A posteriori error analysis of the virtual element method for second-order quasilinear elliptic PDE

Scott Congreve

Department of Numerical Mathematics,
Faculty of Mathematics & Physics,
Charles University

Joint work with
Alice Hodson (Charles University)

POlytopal Element Methods in Mathematics and Engineering, Inria Paris

CHARLES UNIVERSITY
Faculty of mathematics
and physics

Quasilinear Problem

Given polygonal domain $\Omega \subset \mathbb{R}^d, d = 2, 3$ and $f \in L^2(\Omega)$, find u such that

$$
\begin{aligned}
-\nabla \cdot \{\mu(\boldsymbol{x}, |\nabla u|)\nabla u\} &= f && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

Assumption

$\mu \in C(\bar{\Omega} \times [0, \infty))$ *and there exists positive constants* m_μ *and* M_μ *such that*

$$
m_\mu(t - s) \leq \mu(\boldsymbol{x}, t)t - \mu(\boldsymbol{x}, s)s \leq M_\mu(t - s), \quad t \geq s \geq 0, \quad \boldsymbol{x} \in \bar{\Omega}.
$$

Nonlinearities of this type appear in continuum mechanics

Quasilinear Problem

Given polygonal domain $\Omega \subset \mathbb{R}^d, d = 2, 3$ and $f \in L^2(\Omega)$, find u such that

$$
\begin{aligned}
-\nabla \cdot \{\mu(\boldsymbol{x}, |\nabla u|)\nabla u\} &= f && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

From the assumption we have that there exists positive constants C_1 and C_2

$$
\begin{aligned}
|\mu(\boldsymbol{x}, |\boldsymbol{v}|)\boldsymbol{v} - \mu(\boldsymbol{x}, |\boldsymbol{w}|)\boldsymbol{w}| &\leq C_1|\boldsymbol{v} - \boldsymbol{w}|, \\
C_2|\boldsymbol{v} - \boldsymbol{w}|^2 &\leq (\mu(\boldsymbol{x}, |\boldsymbol{v}|)\boldsymbol{v} - \mu(\boldsymbol{x}, |\boldsymbol{w}|)\boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w})
\end{aligned}
$$

for any $v, w \in \mathbb{R}^2$ and $\boldsymbol{x} \in \overline{\Omega}$. [Barrett & Liu, 1994]

Quasilinear Problem

Given polygonal domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ and $f \in L^2(\Omega)$, find u such that

$$
\begin{aligned}
-\nabla \cdot \{\mu(\boldsymbol{x}, |\nabla u|)\nabla u\} &= f && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

Weak formulation: Find $u \in H_0^1(\Omega)$ such that

$$
a(u; u, v) := \int_\Omega \mu(|\nabla u|)\nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} = \int_\Omega f v \, \mathrm{d}\boldsymbol{x},
$$

for all $v \in H_0^1(\Omega)$.

Why VEM/Polytopal Elements?

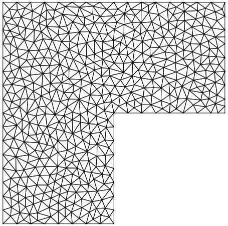
- Nonlinear problem $\implies$ solving via iteration (fixed point, Newton, etc.).
- Depending on the number of iterations and DoFs could be computationally expensive.
- Reduce computational expense — two-grid method: Solve nonlinear problem on a coarse mesh, and use to linearise on a fine mesh

  Xu 1992, 1994, 1996; Xu & Zhou 1999; Axelsson & Layton 1996; Dawson, Wheeler & Woodward 1998; Utnes 1997; Marion & Xu 1995; Wu & Allen 1999

Why VEM/Polytopal Elements?

- Nonlinear problem $\Longrightarrow$ solving via iteration (fixed point, Newton, etc.).
- Depending on the number of iterations and DoFs could be computationally expensive.
- Reduce computational expense — two-grid method: Solve nonlinear problem on a coarse mesh, and use to linearise on a fine mesh

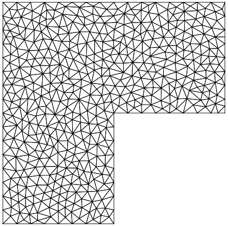
Xu 1992, 1994, 1996; Xu & Zhou 1999; Axelsson & Layton 1996; Dawson, Wheeler & Woodward 1998; Utnes 1997; Marion & Xu 1995; Wu & Allen 1999

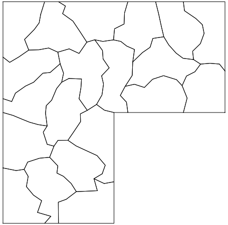


Why VEM/Polytopal Elements?

- Nonlinear problem $\implies$ solving via iteration (fixed point, Newton, etc.).
- Depending on the number of iterations and DoFs could be computationally expensive.
- Reduce computational expense — two-grid method: Solve nonlinear problem on a coarse mesh, and use to linearise on a fine mesh

Xu 1992, 1994, 1996; Xu & Zhou 1999; Axelsson & Layton 1996; Dawson, Wheeler & Woodward 1998; Utnes 1997; Marion & Xu 1995; Wu & Allen 1999

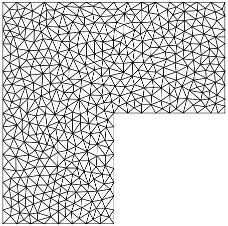


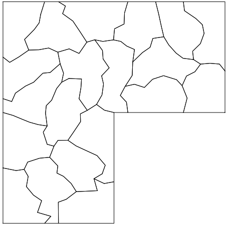


Why VEM/Polytopal Elements?

- Nonlinear problem $\implies$ solving via iteration (fixed point, Newton, etc.).
- Depending on the number of iterations and DoFs could be computationally expensive.
- Reduce computational expense — two-grid method: Solve nonlinear problem on a coarse mesh, and use to linearise on a fine mesh

Xu 1992, 1994, 1996; Xu & Zhou 1999; Axelsson & Layton 1996; Dawson, Wheeler & Woodward 1998; Utnes 1997; Marion & Xu 1995; Wu & Allen 1999





- How do we optimally construct the coarse mesh? Agglomeration and adaptive refinement. . .

PolyDG: C. & Houston 2022

Mesh

Construct mesh $\mathcal{T}_h$ of Ω consisting of simple polygons, with element diameter h_E, $E \in \mathcal{T}_h$.

Assumption (Mesh Regularity)

There exists $\rho > 0$ such that

- *each element $E \in \mathcal{T}_h$ star-shaped w.r.t ball of radius ρh_E*
- *$h_e \geq \rho h_E$ for every $E \in \mathcal{T}_h$ and $e \subset \partial E$*

Remark

As consequence each element $E \in \mathcal{T}_h$ admits a sub-triangulation into triangles.

On each element we consider a order of approximation ℓ.

VEM Space

Given a local enlarged VEM space

$$\widetilde{V}_{h,\ell}^{E} := \left\{ v_h \in H^1(\Omega) : \Delta v_h \in \mathbb{P}_\ell(E) \text{ and } v_h|_e \in \mathbb{P}_\ell(e)\ \forall e \subset \partial E \right\}$$

Given a local enlarged VEM space

$$\widetilde{V}_{h,\ell}^{E} := \left\{ v_h \in H^1(\Omega) : \Delta v_h \in \mathbb{P}_\ell(E) \text{ and } v_h|_e \in \mathbb{P}_\ell(e)\ \forall e \subset \partial E \right\}$$

and a value projection $\Pi_0^E : \widetilde{V}_{h,\ell}^{E} \to \mathbb{P}_\ell$ we define the local virtual element space $V_{h,\ell}^{E}$ as

$$V_{h,\ell}^{E} := \left\{ v_h \in \widetilde{V}_{h,\ell}^{E} : (v_h - \Pi_0^E v_h, p)_E = 0 \quad \forall p \in \mathbb{P}_\ell(E) \backslash \mathbb{P}_{\ell-2}(E) \right\}$$

Ahmad, Alsaedi, Brezzi, Marini, & Russo, 2013

Given a local enlarged VEM space

$$\widetilde{V}_{h,\ell}^E := \left\{ v_h \in H^1(\Omega) : \Delta v_h \in \mathbb{P}_\ell(E) \text{ and } v_h|_e \in \mathbb{P}_\ell(e)\ \forall e \subset \partial E \right\}$$

and a value projection $\Pi_0^E : \widetilde{V}_{h,\ell}^E \to \mathbb{P}_\ell$ we define the local virtual element space $V_{h,\ell}^E$ as

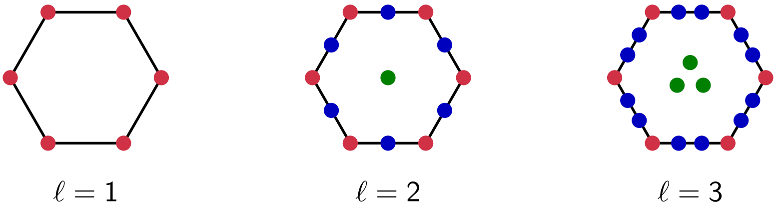
$$V_{h,\ell}^E := \left\{ v_h \in \widetilde{V}_{h,\ell}^E \ :\ (v_h - \Pi_0^E v_h, p)_E = 0 \quad \forall p \in \mathbb{P}_\ell(E) \backslash \mathbb{P}_{\ell-2}(E) \right\}$$

Ahmad, Alsaedi, Brezzi, Marini, & Russo, 2013

The global VEM space $V_{h,\ell}$ is defined as

$$V_{h,\ell} := \left\{ v_h \in H_0^1(\Omega) \ :\ v_h|_E \in V_{h,\ell}^E \quad \forall E \in \mathcal{T}_h \right\}$$

Local Degrees of Freedom



The local space $V_{h,\ell}^E$ is characterised by the degrees of freedom:

(D1) The value of v_h at each vertex of E

(D2) For $\ell > 1$, the moments of v_h up to order $\ell - 2$ on each edge $e \subset \partial E$

$$\frac{1}{|e|}\int_e v_h p \, \mathrm{d}s \quad \forall p \in \mathbb{P}_{\ell-2}(e)$$

(D3) For $\ell > 1$, the moments of v_h up to order $\ell - 2$ inside E

$$\frac{1}{|E|}\int_E v_h p \, \mathrm{d}\boldsymbol{x} \quad \forall p \in \mathbb{P}_{\ell-2}(E)$$

Value projection ($\Pi_0^E : \widetilde{V}_{h,\ell}^E \to \mathbb{P}_\ell(E)$) $\Pi_0^E v_h$ linear combination of dofs, and satisfies

$$\int_E \Pi_0^E v_h p \, \mathrm{d}\boldsymbol{x} = \int_E v_h p \, \mathrm{d}\boldsymbol{x} \quad \forall p \in \mathbb{P}_{\ell-2}(E), \qquad \text{and} \qquad \Pi_0^E q = q \quad \forall q \in \mathbb{P}_\ell(E).$$

Edge projection ($\Pi_0^e : \widetilde{V}_{h,\ell}^E \to \mathbb{P}_\ell(e)$) $\Pi_0^e v_h$ linear combination of dofs, and satisfies

$$\Pi_0^e v_h(e^\pm) = v_h(e^\pm),$$

$$\int_e \Pi_0^e v_h p \, \mathrm{d}s = \int_e v_h p \, \mathrm{d}s \quad \forall p \in \mathbb{P}_{\ell-2}(e), \qquad \text{and} \qquad \Pi_0^e q = q|_e \quad \forall q \in \mathbb{P}_\ell(E).$$

Gradient projection ($\Pi_1^E : \widetilde{V}_{h,\ell}^E \to [\mathbb{P}_\ell(E)]^2$)

$$\int_E \Pi_1^E v_h \cdot \boldsymbol{p} \, \mathrm{d}\boldsymbol{x} = -\int_E \Pi_0^E v_h \nabla \cdot \boldsymbol{p} \, \mathrm{d}\boldsymbol{x} + \sum_{e \subset \partial E} \int_e \Pi_0^e v_h \boldsymbol{p} \cdot \boldsymbol{n}_e \, \mathrm{d}s \quad \forall \boldsymbol{p} \in [\mathbb{P}_{\ell-1}(E)]^2.$$

Here, $e \subset E$ is an element edge, and $e^\pm$ denotes the vertices of e.

Use CLS for choice of projections: Dedner & Hodson 2024

VEM Formulation

Find $u_h \in V_{h,\ell}$ such that

$$a_h(u_h; u_h, v_h) = L_h(v_h) \qquad \text{for all } v_h \in V_{h,\ell}.$$

VEM Formulation

Find $u_h \in V_{h,\ell}$ such that

$$a_h(u_h; u_h, v_h) = L_h(v_h) \qquad \text{for all } v_h \in V_{h,\ell}.$$

Here,

$$a_h(z_h, v_h, w_h) = \sum_{E \in \mathcal{T}_h} a_h^E(z_h, v_h, w_h),$$

$$a_h^E(z_h, v_h, w_h) = \int_E \mu(|\Pi_1^E z_h|)\Pi_1^E v_h \cdot \Pi_1^E w_h \,\mathrm{d}\boldsymbol{x} + S^E(z_h; (I - \Pi_0^E)v_h, (I - \Pi_0^E)w_h),$$

$$L_h(v_h) = \sum_{E \in \mathcal{T}_h} \int_E \Pi_0^E f v_h \,\mathrm{d}\boldsymbol{x},$$

where S^E is a stabilisation to be defined.

Theorem (Existence and Uniqueness)

For $f \in L^2(\Omega)$ there exists a unique solution $u_h \in V_{h,\ell}$ to the VEM formulation.

Theorem (Existence and Uniqueness)

For $f \in L^2(\Omega)$ there exists a unique solution $u_h \in V_{h,\ell}$ to the VEM formulation.

Proof:

- Prove a_h is strongly monotone

$$a_h(w_h; w_h, w_h - z_h) - a_h(z_h; z_h, w_h - z_h) \geq C|w_h - z_h|_1^2 \qquad \forall w_h, z_h \in V_{h,\ell}$$

and Lipschitz continuous

$$|a_h(w_h; w_h, v_h) - a_h(z_h; z_h, v_h)| \leq C|w_h - z_h|_1 |v_h|_1 \qquad \forall v_h, w_h, z_h \in V_{h,\ell}$$

Theorem (Existence and Uniqueness)

For $f \in L^2(\Omega)$ there exists a unique solution $u_h \in V_{h,\ell}$ to the VEM formulation.

Proof:

- Prove a_h is strongly monotone

$$a_h(w_h; w_h, w_h - z_h) - a_h(z_h; z_h, w_h - z_h) \geq C|w_h - z_h|_1^2 \qquad \forall w_h, z_h \in V_{h,\ell}$$

  and Lipschitz continuous

$$|a_h(w_h; w_h, v_h) - a_h(z_h; z_h, v_h)| \leq C|w_h - z_h|_1 |v_h|_1 \qquad \forall v_h, w_h, z_h \in V_{h,\ell}$$

- Result follows from theory of monotone operators
  C. & Hodson (Submitted); Houston, Robson, & Süli 2005

CHARLES UNIVERSITY
Faculty of mathematics
and physics

Theorem (Existence and Uniqueness)

For $f \in L^2(\Omega)$ there exists a unique solution $u_h \in V_{h,\ell}$ to the VEM formulation.

Proof:

- Prove a_h is strongly monotone

$$a_h(w_h; w_h, w_h - z_h) - a_h(z_h; z_h, w_h - z_h) \geq C|w_h - z_h|_1^2 \qquad \forall w_h, z_h \in V_{h,\ell}$$

  and Lipschitz continuous

$$|a_h(w_h; w_h, v_h) - a_h(z_h; z_h, v_h)| \leq C|w_h - z_h|_1 |v_h|_1 \qquad \forall v_h, w_h, z_h \in V_{h,\ell}$$

- Result follows from theory of monotone operators

C. & Hodson (Submitted); Houston, Robson, & Süli 2005

Prove of the volume term follows from properties of μ, so only need to prove for stabilisation.

The stabilisation must satisfy the following:

- admissible stabilisation; i.e., $\exists C_*, C^*$, independent of h, E, such that,

$$C_* a^E(z_h; v_h, v_h) \leq S^E(z_h; v_h, v_h) \leq C^* a^E(z_h; v_h, v_h) \qquad \forall z_h, v_h \in V_{h,\ell}^E, \forall E \in \mathcal{T}_h.$$

The stabilisation must satisfy the following:

- admissible stabilisation; i.e., $\exists C_*, C^*$, independent of h, E, such that,

$$C_* a^E(z_h; v_h, v_h) \leq S^E(z_h; v_h, v_h) \leq C^* a^E(z_h; v_h, v_h) \qquad \forall z_h, v_h \in V_{h,\ell}^E, \forall E \in \mathcal{T}_h.$$

- and either
    - S^E is independent of the first argument and linear in the other two, or

Stabilisation

The stabilisation must satisfy the following:

- admissible stabilisation; i.e., $\exists C_*, C^*$, independent of h, E, such that,

$$C_* a^E(z_h; v_h, v_h) \leq S^E(z_h; v_h, v_h) \leq C^* a^E(z_h; v_h, v_h) \qquad \forall z_h, v_h \in V^E_{h,\ell}, \forall E \in \mathcal{T}_h.$$

- and either
  - S^E is independent of the first argument and linear in the other two, or
  - it is strongly monotone and Lipschitz continuous in the sense that

$$\begin{aligned} &S^E(w_h; (I - \Pi^E_0) w_h, (I - \Pi^E_0)(w_h - z_h)) \\ &\quad - S^E(z_h; (I - \Pi^E_0) z_h, (I - \Pi^E_0)(w_h - z_h)) \geq C|w_h - z_h|_1^2 \qquad \forall w_h, z_h \in V_{h,\ell} \\ &|S^E(w_h; (I - \Pi^E_0) w_h, (I - \Pi^E_0) v_h) \\ &\quad - S^E(z_h; (I - \Pi^E_0) z_h, (I - \Pi^E_0) v_h)| \leq C|w_h - z_h|_1 |v_h|_1 \quad \forall v_h, w_h, z_h \in V_{h,\ell} \end{aligned}$$

Stabilisation

We use dofi-dofi as the basis and propose several stabilisations:

- weighted by the constants from the non-linearity; e.g.,

$$S^E(z_h; v_h, w_h) := M_\mu m_\mu \sum_{\lambda \in \Lambda^E} \lambda(v_h)\lambda(w_h).$$

  This is admissible, and results in well-posed formulation.

We use dofi-dofi as the basis and propose several stabilisations:

- weighted by the constants from the non-linearity; e.g.,

$$S^E(z_h; v_h, w_h) := M_\mu m_\mu \sum_{\lambda \in \Lambda^E} \lambda(v_h)\lambda(w_h).$$

  This is admissible, and results in well-posed formulation.

- weighted by the average over the element; i.e.,

$$S^E(z_h; v_h, w_h) := \mu_E(\boldsymbol{x}, |\Pi_1^{E,0} z_h|) \sum_{\lambda \in \Lambda^E} \lambda(v_h)\lambda(w_h),$$

  where $\Pi_1^{E,0}$ is gradient projection onto constants, and $\mu_E(\cdot)$ denotes the average of μ.
  Adak, Arrutselvi, Natarajan, Natarajan, 2022; Cangiani, Chatzipantelidis, Diwan, Georgoulis, 2020
  This is admissible, but unable to prove well-posed without additional assumptions on μ.

We use dofi-dofi as the basis and propose several stabilisations:

- weighted by the constants from the non-linearity; e.g.,

$$S^E(z_h; v_h, w_h) := M_\mu m_\mu \sum_{\lambda \in \Lambda^E} \lambda(v_h)\lambda(w_h).$$

  This is admissible, and results in well-posed formulation.

- weighted by the average over the element; i.e.,

$$S^E(z_h; v_h, w_h) := \mu_E(\boldsymbol{x}, |\Pi_1^{E,0} z_h|) \sum_{\lambda \in \Lambda^E} \lambda(v_h)\lambda(w_h),$$

  where $\Pi_1^{E,0}$ is gradient projection onto constants, and $\mu_E(\cdot)$ denotes the average of μ.
  Adak, Arrutselvi, Natarajan, Natarajan, 2022; Cangiani, Chatzipantelidis, Diwan, Georgoulis, 2020
  This is admissible, but unable to prove well-posed without additional assumptions on μ.

- multiplied by nonlinearity applied to dof; i.e.,

$$S^E(z_h; v_h, w_h) := \sum_{\lambda \in \Lambda^E} \mu(|\lambda(z_h)|)\lambda(v_h)\lambda(w_h).$$

We first quote a key result:

Theorem (Approximation using VEM functions)

Under the mesh regularity assumptions, for any $w \in H^1(\Omega)$ there exists a $w_I \in V_{h,\ell}$ such that for all $E \in \mathcal{T}_h$

$$\|w - w_I\|_{0,E} + h_E|w - w_I|_{1,E} \leq Ch_E|w|_{1,E}$$

where C depends only on ℓ and mesh regularity.

Mora, Rivera, & Rodríguez, 2015; Cangiani, Georgoulis, Pryer, & Sutton, 2017

We also note that

$$\Pi_1^E v_h = \mathcal{P}_{\ell-1}^E(\nabla v_h)$$

where $\mathcal{P}_{\ell-1}^E$ is the L^2-orthogonal projection onto $\mathbb{P}_{\ell-1}$.

Dedner & Hodson, 2022

Theorem (Upper bound [C. & Hodson (Submitted)])

$$|u - u_h|_1^2 \leq C \sum_{E \in \mathcal{T}_h} (\eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2)$$

where

$$\eta_E^2 := h_E^2 \| f_h + \nabla \cdot \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h \|_{0,E}^2 + \sum_{e \subset \partial E} h_e \| [\![\mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h]\!] \|_{0,e}^2,$$

$$\Theta_E^2 := h_E^2 \| f - f_h + \nabla \cdot (\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h - \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h) \|_{0,E}^2 + h_E^2 \| f - f_h \|_{0,E}^2 + \sum_{e \subset \partial E} h_e \| [\![(\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|) - \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|)) \mathcal{P}_{\ell-1}^E \nabla u_h]\!] \|_{0,e}^2,$$

$$\mathcal{S}_E^2 := S^E(u_h; (I - \mathcal{P}_\ell^E) u_h, (I - \mathcal{P}_\ell^E) u_h),$$

$$\Psi_E^2 := \| (\mathcal{P}_{\ell-1}^E - I)(\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h) \|_{0,E}^2.$$

CHARLES UNIVERSITY
Faculty of mathematics and physics

Theorem (Upper bound [C. & Hodson (Submitted)])

$$|u - u_h|_1^2 \leq C \sum_{E \in \mathcal{T}_h} (\eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2)$$

where

$$\eta_E^2 := h_E^2 \| f_h + \nabla \cdot \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h \|_{0,E}^2 + \sum_{e \subset \partial E} h_e \| [\![\mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h]\!] \|_{0,e}^2,$$

$$\Theta_E^2 := h_E^2 \| f - f_h + \nabla \cdot (\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h - \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h) \|_{0,E}^2$$
$$+ h_E^2 \| f - f_h \|_{0,E}^2 + \sum_{e \subset \partial E} h_e \| [\![(\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|) - \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|))\mathcal{P}_{\ell-1}^E \nabla u_h]\!] \|_{0,e}^2,$$

$$\mathcal{S}_E^2 := S^E(u_h; (I - \mathcal{P}_\ell^E)u_h, (I - \mathcal{P}_\ell^E)u_h),$$

$$\Psi_E^2 := \| (\mathcal{P}_{\ell-1}^E - I)(\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h) \|_{0,E}^2.$$

Theorem (Upper bound [C. & Hodson (Submitted)])

$$|u - u_h|_1^2 \leq C \sum_{E \in \mathcal{T}_h} (\eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2)$$

where

$$\eta_E^2 := h_E^2 \|f_h + \nabla \cdot \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h\|_{0,E}^2 + \sum_{e \subset \partial E} h_e \|[\![\mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h]\!]\|_{0,e}^2,$$

$$\Theta_E^2 := h_E^2 \|f - f_h + \nabla \cdot (\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h - \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h)\|_{0,E}^2 + h_E^2 \|f - f_h\|_{0,E}^2 + \sum_{e \subset \partial E} h_e \|[\![(\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|) - \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|))\mathcal{P}_{\ell-1}^E \nabla u_h]\!]\|_{0,e}^2,$$

$$\mathcal{S}_E^2 := S^E(u_h; (I - \mathcal{P}_\ell^E)u_h, (I - \mathcal{P}_\ell^E)u_h),$$

$$\Psi_E^2 := \|(\mathcal{P}_{\ell-1}^E - I)(\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h)\|_{0,E}^2.$$

Theorem (Upper bound [C. & Hodson (Submitted)])

$$|u - u_h|_1^2 \leq C \sum_{E \in \mathcal{T}_h} (\eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2)$$

where

$$\eta_E^2 := h_E^2 \| f_h + \nabla \cdot \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h \|_{0,E}^2 + \sum_{e \subset \partial E} h_e \| [\![\mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h]\!] \|_{0,e}^2,$$

$$\Theta_E^2 := h_E^2 \| f - f_h + \nabla \cdot (\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h - \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h) \|_{0,E}^2 + h_E^2 \| f - f_h \|_{0,E}^2 + \sum_{e \subset \partial E} h_e \| [\![(\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|) - \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|)) \mathcal{P}_{\ell-1}^E \nabla u_h]\!] \|_{0,e}^2,$$

$$\mathcal{S}_E^2 := S^E(u_h; (I - \mathcal{P}_\ell^E) u_h, (I - \mathcal{P}_\ell^E) u_h),$$

$$\Psi_E^2 := \| (\mathcal{P}_{\ell-1}^E - I)(\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h) \|_{0,E}^2.$$

Theorem (Upper bound [C. & Hodson (Submitted)])

$$|u - u_h|_1^2 \leq C \sum_{E \in \mathcal{T}_h} (\eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2)$$

where

$$\eta_E^2 := h_E^2 \|f_h + \nabla \cdot \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h\|_{0,E}^2 + \sum_{e \subset \partial E} h_e \|[\![\mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h]\!]\|_{0,e}^2,$$

$$\Theta_E^2 := h_E^2 \|f - f_h + \nabla \cdot (\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h - \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h)\|_{0,E}^2$$
$$+ h_E^2 \|f - f_h\|_{0,E}^2 + \sum_{e \subset \partial E} h_e \|[\![(\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|) - \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|))\mathcal{P}_{\ell-1}^E \nabla u_h]\!]\|_{0,e}^2,$$

$$\mathcal{S}_E^2 := S^E(u_h; (I - \mathcal{P}_\ell^E)u_h, (I - \mathcal{P}_\ell^E)u_h),$$

$$\Psi_E^2 := \|(\mathcal{P}_{\ell-1}^E - I)(\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|)\mathcal{P}_{\ell-1}^E \nabla u_h)\|_{0,E}^2.$$

Theorem (Upper bound [C. & Hodson (Submitted)])

$$|u - u_h|_1^2 \leq C \sum_{E \in \mathcal{T}_h} (\eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2)$$

where

$$\eta_E^2 := h_E^2 \| f_h + \nabla \cdot \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h \|_{0,E}^2 + \sum_{e \subset \partial E} h_e \| [\![\mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h]\!] \|_{0,e}^2,$$

$$\Theta_E^2 := h_E^2 \| f - f_h + \nabla \cdot (\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h - \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h) \|_{0,E}^2 + h_E^2 \| f - f_h \|_{0,E}^2 + \sum_{e \subset \partial E} h_e \| [\![(\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|) - \mu_h(|\mathcal{P}_{\ell-1}^E \nabla u_h|)) \mathcal{P}_{\ell-1}^E \nabla u_h]\!] \|_{0,e}^2,$$

$$\mathcal{S}_E^2 := S^E(u_h; (I - \mathcal{P}_\ell^E) u_h, (I - \mathcal{P}_\ell^E) u_h),$$

$$\Psi_E^2 := \| (\mathcal{P}_{\ell-1}^E - I)(\mu(|\mathcal{P}_{\ell-1}^E \nabla u_h|) \mathcal{P}_{\ell-1}^E \nabla u_h) \|_{0,E}^2.$$

Corollary

$$|u - \Pi_0^h u_h|_1^2 \leq \overline{C} \sum_{E \in \mathcal{T}_h} (\eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2)$$

$$\|\nabla u - \Pi_1^h u_h\|_0^2 \leq \widehat{C} \sum_{E \in \mathcal{T}_h} (\eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2)$$

CHARLES UNIVERSITY
Faculty of mathematics and physics

Corollary

$$|u - \Pi_0^h u_h|_1^2 \leq \overline{C} \sum_{E \in \mathcal{T}_h} (\eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2)$$

$$\|\nabla u - \Pi_1^h u_h\|_0^2 \leq \widehat{C} \sum_{E \in \mathcal{T}_h} (\eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2)$$

Theorem (Local lower bound [C. & Hodson (Submitted)])

For each element $E \in \mathcal{T}_h$

$$\eta_E^2 \leq C \sum_{E' \in \omega_E} \left(\|\nabla (u - u_h)\|_{0,E'}^2 + \mathcal{S}_{E'}^2 + \Theta_{E'}^2 \right)$$

where ω_E denotes the patch of elements containing E and its neighbouring elements.

Adaptive Refinement

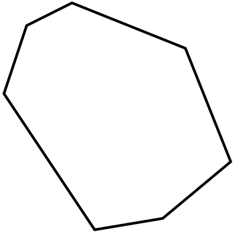
- Mark for refinement elements $E \in \mathcal{T}_h$ based on error indicators using Dörfler marking; i.e., construct the smallest subset of elements $\mathcal{T}_h^M \subset \mathcal{T}_h$ such that, for given $\theta \in (0, 1)$,

$$\left(\sum_{E \in \mathcal{T}_h^M} \eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2 \right)^{1/2} \geq \theta \left(\sum_{E \in \mathcal{T}_h} \eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2 \right)^{1/2},$$

- Mark for refinement elements $E \in \mathcal{T}_h$ based on error indicators using Dörfler marking; i.e., construct the smallest subset of elements $\mathcal{T}_h^M \subset \mathcal{T}_h$ such that, for given $\theta \in (0,1)$,

$$\left(\sum_{E \in \mathcal{T}_h^M} \eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2 \right)^{1/2} \geq \theta \left(\sum_{E \in \mathcal{T}_h} \eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2 \right)^{1/2},$$

- Refine polygon by joining midpoint of each edge to the barycentre of the element

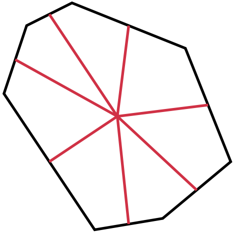


CHARLES UNIVERSITY
Faculty of mathematics and physics

- Mark for refinement elements $E \in \mathcal{T}_h$ based on error indicators using Dörfler marking; i.e., construct the smallest subset of elements $\mathcal{T}_h^M \subset \mathcal{T}_h$ such that, for given $\theta \in (0,1)$,

$$\left(\sum_{E \in \mathcal{T}_h^M} \eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2 \right)^{1/2} \geq \theta \left(\sum_{E \in \mathcal{T}_h} \eta_E^2 + \Theta_E^2 + \mathcal{S}_E^2 + \Psi_E^2 \right)^{1/2},$$

- Refine polygon by joining midpoint of each edge to the barycentre of the element

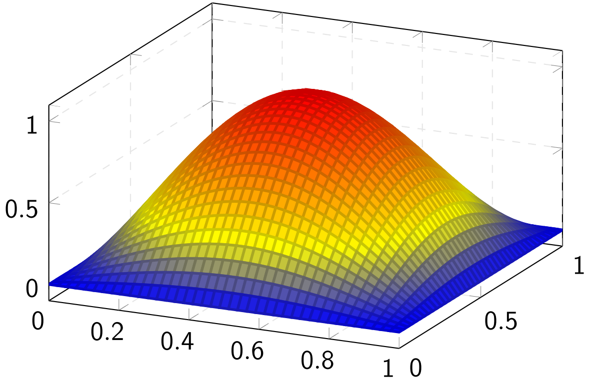


(assumes convex, but can use any point the element is star-shaped w.r.t.)

CHARLES UNIVERSITY
Faculty of mathematics and physics

We let $\Omega = (0,1)^2$, define $\mu(\boldsymbol{x}, |\nabla u|) = 2 + (1 + |\nabla u|^2)^{-1}$ and select f such that

$$u(x, y) = \sin(\pi x)\sin(\pi y).$$

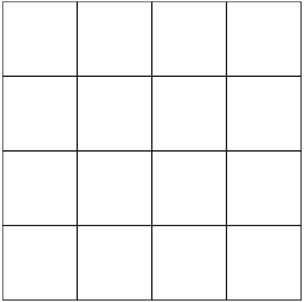


Numerical Experiments

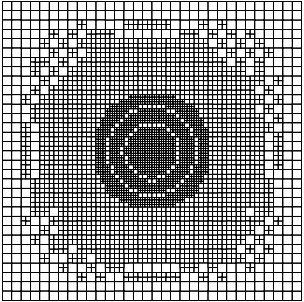
We let $\Omega = (0,1)^2$, define $\mu(\boldsymbol{x}, |\nabla u|) = 2 + (1 + |\nabla u|^2)^{-1}$ and select f such that

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

First consider quadrilateral elements:



Initial mesh

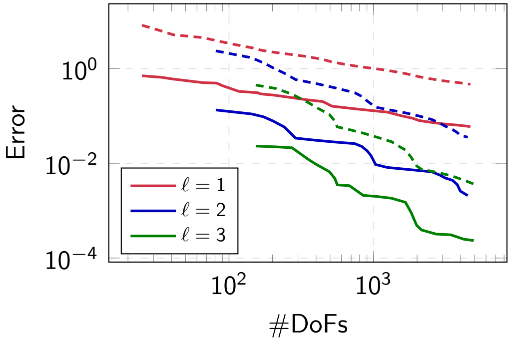


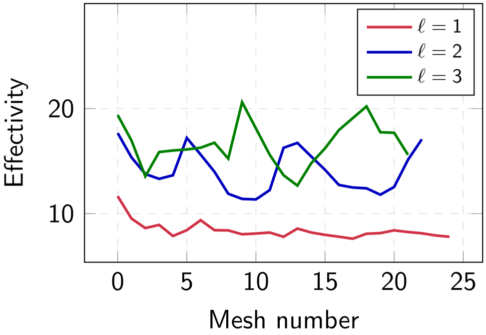
After 23 refinements

We let $\Omega = (0,1)^2$, define $\mu(\boldsymbol{x}, |\nabla u|) = 2 + (1 + |\nabla u|^2)^{-1}$ and select f such that

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

First consider quadrilateral elements:





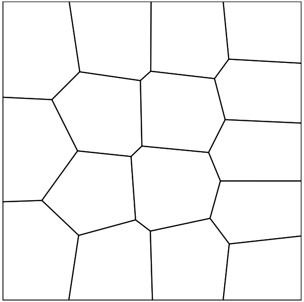
Error Bound & H^1-error ($\|\nabla u - \Pi_1^h u_h\|_{0,\Omega}$)

Effectivity

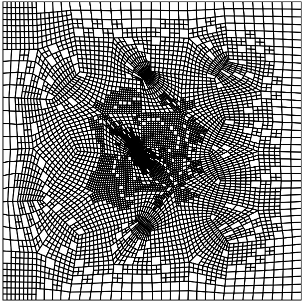
We let $\Omega = (0,1)^2$, define $\mu(\boldsymbol{x}, |\nabla u|) = 2 + (1 + |\nabla u|^2)^{-1}$ and select f such that

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

Now consider voronoi elements:



Initial mesh

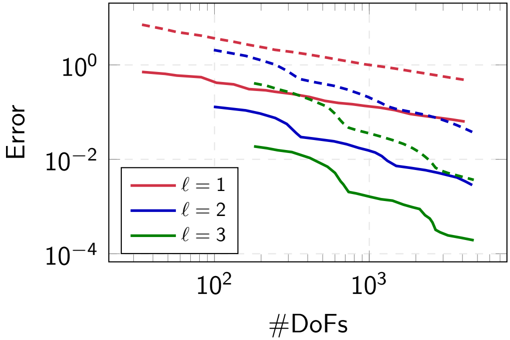


After 23 refinements

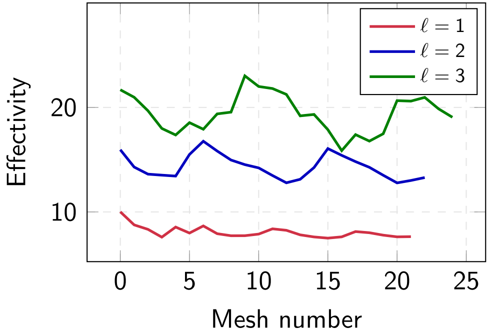
We let $\Omega = (0,1)^2$, define $\mu(\boldsymbol{x}, |\nabla u|) = 2 + (1 + |\nabla u|^2)^{-1}$ and select f such that

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

Now consider voronoi elements:



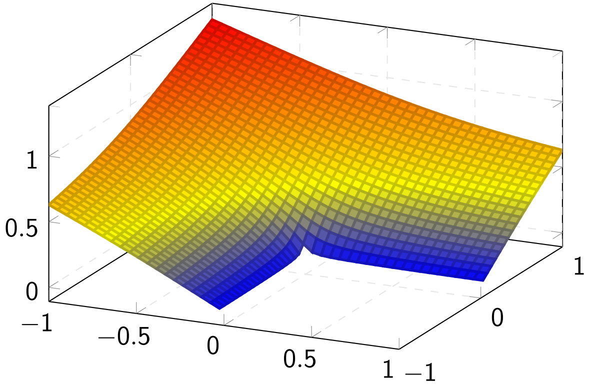
Error Bound & H^1-error ($\|\nabla u - \Pi_1^h u_h\|_{0,\Omega}$)



Effectivity

We let $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$, define $\mu(\boldsymbol{x}, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2}$ and select f such that

$$u(r,\theta) = r^{2/3} \sin\left({}^{2\theta}/_{3}\right).$$

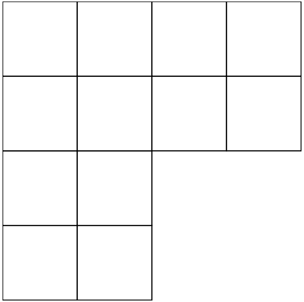


CHARLES UNIVERSITY
Faculty of mathematics and physics

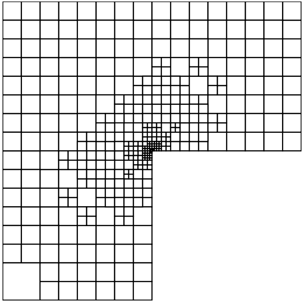
We let $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$, define $\mu(\boldsymbol{x}, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2}$ and select f such that

$$u(r,\theta) = r^{2/3} \sin\left(2\theta/3\right).$$

First consider quadrilateral elements:



Initial mesh

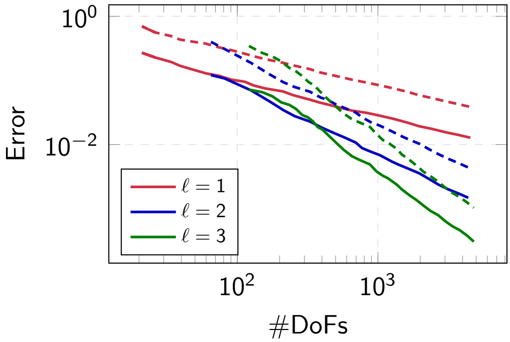


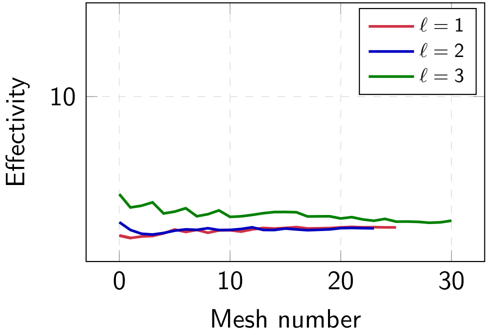
After 15 refinements

We let $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$, define $\mu(\boldsymbol{x}, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2}$ and select f such that

$$u(r,\theta) = r^{2/3} \sin\left({}^{2\theta}/_{3}\right).$$

First consider quadrilateral elements:



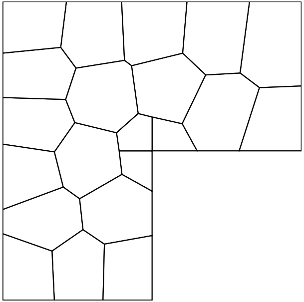


Error Bound & H^1-error ($\|\nabla u - \Pi_1^h u_h\|_{0,\Omega}$) Effectivity

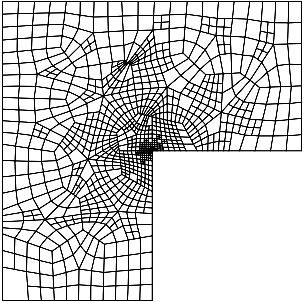
We let $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$, define $\mu(\boldsymbol{x}, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2}$ and select f such that

$$u(r,\theta) = r^{2/3} \sin\left({}^{2\theta}/_{3}\right).$$

Now consider voronoi elements:



Initial mesh

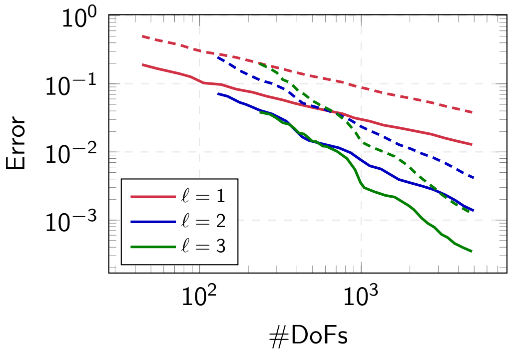


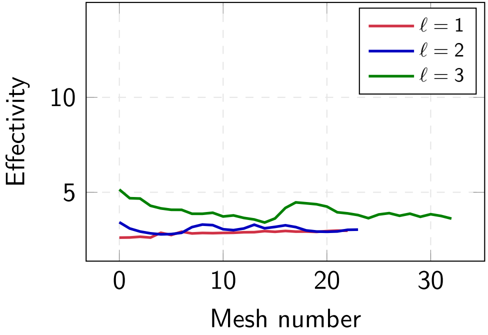
After 15 refinements

We let $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$, define $\mu(\boldsymbol{x}, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2}$ and select f such that

$$u(r,\theta) = r^{2/3} \sin\left({}^{2\theta}/{}_{3}\right).$$

Now consider voronoi elements:





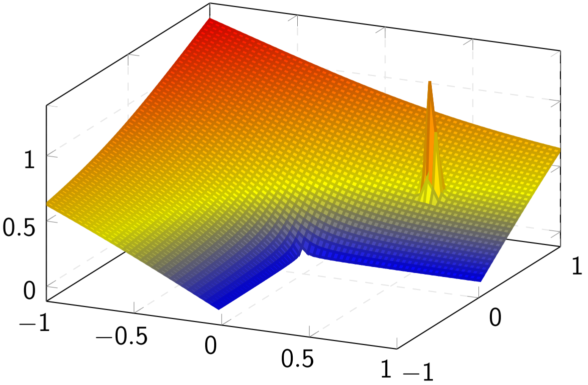
Error Bound & H^1-error ($\|\nabla u - \Pi_1^h u_h\|_{0,\Omega}$)

Effectivity

Numerical Experiments

We let $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$, define $\mu(\boldsymbol{x}, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2}$ and select f such that

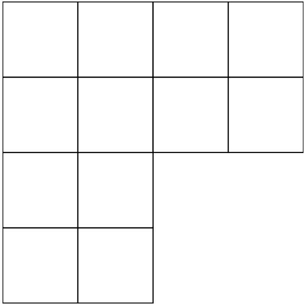
$$u(r,\theta) = r^{2/3} \sin\left({}^{2\theta}/_{3}\right) + \mathrm{e}^{-(1000(x-0.5)^2 + 1000(y-0.5)^2)}.$$



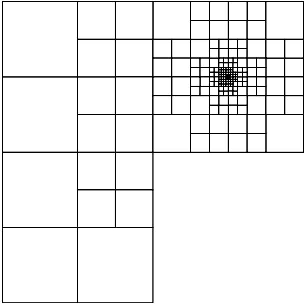
We let $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$, define $\mu(\boldsymbol{x}, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2}$ and select f such that

$$u(r,\theta) = r^{2/3} \sin\left({}^{2\theta}/_{3}\right) + \mathrm{e}^{-(1000(x-0.5)^2+1000(y-0.5)^2)}.$$

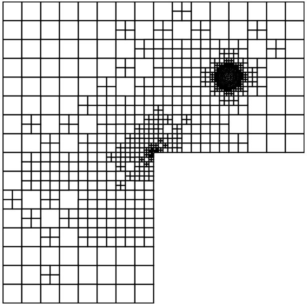
First consider quadrilateral elements:



Initial mesh



After 20 refinements



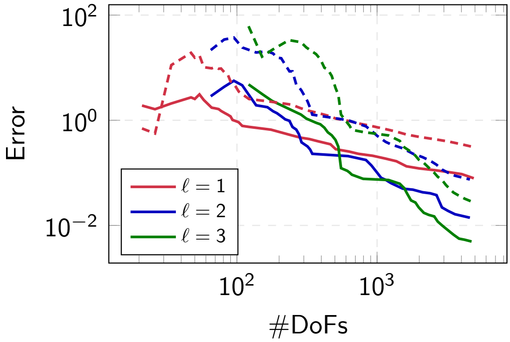
After 37 refinements

CHARLES UNIVERSITY
Faculty of mathematics
and physics

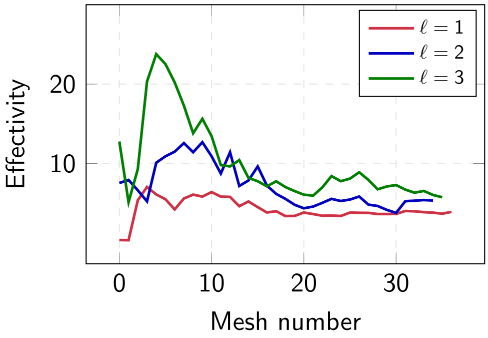
We let $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$, define $\mu(\boldsymbol{x}, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2}$ and select f such that

$$u(r,\theta) = r^{2/3} \sin\left(2\theta/3\right) + \mathrm{e}^{-(1000(x-0.5)^2 + 1000(y-0.5)^2)}.$$

First consider quadrilateral elements:



Error Bound & H^1-error ($\|\nabla u - \Pi_1^h u_h\|_{0,\Omega}$)



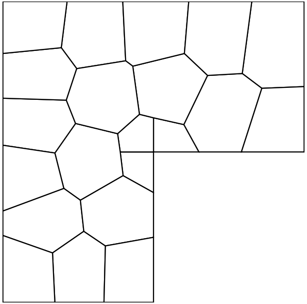
Effectivity

Numerical Experiments

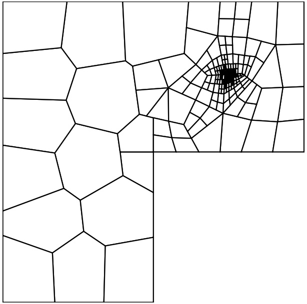
We let $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$, define $\mu(\boldsymbol{x}, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2}$ and select f such that

$$u(r,\theta) = r^{2/3} \sin\left(2\theta/3\right) + \mathrm{e}^{-(1000(x-0.5)^2 + 1000(y-0.5)^2)}.$$

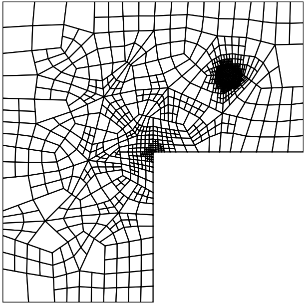
Now consider voronoi elements:



Initial mesh



After 13 refinements

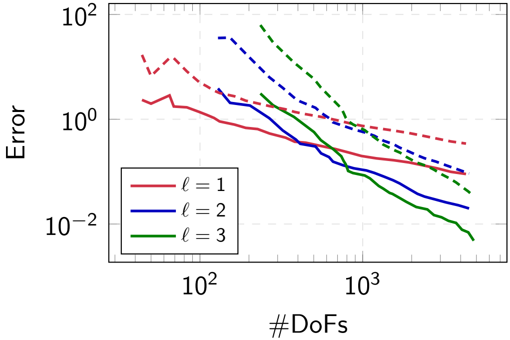


After 27 refinements

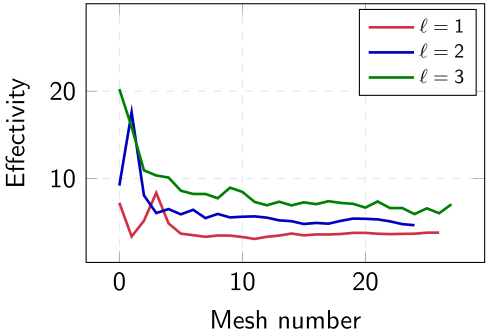
We let $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$, define $\mu(\boldsymbol{x}, |\nabla u|) = 1 + e^{-|\nabla u|^2}$ and select f such that

$$u(r,\theta) = r^{2/3} \sin\left(2\theta/3\right) + e^{-(1000(x-0.5)^2 + 1000(y-0.5)^2)}.$$

Now consider voronoi elements:



Error Bound & H^1-error ($\|\nabla u - \Pi_1^h u_h\|_{0,\Omega}$)



Effectivity

Conclusions

- Conforming VEM for quasilinear PDE
- Well-posedness & implication to stabilisations
- Energy norm residual based error bounds and indicators

Conclusions & Future

Conclusions

- Conforming VEM for quasilinear PDE
- Well-posedness & implication to stabilisations
- Energy norm residual based error bounds and indicators

Future

- *hp*-VEM
- quasi-Newtonian
- Two-grid
- Interpolation result for agglomerated elements