygons and fine mesh of simplices.
- We can adaptively refine the coarse mesh based on the error estimates.

Future Aims:

- Extend to general nonlinearities.
- Non-Newtonian fluids.