Nonconforming virtual element method for the Monge-Ampère equation

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Monge-Ampère

Given a convex domain $\Omega \subset \mathbb{R}^2$ with smooth/piecewise boundary and $f = f(\nabla u, u, \boldsymbol{x})$, find u such that

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For non-strictly convex domains it is known that the above equation does not have classical solutions in general. However, for $f > 0$ we have unique generalized solution in class of convex functions (may still have non-convex solutions). [Aleksandrov, 1961]

Vanishing Moment Method

We approximate the Monge-Ampère equation by a sequence of higher order PDEs:

$$-\varepsilon\Delta^2 u^\varepsilon + \det(D^2 u^\varepsilon) = f, \qquad \text{in } \Omega, \tag{1}$$

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Definition (Vanishing Moment Method [Feng & Neilan, 2007])

Suppose that u^ε solves (1) for each $\varepsilon > 0$, we call $\lim_{\varepsilon\to 0^+} u^\varepsilon$ a moment solution of the Monge-Ampère equation provided that the limit exists.

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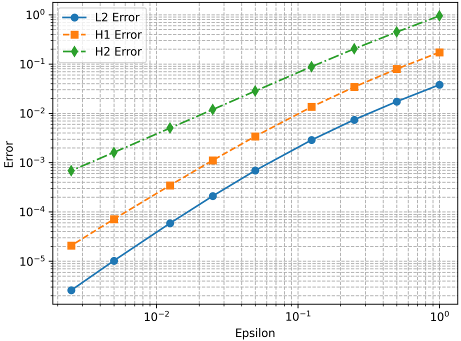
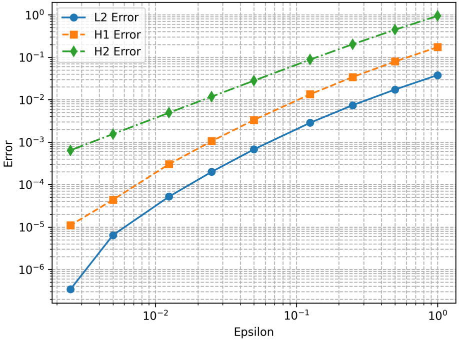
From [Neilan, PhD Thesis], we have

$$\|u^\varepsilon\|_{H^j} = \mathcal{O}\left(\varepsilon^{\frac{1-j}{2}}\right), \quad \|u^\varepsilon\|_{W^{j,\infty}} = \mathcal{O}\left(\varepsilon^{-j}\right), \quad \|\Phi^\varepsilon\|_{L^2} = \mathcal{O}\left(\varepsilon^{-\frac{1}{2}}\right) \quad \|\Phi^\varepsilon\|_{L^\infty} = \mathcal{O}\left(\varepsilon^{-1}\right).$$

To demonstrate this, we consider a simple numerical experiment (cf. Neilan, 2010) using the VEM method we will discuss shortly. Let $h \approx 0.0277$ be fixed, define $f = 4$ and g such that $u = x^2 + y^2$ on $\Omega = (0, 1)$, and consider the error $u - u_h^\varepsilon$ as $\varepsilon \to 0$.



Triangle (Structured)



Voronoi

Let $V := H^2(\Omega)$ and $W := H^2(\Omega) \cap H^1_0(\Omega)$. Find $u^\varepsilon \in V$ such that

$$A_{QL}(u^\varepsilon, v) = \int_\Omega fv \, \mathrm{d}\boldsymbol{x} + \varepsilon \int_{\partial\Omega} \left(\frac{\partial^2 g}{\partial \boldsymbol{t}^2} - \varepsilon \right) \frac{\partial v}{\partial \boldsymbol{n}} \, \mathrm{d}s \qquad \text{for all } v \in W.$$

where

$$A_{QL}(u^\varepsilon, v) = \underbrace{-\varepsilon \int_\Omega D^2 u^\varepsilon : D^2 v \, \mathrm{d}\boldsymbol{x}}_{a_{QL}(u^\varepsilon, v)} + \underbrace{\int_\Omega \det(D^2 u^\varepsilon) v \, \mathrm{d}\boldsymbol{x}}_{b_{QL}(u^\varepsilon, v)},$$

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Lemma

Let $v = (v_1, v_2, \ldots, v_n) : \Omega \to \mathbb{R}^n$ *be a vector-valued function, and assume* $\mathbf{v} \in [C^2(\Omega)]^n$*. Then,*

$$\nabla \cdot (\mathrm{cof}(\nabla \mathbf{v}))_i = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (\mathrm{cof}(\nabla \mathbf{v}))_{ij} = 0 \quad \textit{for} \quad i = 1, 2, \ldots, n$$

This allows us to linearize the vanishing moment PDE:

$$\begin{aligned} L_{u^\epsilon}(v) &= \varphi && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega, \\ \Delta v &= \psi && \text{on } \partial\Omega. \end{aligned}$$

where

$$L_{u^\varepsilon}(v) := \varepsilon \Delta^2 v - \Phi^\varepsilon : D^2 v = \varepsilon \Delta^2 v - \nabla \cdot (\Phi^\varepsilon \nabla v), \qquad \text{and} \qquad \Phi^\varepsilon = \mathrm{cof}(D^2 u^\epsilon).$$

Find $v \in W$ such that

$$A_L(v, w) = \int_\Omega \varphi w \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \psi \frac{\partial w}{\partial \boldsymbol{n}} \, \mathrm{d}s \qquad \text{for all } w \in W,$$

where

$$A_L(v, w) := \epsilon \int_\Omega D^2 v : D^2 w \, \mathrm{d}\boldsymbol{x} + \int_\Omega \Phi^\epsilon \nabla v \cdot \nabla w \, \mathrm{d}\boldsymbol{x}.$$

Why VEM/Polytopal Elements?

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- Nonlinear problem $\implies$ solving via iteration (fixed point, Newton, etc.).
- Depending on the number of iterations and DoFs could be computationally expensive.
- Reduce computational expense — two-grid method: Solve nonlinear problem on a coarse mesh, and use to linearise on a fine mesh

Xu 1992, 1994, 1996; Xu & Zhou 1999; Axelsson & Layton 1996;
Dawson, Wheeler & Woodward 1998; Utnes 1997; Marion & Xu 1995; Wu & Allen 1999
Awanou, Li & Malitz 2020 (C^0-IP for Monge-Ampère)

Mesh

Construct mesh $\mathcal{T}_h$ of Ω consisting of simple polygons, with element diameter h_E, $E \in \mathcal{T}_h$.

Assumption (Mesh Regularity)

There exists $\rho > 0$ such that

- *each element $E \in \mathcal{T}_h$ star-shaped w.r.t ball of radius ρh_E*
- *$h_e \geq \rho h_E$ for every $E \in \mathcal{T}_h$ and $e \subset \partial E$*

Additionally, we define $\mathcal{E}_h$ as the set of all faces.
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On each element we consider a order of approximation ℓ.
For $s > 0$ we define the broken space

$$H^s(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) : v|_E \in H^2(E), \quad \forall E \in \mathcal{T}_h \right\}.$$

and

$$H^{2,\mathrm{nc}}_\ell(\mathcal{T}_h) := \left\{ v \in H^2(\mathcal{T}_h) \cap H^1_0(\Omega) : \int_e [\![\nabla v \cdot \boldsymbol{n}]\!] p \,\mathrm{d}s = 0 \; \forall p \in \mathbb{P}_{\ell-2}(e), \forall e \in \mathcal{E}_h \right\}.$$

Given a local enlarged VEM space

$$\widetilde{V}_{h,\ell}^{E} := \left\{ v_h \in H^2(E) : \Delta^2 v_h \in \mathbb{P}_\ell(E), v_h|_e \in \mathbb{P}_\ell(e), \Delta^2 v_h|_e \in \mathbb{P}_{\ell-2}(E)\ \forall e \subset \partial E \right\}$$

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and a value projection $\Pi_0^E : \widetilde{V}_{h,\ell}^{E} \to \mathbb{P}_\ell$ we define the local virtual element space $V_{h,\ell}^{E}$ as

$$V_{h,\ell}^{E} := \left\{ v_h \in \widetilde{V}_{h,\ell}^{E} : (v_h - \Pi_0^E v_h, p)_E = 0 \quad \forall p \in \mathbb{P}_\ell(E) \backslash \mathbb{P}_{\ell-4}(E) \right\}$$

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The global VEM space $V_{h,\ell}$ is defined as

$$V_{h,\ell} := \left\{ v_h \in H_\ell^{2,\mathrm{nc}}(\Omega) \ : \ v_h|_E \in V_{h,\ell}^E \quad \forall E \in \mathcal{T}_h \right\}$$

VEM Space

Given a local enlarged VEM space

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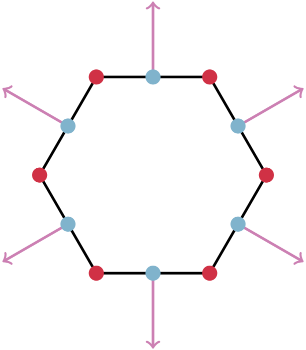
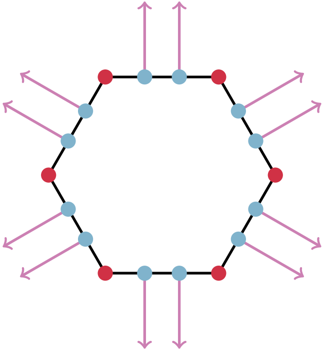
We note that $V_{h,\ell} \not\subset H^2(\Omega)$ but $V_{h,\ell} \subset H^1(\Omega)$. Hence, we have a C^1-nonconforming, C^0-conforming space. [Zhao et al., 2016]

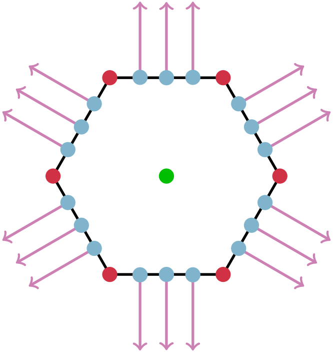
Local Degrees of Freedom



$\ell = 2$



$\ell = 3$



$\ell = 4$

Local Degrees of Freedom

The local space $V_{h,\ell}^E$ is characterised by the degrees of freedom:

(D1) The value of v_h at each vertex of E

(D2) For $\ell > 1$, the moments of v_h up to order $\ell - 2$ on each edge $e \subset \partial E$

$$\frac{1}{|e|}\int_e v_h p \,\mathrm{d}s \quad \forall p \in \mathbb{P}_{\ell-2}(e)$$

(D3) For $\ell > 1$, the normal moments of v_h up to order $\ell - 2$ on each edge $e \subset \partial E$

$$\int_e \partial_n v_h p \,\mathrm{d}s \quad \forall p \in \mathbb{P}_{\ell-2}(E)$$

(D4) For $\ell > 3$, the moments of v_h up to order $\ell - 4$ inside E

$$\frac{1}{|E|}\int_E v_h p \,\mathrm{d}\mathbf{x} \quad \forall p \in \mathbb{P}_{\ell-4}(E)$$

Value projection ($\Pi_0^E : \widetilde{V}_{h,\ell}^E \to \mathbb{P}_\ell(E)$) $\Pi_0^E v_h$ linear combination of dofs, and satisfies

$$\int_E \Pi_0^E v_h p \, \mathrm{d}\boldsymbol{x} = \int_E v_h p \, \mathrm{d}\boldsymbol{x} \quad \forall p \in \mathbb{P}_{\ell-4}(E), \qquad \text{and} \qquad \Pi_0^E q = q \quad \forall q \in \mathbb{P}_\ell(E).$$

Edge projection ($\Pi_0^e : \widetilde{V}_{h,\ell}^E \to \mathbb{P}_\ell(e)$) $\Pi_0^e v_h$ linear combination of dofs, and satisfies $\Pi_0^e v_h(e^\pm) = v_h(e^\pm)$,

$$\int_e \Pi_0^e v_h p \, \mathrm{d}s = \int_e v_h p \, \mathrm{d}s \quad \forall p \in \mathbb{P}_{\ell-2}(e), \qquad \text{and} \qquad \Pi_0^e q = q|_e \quad \forall q \in \mathbb{P}_\ell(E).$$

Edge normal projection ($\Pi_1^e : \widetilde{V}_{h,\ell}^E \to \mathbb{P}_{\ell-1}(e)$) $\Pi_1^e v_h$ linear combination of dofs, and satisfies

$$\int_e \Pi_1^e v_h p \, \mathrm{d}s = \int_e \partial_n v_h p \, \mathrm{d}s \quad \forall p \in \mathbb{P}_{\ell-2}(e), \qquad \text{and} \qquad \Pi_1^e q = \partial_n q|_e \, \forall q \in \mathbb{P}_\ell(E).$$

Gradient projection ($\Pi_1^E : \widetilde{V}_{h,\ell}^E \to [\mathbb{P}_\ell(E)]^2$)

$$\int_E \Pi_1^E v_h \cdot \boldsymbol{p} \,\mathrm{d}\boldsymbol{x} = -\int_E \Pi_0^E v_h \nabla \cdot \boldsymbol{p} \,\mathrm{d}\boldsymbol{x} + \sum_{e \subset \partial E} \int_e \Pi_0^e v_h \boldsymbol{p} \cdot \boldsymbol{n} \,\mathrm{d}s \quad \forall \boldsymbol{p} \in [\mathbb{P}_{\ell-1}(E)]^2.$$

Hessian projection ($\Pi_2^E : \widetilde{V}_{h,\ell}^E \to [\mathbb{P}_\ell(E)]^{2\times 2}$) For all $\boldsymbol{p} \in [\mathbb{P}_{\ell-2}(E)]^{2\times 2}$

$$\int_E \Pi_2^E v_h \cdot p \,\mathrm{d}\boldsymbol{x} = -\int_E \Pi_1^E v_h \nabla p \,\mathrm{d}\boldsymbol{x} + \sum_{e \subset \partial E} \int_e (\Pi_1^e v_h \boldsymbol{n} \otimes p\boldsymbol{n} + \partial_t (\Pi_0^e v_h) \boldsymbol{t} \otimes p\boldsymbol{n}) \,\mathrm{d}s$$

.

Here, $e \subset E$ is an element edge, and $e^{\pm}$ denotes the vertices of e.

Use CLS for choice of projections: Dedner & Hodson 2024,

VEM Forms

We now define the forms necessary for the VEM formulation:

$$A^E_{QL,h}(u_h, v_h) := -\varepsilon \int_E \Pi^E_2 u_h : \Pi^E_2 v_h \,\mathrm{d}\boldsymbol{x} + \int_E \det(\Pi^E_2 u_h)\Pi^E_0 v_h \,\mathrm{d}\boldsymbol{x} + S^E_\rho(u_h - \Pi^E_0 u_h, v_h - \Pi^E_0 v_h)$$

$$A^E_{L,h}(u_h, v_h) := \varepsilon \int_E \Pi^E_2 u_h : \Pi^E_2 v_h \,\mathrm{d}\boldsymbol{x} + \int_E (\Phi^\varepsilon \Pi^E_1 u_h) \cdot \Pi^E_1 v_h \,\mathrm{d}\boldsymbol{x} + S^E_\kappa(u_h - \Pi^E_0 u_h, v_h - \Pi^E_0 v_h)$$

where

$$S^E_\rho(u, v) := (-\varepsilon h_E^2 + \gamma_E) \sum_{i=1}^N \mathrm{dof}_i(u)\mathrm{dof}_i(v)$$

$$S^E_\kappa(u, v) := (\varepsilon h_E^2 + \Phi^\varepsilon) \sum_{i=1}^N \mathrm{dof}_i(u)\mathrm{dof}_i(v)$$

There must exist constants c_*, c^*, d_*, d^* such that

$$c_* A^E_L(v_h, v_h) \le S^E_\kappa(v_h, v_h) \le c^* A^E_L(v_h, v_h) \qquad d_* A^E_{QL}(v_h, v_h) \le S^E_\kappa(v_h, v_h) \le d^* A^E_{QL}(v_h, v_h)$$

Linear VEM Formulation

Theorem (Existence and Uniqueness of Linearized VEM)

There exists a unique $v_h \in V_{h,\ell}$ such that

$$A_{L,h}(v_h, w_h) = \int_\Omega \varphi_h w_h \,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \psi \frac{\partial w_h}{\partial \boldsymbol{n}} \,\mathrm{d}s \qquad \text{for all } w_h \in V_{h,\ell}.$$

Here

$$A_{L,h}(u, v) = \sum_{E \in \mathcal{T}_h} A^E_{L,h}(u, v).$$

Lemma (Strang-type Estimate [C., Hodson, Pradhan (In Prep.)])

For every approximation v_I of v in $V_{h,\ell}$

$$\alpha_* \|v - v_h\|_{2,h} \leq C_3(\epsilon)^{-1} \Bigg\{ (1 - C_3(\epsilon)\alpha_*)\|v - v_I\|_{2,h} + \|\varphi - \varphi_h\|_{V'_{h,\ell}} + \sup_{\delta_h \in V_{h,\ell}} \frac{|E(v,\delta_h)|}{\|\delta\|_{2,h}} + \inf_{p \in \mathbb{P}_\ell(\mathcal{T}_h)} \left(\|v - p\|_{2,h} + \sum_{K \in \mathcal{T}_h} \sup_{\delta_h \in V_{h,l}} \frac{|PE(p,\delta_h)|}{\|\delta\|_{2,h}} \right) \Bigg\}$$

where $\delta_h := v_h - v_I \neq 0$, $\|\varphi - \varphi_h\|_{V'_{h,\ell}} := \sup_{\delta_h \in V_{h,\ell}} \frac{|\langle \varphi - \varphi_h, \delta_h \rangle|}{\|\delta\|_{2,h}}$, *the polynomial consistency error* $PE(p,\delta_h) := A_L^K(p,\delta_h) - A_{L,h}^K(p,\delta_h)$ *and the nonconformity error is given by*

$$E(v,\delta_h) = (\varphi,\delta_h) + \langle \psi, \partial_n \delta_h \rangle_{\partial\Omega} - A_L(v,\delta_h).$$

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Theorem (A priori Error Bound [C., Hodson, Pradhan (In Prep.)])

Suppose that mesh regularity assumptions are satisfied. Let $\ell \geq 2$ *be a positive integer and let* $v \in H^{s+1}(\Omega)$ *be the solution of the linearized PDE for some positive integer* s. *Define* $r = \min(\ell, s)$ *and assume that* $\varphi \in H^{r-3}(\Omega)$. *Let* $v_h \in V_{h,\ell}$ *be the corresponding virtual element solution. Then, there exists a constant* $C_5(\varepsilon) > 0$, *independent of* h, *such that*

$$\|v - v_h\|_{2,h} \leq C_5(\varepsilon) h^{r-1}(\|v\|_{r+1} + \|\varphi\|_{r-3}).$$

where $C_5(\varepsilon) = C(1 - C_3(\varepsilon)\alpha_*)^{-1} \max(1 - C_3(\varepsilon)\alpha_*, \varepsilon, C_4(\varepsilon), 1)$.

Vanishing Moment VEM Formulation

Find $u_h^\varepsilon \in V_{h,\ell}$ such that

$$A_{QL,h}(u_h^\varepsilon, v_h) = \int_\Omega f_h v_h \,\mathrm{d}\boldsymbol{x} + \varepsilon \sum_{e \in \mathcal{E}_h^B} \int_e \left(\frac{\partial^2 g}{\partial \boldsymbol{t}^2} - \varepsilon \right) \Pi_1^e v_h \,\mathrm{d}s \qquad \text{for all } v_h \in V_{h,\ell}.$$

Here

$$A_{QL,h}(u, v) = \sum_{E \in \mathcal{T}_h} A_{QL,h}^E(u, v).$$

Well-posedness

Theorem (Existence and uniqueness [C., Hodson, Pradhan (In Prep.)])

For all $\varepsilon > 0$ and sufficently small h there exists a unique solution $u_h^\varepsilon \in V_{h,\ell}$ to the VEM formulation of the vanishing moment method.

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Proof.

To show well-posedness, we first define an operator $T_h : V_{h,\ell} \to V_{h,\ell}$ such that for any $v_h \in V_{h,\ell}$, $T_h(v_h)$ is the solution of the problem

$$A_{L,h}(v_h - T_h(v_h), w_h) = A_{QL,h}(v_h, w_h) - \int_\Omega f_h w_h \,\mathrm{d}\boldsymbol{x} + \varepsilon \sum_{e \in \mathcal{E}_h^B} \int_e \left(\frac{\partial^2 g}{\partial \boldsymbol{t}^2} - \varepsilon \right) \Pi_1^e w_h \,\mathrm{d}s.$$

$T_h(v_h)$ exists and is unique by the well-posedness of the linear VEM. Furthermore, the solution u_h^ε of the nonlinear formulation is equivalent to the fixed point of the mapping T_h. Therefore, it is sufficient to show existence and uniqueness of this fixed point (i.e. by Banach). □

Well-posedness

Defining $u_I^\varepsilon \in V_{h,\ell}$ as the interpolation of u^ε we show the existing of a fixed point to T_h in the ball

$$B(u_I^\varepsilon, \zeta) := \{v_h \in V_{h,\ell} : \|v_h - u_I^\varepsilon\|_{2,h} \leq \zeta\}$$

Lemma

For $u_I^\epsilon \in V_{h,\ell}$, there exists $C_6(\epsilon) > 0$ such that

$$\|u_I^\epsilon - T_h(u_I^\epsilon)\|_{2,h} \leq Ch^{r-1}(C_6(\epsilon)\|u^\epsilon\|_{r+1} + \|f\|_{r-3}). \tag{2}$$

Lemma (Contraction mapping)

For any $w_h, v_h \in V_{h,\ell}$, there exists $C_7(\epsilon, h) > 0$ such that

$$\|T_h(w_h) - T_h(v_h)\|_{2,h} \leq C_7(\epsilon, h)\|w_h - v_h\|_{2,h}. \tag{3}$$

Lemma

There exists a $h_1 > 0$ and $\zeta > 0$ such that for all $h < h_1$ T_h has a unique fixed point.

Proof.

By the previous two lemmas we can show that there exists a h_1 such that for all $h < h_1$ and $v_h \in B(u_I^\varepsilon, \zeta)$

$$\|T_h(u_I^\varepsilon) - T_h(v_h)\|_{2,h} \leq \frac{1}{2}\|u_I^\varepsilon - v_h\|_{2,h};$$

and

$$\|u_I^\varepsilon - T_h(v_h)\|_{2,h} \leq \|u_I^\varepsilon - T_h(u_I^\varepsilon)\|_{2,h} + \|T_h(u_I^\varepsilon) - T_h(v_h)\|_{2,h} \leq \frac{\zeta}{2} + \frac{1}{2}\|u_I^\varepsilon - v_h\|_{2,h} \leq \zeta.$$

Hence $T(B(u_I^\varepsilon, \zeta)) \subset B(u_I^\varepsilon, \zeta)$ and as T_h is a contraction (by previous lemma) we can apply Banach's fixed point theorem. □

a priori Error Estimate

Theorem (*A priori* Error Bound [C., Hodson, Pradhan (In Prep.)])

Suppose that mesh regularity assumptions are satisfied. Let $\ell \geq 2$ *be a positive integer and let* $u^\varepsilon \in H^{s+1}(\Omega)$ *be the solution of the vanishing moment method for some positive integer* s*. Define* $r = \min(\ell, s)$ *and assume that* $f \in H^{r-3}(\Omega)$*. Let* $u_h^\epsilon \in V_{h,\ell}$ *be the corresponding virtual element solution. Then, there exists a contant* $C_8(\epsilon) > 0$*, independent of* h*, such that*

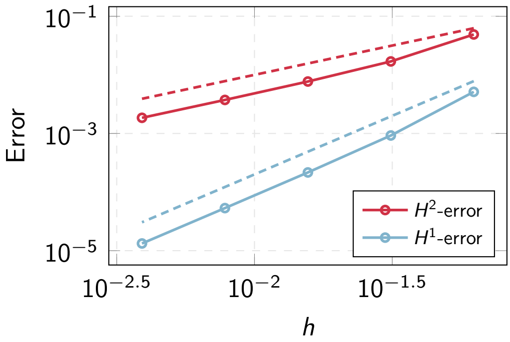
$$\|u^\epsilon - u_h^\epsilon\|_{2,h} \leq C_8(\epsilon) h^{r-1} (\|u^\epsilon\|_{r+1} + \|f\|_{r-3}). \tag{4}$$

where $C_8(\varepsilon) = C \max\{C_6(\epsilon), 1\}$.

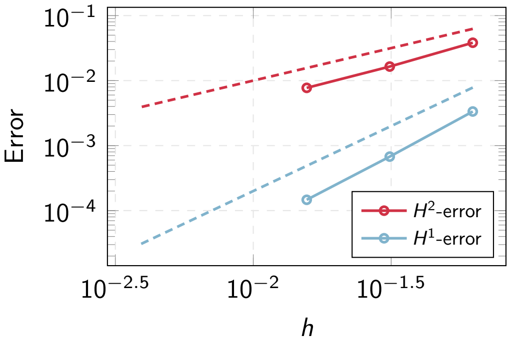
Numerical Experiments

We let $\Omega = (0,1)^2$, $\ell = 2$, and define $f = x^2y^2 - 4\varepsilon$ and g such that

$$u^\varepsilon = \frac{1}{12}(x^4 + y^4).$$



Uniform Quad Elements



Voronoi Mesh

Conclusions & Future

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- C^0-conforming C^1-nonconforming VEM for the vanishing moment method for Monge-Ampère'
- Well-posedness
- *a priori* error results

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Future

- VEM for Monge-Ampère without vabishing moment
- *a posteriori* error estimates
- Two-grid