EXERCISES 1 (30.9.2022)

1. Easier and essential exercises: Prove the following assertions (once you prove those, we will use those as "known facts")

Fact 1. *Let X be a vector space and $A \subset X$. Then A is absolutely convex if and only if $\alpha x + \beta y \in A$ for every $x, y \in A$ and $\alpha, \beta \in \mathbb{K}$ with $|\alpha| + |\beta| \le 1$. Moreover, we have*

$$\operatorname{aconv} A = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \ x_1, \dots, x_n \in A, \lambda_1, \dots, \lambda_n \in \mathbb{K}, \sum_{i=1}^{n} |\lambda_i| \le 1, n \in \mathbb{N} \right\}.$$

Fact 2. *Let X be TVS, $a \in X$ and $\lambda \in \mathbb{K} \setminus \{0\}$. Then the operations $x \mapsto x + a$ and $x \mapsto \lambda x$ are homeomorphisms of X onto X. Moreover, for every $x \in X$ we have $\tau(x) = x + \tau(0)$.*

Fact 3. *Let X be TVS.*

(a) If $G \subset X$ is open and $A \subset X$ arbitrary, then $A + G$ is open.

(b) If $F \subset X$ is closed and $K \subset X$ compcat, then $F + K$ is closed.

(c) If $K, L \subset X$ are compact, then $K + L$ is compact.

Fact 4. *Let X be TVS and $A, B \subset X$. Then $\overline{A} = \bigcap \{A + U : \ U \in \tau(0)\}$.*

Fact 5. *Let X be TVS and $A, B \subset X$. Then*

(a) $\overline{A} + \overline{B} \subset \overline{A + B}$ and $\operatorname{Int} A + \operatorname{Int} B \subset \operatorname{Int}(A + B)$.

(b) $\lambda \overline{A} = \overline{\lambda A}$ for every $\lambda \in \mathbb{K} \setminus \{0\}$ and if A is subspace, then $\overline{A}$ is subspace.

Fact 6. *Let X be TVS and $A \subset X$. Then $\overline{\operatorname{span}} A = \overline{\operatorname{span} A}$, $\overline{\operatorname{conv}} A = \overline{\operatorname{conv} A}$ and $\overline{\operatorname{aconv}} A = \overline{\operatorname{aconv} A}$.*

2. Further exercises: a) Let $X \neq \{0\}$ be a vector space and τ be the discrete topology on X. Prove that then addition is continuous, but multiplication is not continuous.
b) Prove that on $\mathbb{R}^2$ there is a topology τ such that addition is separately continuous, but not continuous. (Hint: consider topology whose basis of neighborhoods of the origin is given by sets $\{(0,0)\} \cup \{(x, y) \colon |y| < |x| < r\}$, $r > 0$).
c) We say that $(X, \|\cdot\|)$ is a quasi-normed linear space, if X is a vector space and $\|\cdot\| : X \to [0, \infty)$ is a mapping satisfying all the axioms on the norm with the exception that triangle inequality is replaced by the following weaker condition

$$\exists C \ge 0 \forall x, y \in X : \quad \|x + y\| \le C(\|x\| + \|y\|).$$

For $x \in X$ and $r > 0$ put $U(x, r) := \{y \in X \colon \|x - y\| < r\}$. Prove that there is a unique topology τ on X such that (X, τ) is HTVS and $\{U(0, r) \colon r > 0\}$ is basis of neighborhoods of 0.
d) Prove that ℓ_p for $0 < p < 1$ is HTVS with respect to topology given by the metric $d(x, y) = \|x - y\|_p^p := \sum_{n=1}^{\infty} |x_n - y_n|^p$ (Hint: in order to check that d is indeed a metric, note that we have $a \le a^p$ for $a \in (0, 1)$ which implies $(t + s)^p = \frac{t}{t+s}(t + s)^p + \frac{s}{t+s}(t + s)^p \le t^p + s^p$ for every $t, s > 0$).
e) Prove that ℓ_p for $0 < p < 1$ is not locally convex. (Hint: realize that for small $\delta > 0$ we have that $\|\delta e_i\|_p$ is small while for the natural convex combinations we obtain that $\| \sum_{i=1}^{n} \frac{1}{n} \delta e_i \|_p$ is big).
f) Prove that $L_p([0, 1]))$ for $0 < p < 1$ is HTVS with respect to topology given by the metric $d(f, g) = \|f - g\|_p^p := \int_0^1 |f(t) - g(t)|^p \, \mathrm{d}t$. Moreover, prove that $L_p([0, 1])$ is not locally convex.
g) Consider the vector space $X = \{f \colon [0, 1] \to \mathbb{K} : \ f \text{ measurable}\}$ with metric $\rho(f, g) = \int_0^1 \min\{|f - g|, 1\} \, \mathrm{d}\lambda$ (we identify functions equal almost everywhere). Prove that X endowed with the topology given by the metric ρ is HTVS, which is not locally convex (Hint: show that $\operatorname{conv} U(0, r) = X$ for every $r > 0$). Moreover, prove that a sequence $\{f_n\} \subset X$ converges to $f \in X$ in metric ρ if and only if $f_n \to f$ in measure.

Suitable for credit: exercises 2.b, 2.f, 2.g

EXERCISES 2 (7.10.2022)

1. Easier and essential exercises:
a) Let $(X, \|\cdot\|)$ be a normed linear space. Prove that $\mu_{U(0,1)}(x) = \|x\| = \mu_{B(0,1)}(x)$ for every $x \in X$.
b) Let X be a vector space, $A \subset X$ such that $\operatorname{span} A = X$ and consider the Minkowski functional $\mu_{\operatorname{aconv} A}$. Prove that for every $x \in X$ we have

$$\mu_{\operatorname{aconv} A}(x) = \inf\Big\{\sum_{i=1}^n |a_i| \colon \sum_{i=1}^n a_i x_i = x,\ a_i \in \mathbb{K},\ x_i \in A,\ n \in \mathbb{N}\Big\}.$$

Now, put $N := \{x \in X \colon \mu_{\operatorname{aconv} A}(x) = 0\}$ and consider the vector space $Z := X/_N$ (quotient of X by points for which $\mu_{\operatorname{aconv} A}(x) = 0$). Prove that $\|\cdot\| : Z \to [0,\infty)$ given by the formula $\|x + N\| := \mu_{\operatorname{aconv} A}(x)$, $x \in X$ defines a norm on the vector space Z.
c) Prove that $\mathbb{K}^I$ is metrizable if and only if I is countable.
2. Further exercises: a) Find an example of a quasi-norm $\|\cdot\|$ and a balanced neighborhood U of 0 in $(\mathbb{R}^2, \|\cdot\|)$ such that the corresponding Minkowski functional μ_U is not continuous.
(*Hint:* Note that given a quasi-norm $\|\cdot\|$ on $\mathbb{R}^2$, we have $\mu_{U(0,1)}(\cdot) = \|\cdot\|$, so it suffices to find a discontinuous quasi-norm. Consider now the quasi-norm given by the fomula $\|(x,y)\| := |x| + |y|$ if $y \neq 0$ and $\|(x,0)\| := 2|x|$)
b) Using Theorem 7, prove that for any TVS X the following holds

(i) X is completely regular;

(ii) if X has countable basis of neighborhoods of 0, then it is metrizable by a translation invariant metric.

c) Let X be TVS, $A \subset X$ balanced neighborhood of 0. Prove that the following conditions are equivalent

(i) μ_A is continuous;

(ii) For every $x \in \overline{A}$ we have $\{tx \colon t \in [0,1)\} \subset \operatorname{Int} A$;

(iii) $\operatorname{Int} A = \{x \colon \mu_A(x) < 1\}$ and $\overline{A} = \{x \colon \mu_A(x) \leq 1\}$.

3. Harder exercises (not intended for exams): a) Prove the following Theorem.

Theorem 7. *Let X be TVS and $(V_n)_{n\in\mathbb{N}}$ a sequence of balanced neighborhoods of 0 satisfying $V_{n+1} + V_{n+1} \subset V_n$, $n \in \mathbb{N}$. Then there exists a continuous mapping $p : X \to [0,\infty)$ such that*

(i) $p(x) = 0$ if and only if $x \in \bigcap_{n\in\mathbb{N}} V_n$;

(ii) $p(\alpha x) \leq p(x)$ whenever $|\alpha| \leq 1$ and $x \in X$;

(iii) $p(x+y) \leq p(x) + p(y)$ for every $x, y \in X$;

(iv) for every $n \in \mathbb{N}$ we have $\{x \in X \colon p(x) < 2^{-n}\} \subset V_n \subset \{x \in X \colon p(x) \leq 2^{-n}\}$.

Sketch of the proof. Given finite nonempty $F \subset \mathbb{N}$ we put $q_F := \sum_{n\in F} 2^{-n}$ and $V_F := \sum_{n\in F} V_n$ and define $p : X \to [0,\infty)$ by the formula

$$p(x) := \begin{cases} \inf\{q_F \colon x \in V_F\} & \text{if } x \in \bigcup_{\emptyset \neq F \subset \mathbb{N} \text{ finite}} V_F, \\ 1 & \textit{otherwise.} \end{cases}$$

First, prove the property (ii). Next, prove that $q_{F_1} < q_{F_2}$ implies $V_{F_1} \subset V_{F_2}$ and deduce properties (i) and (iv). Finally, prove that $q_{F_1} + q_{F_2} = q_F$ implies $V_{F_1} + V_{F_2} \subset V_F$ (inductively with respect to $|F|$) and deduce property (iii) and continuity of p. $\square$

b) Let $0 \notin A \subset \mathbb{R}^n$ be a finite set satisfying $\operatorname{span} A = \mathbb{R}^n$ such that no two elements of A are scalar multiples of each other. Let $p : \mathbb{R}^n \to [0,\infty)$ be a pseudonorm. Prove that for every $\varepsilon > 0$ there exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ satisfying that $\max_{a\in A} \big|\|a\| - p(a)\big| < \varepsilon$ and $\|a\| \in \mathbb{Q}$ for every $a \in A$.

Suitable for credit: exercises 2.a, 2.b, 2.c, 3.b

EXERCISES 3 (14.10.2022)

1. Easier and essential exercises:
a) Let X be a normed linear space and $A \subset X$. Prove that A is bounded as a subset of TVS X, if and only if it is bounded with respect to the metric generated by the norm.
b) Prove that $\mathbb{K}^I$ is normable if and only if I is finite.
c) Let X be a TVS and $A \subset X$. Prove that

(i) If A is compact, then it is bounded.

(ii) If A is bounded, then $\overline{A}$ is bounded.

(iii) If A is bounded and X is LCS, then $\operatorname{conv} A$ and $\operatorname{aconv} A$ are bounded.

2. Further exercises: a) For $p \in (0,1)$ find a sequence $(c_n) \in \mathbb{R}^{\mathbb{N}}$ such that the set $\{c_n e_n\} \cup \{0\} =: K \subset \ell_p$ is compact (and therefore bounded), but $\operatorname{conv} K$ is not bounded (Hint: consider convex combinations $\sum_{n=1}^m \frac{1}{m} c_n e_n$).
b) Consider the vector space $X = C^\infty([0,1])$ endowed with the topology τ generated by pseudonorms

$$\nu_N(f) := \max_{n \le N} \|f^{(n)}\|_\infty, \qquad N \in \mathbb{N} \cup \{0\}.$$

Prove that (X, τ) is metrizable LCS which is not normable.
c) Prove that $L_p([0,1])^* = \{0\}$ for every $p \in (0,1)$ (Hint: given $0 \neq \phi \in L_p([0,1])^*$, the set $\phi^{-1}(-1,1) \neq L_p([0,1])$ is convex open neighborhood of 0; so it suffices to prove that for any $r > 0$ we have $\operatorname{conv} U(0,r) = L_p([0,1])$).
d) Fix $p \in (0,1)$. Consider the mapping $I : \ell_\infty \to (\ell_p)^*$ defined as $I(x)(y) := \sum_{n=1}^\infty x_n y_n$ for $x \in \ell_\infty$ and $y \in \ell_p$. Prove that I is isometry onto ℓ_p and show that $(\ell_p)^*$ separate the points of ℓ_p.

3. Bonus exercises (not intended for exams): a) Pick $p \in (0,1)$. We say that $(X, \|\cdot\|)$ is a p-normed linear space, if X is a vector space and $\|\cdot\| : X \to [0,\infty)$ is a mapping satisfying all the axioms on the norm with the exception that triangle inequality is replaced by the following weaker condition

$$\forall x, y \in X : \quad \|x+y\|^p \le \|x\|^p + \|y\|^p.$$

If $(X, \|\cdot\|^p)$ is complete metric space (where by $\|\cdot\|^p$ we denote the metric $(x,y) \mapsto \|x-y\|^p$), we say $(X, \|\cdot\|)$ is a p-Banach space. Prove that any p-normed linear space is quasi-normed space and that $(\ell_p, \|\cdot\|_p)$ is p-Banach space, where $\|x\|_p := \sqrt[p]{\sum_{n=1}^\infty |x_n|^p}$.
b) Let $p \in (0,1)$ and $(X, \|\cdot\|)$ be a p-Banach space such that X^* separates the points of X. Let $|\cdot|$ be the Minkowski functional of the set $\operatorname{aconv} U_X(0,1)$.

(i) Prove that $|\cdot|$ is a norm on X.

(ii) Let us denote by $\widehat{X}$ the completion of $(X, |\cdot|)$. Prove that the mapping $I : X \to \widehat{X}$ defined by $I(x) = x$, $x \in X$ is continuous and $|I(x)| \le \|x\|$.

(iii) Prove that whenever Y is a Banach space and $T : X \to Y$ is linear and continuous satisfying $\|Tx\| \le C\|x\|$ for $x \in X$, then there exists a unique $\widehat{T} : \widehat{X} \to Y$ satisfying $\widehat{T} \circ I = T$ and $\|\widehat{T}\| \le C$.

(iv) Prove that the property (iii) characterizes the Banach space $\widehat{X}$ up to isometry. That is, if $\widetilde{X}$ is a Banach space for which there exists $\widetilde{I} : X \to \widehat{X}$ continuous onto dense subspace such that for any Y Banach and $T : X \to Y$ there is $T' : \widetilde{X} \to Y$ satisfying $\|T'\| = \|T\|$, then $\widetilde{X}$ is linearly isometric to $\widehat{X}$.

We say that $\widehat{X}$ is the *Banach envelope* of X.
c) Prove that the Banach envelope of the p-Banach space ℓ_p is the Banach space ℓ_1.

Suitable for credit: exercises 2.b, 2.a+d, 3.b, 3.c

1. Easier and essential exercises:
a) Prove that any cauchy net in a complete metric space is convergent.
b) Let X be TVS and $A, B \subset X$ totally bounded. Prove that $A \cup B$, $A + B$, $\overline{A}$ are totally bounded.
c) Let X be TVS. Prove that $A \subset X$ is totally bounded if and only if for every $U \in \tau(0)$ there exists a finite set $F \subset X$ such that $A \subset F + U$. Deduce that subsets of totally bounded sets are totally bounded.
d) Let X be LCS and $A \subset X$ totally bounded. Then $\operatorname{conv} A$ and $\operatorname{aconv} A$ are totally bounded.

2. Further exercises: In exercises below work only with spaces over $\mathbb{R}$.
a) Let X be an infinite-dimensional normed linear space. Find two disjoint convex sets $A, B \subset X$ which are both dense and deduce that there does not exist $x^* \in X^* \setminus \{0\}$ satisfying $\sup_A \operatorname{Re} x^* \leq \inf_B \operatorname{Re} x^*$. (Hint: use the existence of discontinuous linear forms)
b) Let $X = c_0$. Put $A = \{z \in c_0 \colon z_n \geq 0 \text{ for every } n\}$ and $B = \{(\frac{t}{n^2} - \frac{1}{n})_{n=1}^{\infty} \colon t \in \mathbb{R}\}$. Prove that A, B are disjoint closed convex sets, but there does not exist $x^* \in X^* \setminus \{0\}$ satisfying $\sup_B x^* \leq \inf_A x^*$. (*Hint: pick $f \in (c_0)^* = \ell_1$ satisfying $\sup_B f \leq \inf_A f$. Prove that $f \geq 0$ on A and so $\inf_A f = 0$, deduce that $f(n) \geq 0$ for every n. Then show that $\sup_B f \leq 0$ implies $f\big((\frac{1}{n^2})_{n=1}^{\infty}\big) = 0$ which in turn implies that $f(n) = 0$ for every n*)
c) If D is a non-empty convex subset of a Banach space X so that $0 \notin \overline{D}$, then there is $x^* \in S_{X^*}$ such that

$$\inf\{x^*(x) \colon x \in D\} = \inf\{\|x\| \colon x \in D\}.$$

(*Hint: put $\eta = \inf\{\|x\| \colon x \in D\}$ and use Hahn-Banach to separate $U(0, \eta)$ from D*)
d) Fix $p \in (0, 1)$. Find a closed subspace $M \subset \ell_p$ and $x^* \in M^*$ such that there does not exist $\varphi \in (\ell_p)^*$ satisfying $\varphi \supset x^*$.
(*Hint: pick a sequence (x_n) in ℓ_p such that the points x_n have disjoint supports, $\|x_n\|_p = 1$ and $\|x_n\|_1 \to 0$. Then prove that $e_n \mapsto x_n$ induces isometry between ℓ_p and $M := \overline{\operatorname{span}}\{e_n \colon n \in \mathbb{N}\}$. Pick $x^* \in M^*$ satisfying $x^*(e_n) = 1$ for every $n \in \mathbb{N}$. Finally, show that for every $\varphi \in (\ell_p)^*$ we have $\varphi(x_n) \to 0$.*)
e) Prove that the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is Fréchet space (for the purpose of this exercise, proof it just for $d = 1$).

3. Bonus exercises (not intended for exams): a) Let (X, d) be a metric TVS with d being translation invariant. Prove that there exists a *completion of* X, that is, an F-space $\widetilde{X}$ whose topology is generated by a translation invariant metric d' and a linear isometry $I : X \to \widetilde{X}$ such that $\overline{I(X)} = \widetilde{X}$.
(*Hint: consider*

$$Y := \{(x_n) \colon (x_n) \text{ is cauchy sequence in } X\}$$

and for $(x_n) \in Y$ put $[(x_n)] := \{(y_n) \in Y \colon \lim d(x_n, y_n) = 0\}$. Then put $\widetilde{X} := \{[(x_n)] \colon (x_n) \in Y\}$, endow it with metric $d([(x_n)], [(y_n)]) := \lim d(x_n, y_n)$ and natural vector operations $+$ and $\cdot$. The mapping $I : X \to \widetilde{X}$ will be given by $I(x) = [(x, x, x, \ldots)]$.)
b) Let (X, τ) be a TVS metrizable by a complete metric. Prove that it is an F-space.
(*Hint: pick a translation invariant metric ρ generating the topology τ. Prove that (X, ρ) is G_δ (and therefore comeager) in its completion $(\widetilde{X}, \rho)$ and deduce that for any $x_0 \in \widetilde{X}$ we have $(x_0 + X) \cap X \neq \emptyset$.*

In order to prove that (X, ρ) is G_δ in its completion, pick ρ' generating τ such that (X, ρ') is complete. For every $n \in \mathbb{N}$ and $x \in X$ pick $r_n(x) < \frac{1}{n}$ satisfying $U_\rho(x, r_n(x)) \subset U_{\rho'}(x, \frac{1}{n})$. Finally, show that $X = \bigcap G_n$, where $G_n = \bigcup_{x \in X} U_\rho(x, r_n(x))$ are open sets in $(\widetilde{X}, \rho)$.)

Suitable for credit: exercises 2.b, 2.c, 2.d

EXERCISES 5 (4.11.2022)

1. Easier and essential exercises:
a) Let X be a Banach space. Prove that the canonical isometry $\varepsilon : X \to X^{**}$ is homeomorphism from (X, w) into (X^{**}, w^*).
b) Let X, Y be Banach spaces, (T_i) a bounded net of linear operators from $\mathcal{L}(X,Y)$, $T \in \mathcal{L}(X,Y)$ and $D \subset X$ such that $\overline{\operatorname{span}}\, D = X$. Prove that then $T_i x \to Tx$ for every $x \in X$ if and only if $T_i x \to Tx$ for every $x \in D$.

As a corollary deduce the following:

(i) Pick $p \in [1, \infty]$ and consider $X = \ell_p(\Gamma)$ as a dual space with respect to the standard duality (that is, $\ell_q(\Gamma)^* = \ell_p(\Gamma)$ for $p \in (1, \infty]$ and $c_0(\Gamma)^* = \ell_1(\Gamma)$). Then for a bounded net (x_i) in X and $x \in X$ we have that $x_i \overset{w^*}{\to} x$ if and only if $x_i(\gamma) \to x(\gamma)$, $\gamma \in \Gamma$.

(ii) Pick $p \in (1, \infty)$ and consider $X = \ell_p(\Gamma)$ or $X = c_0$. Then for a bounded net (x_i) in X and $x \in X$ we have that $x_i \overset{w}{\to} x$ if and only if $x_i(\gamma) \to x(\gamma)$, $\gamma \in \Gamma$.

c) Let $X = C([0,1])$ and consider three topologies on X - norm topology $\|\cdot\|$, weak topology w and the topology of pointwise convergence τ_p. Prove that

(i) There exists a sequence in X which is τ_p-convergent, but not bounded in norm.

(ii) A sequence in X is weak convergent if and only if it is norm bounded and τ_p-convergent.

2. Further exercises:
a) Let $p \in (1, \infty)$. Find an example of a sequence (x_n) in ℓ_p such that $x_n(k) \to 0$ for every $k \in \mathbb{N}$, but x_n does not converge to 0 weakly. (*Hint: consider* $x_n = \exp(n) \cdot e_n$)
b) Let X, Y be Banach spaces and $T \in \mathcal{L}(Y^*, X^*)$. Prove that $T = S^*$ for some $S \in \mathcal{L}(X,Y)$ if and only if T is w^*-w^* continuous.
c) Let X be an infinite-dimensional Banach space. Prove that any neighborhood of 0 in the weak topology contains a non-trivial subspace of X. Deduce that $\overline{S_X}^w = B_X$ and then deduce that $\overline{S_{X^*}}^{w^*} = B_{X^*}$.
d) Let X be a Banach space. Prove that $\dim X < \infty$ if and only if weak topology on X coincides with the norm topology if and only if weak star topology on X^* coincides with the norm topology.
e) Let X be an infinite-dimensional Banach space. Find a net (x_i) in X which is weakly convergent to 0, but not bounded.
(*Hint: let us denote the weak topology by* τ_w. *Using 2.c above, for any* $U \in \tau_w(0)$ *pick* $f_U \in X^* \setminus \{0\}$ *with* $\mathbb{R}f \subset U$. *Consider the partially ordered set* $\mathcal{I} = \{(U,n) \colon U \in \tau_w(0),\ n \in \mathbb{N}\}$ *such that* $(U,n) \le (U',n')$ *iff* $U \supset U'$ *and* $n \le n'$. *Finally, consider the net* $(nf_U)_{(U,n)\in\mathcal{I}}$.)

3. Bonus exercises (not intended for exams):
a) Let X be a Banach space, $C > 0$ and $f, g \in S_{X^*}$. Suppose that $\|f|_{\ker g}\| \le C$. Prove that there exists $\alpha \in \mathbb{K}$ with $|\alpha| = 1$ such that $\|f - \alpha g\| \le 2C$.
(*Hint: for* $C \ge 1$ *it is trivial, so suppose* $C < 1$. *Pick the Hahn-Banach extension* $x^* \in X^*$ *of* $f|_{\ker g} \in (\ker g)^*$. *Because* $\ker g \subset \ker(f - x^*)$, *there is* $\beta \in \mathbb{K}$ *satisfying* $f - x^* = \beta g$. *Show that it suffices to put* $\alpha = \frac{\beta}{|\beta|}$.)
b) Let X be a Banach space and $f \in X^{**}$. Prove that $f \in \varepsilon(X)$ if and only if $f|_{B_{X^*}}$ is w^*-continuous.
(*Hint: One implication follows directly from a theorem from the lecture. For the other one, assume that* $f|_{B_{X^*}}$ *is* w^*-*continuous, without loss of generality assume that* $\|f\| = 1$. *For* $\eta \in (0,1)$ *consider the sets* $A_\eta := \{x^* \in B_{X^*} \colon \operatorname{Re} f(x^*) \ge \eta\}$ *and* $B_\eta := \{x^* \in B_{X^*} \colon \operatorname{Re} f(x^*) \le -\eta\}$, *those sets are* w^*-*compact, disjoint and convex, so there is* $x \in X$ *such that for* $g = \varepsilon(x)$ *we have* $\sup_{A_\eta} \operatorname{Re} g < \inf_{B_\eta} \operatorname{Re} g$. *Deduce that* $\|f|_{\ker g}\| \le \eta$ *and use the previous exercise to show that* f *is in the closure of* $\kappa(X)$, *so it is in* $\kappa(X)$.)

Suitable for credit: exercises 2.c+d, 2.a+e, 3.a+b

EXERCISES 6 (11.11.2022)

If not said otherwise, $\Omega \subset \mathbb{R}^d$ is a an open nonempty set and on $\mathcal{D}(\Omega)$ we consider the topology τ from Theorem 64.

Definition. Say that a sequence (x_n) in a TVS X is *cauchy* if for every $U \in \tau(0)$ there exists $n_0 \in \mathbb{N}$ satisfying $x_n - x_m \in U$ for every $n, m \geq n_0$. We say X is *sequentially complete* if every cauchy sequence is convergent.

1. Easier and essential exercises:
a) Prove that τ is the biggest locally convex topology on $\mathscr{D}(\Omega)$ such that the inclusion $i_K : (\mathscr{D}(K), \tau_K) \to (\mathscr{D}(\Omega), \tau)$ is continuous mapping for any compact set $K \subset \Omega$.
b) Prove that the inclusion $i : (\mathscr{D}(\mathbb{R}^d), \tau) \to (C^\infty(\mathbb{R}^d), \tau_{C^\infty})$ is continuous mapping.

2. Further exercises:
a) Find a sequence (f_n) in $\mathscr{D}(\mathbb{R})$ such that $f_n \overset{\tau_{C^\infty}}{\to} 0$, but f_n is not convergent in $\mathscr{D}(\mathbb{R})$.
(*Hint: Pick some* $\psi \in \mathscr{D}(\mathbb{R})$ *with* $\operatorname{supp} \psi \supset [-1, 1]$ *and put* $f_n(x) := \frac{1}{n}\psi(\frac{x}{n})$)
b) Find a sequence (f_n) in $\mathscr{D}(\mathbb{R})$ and $f \in C^\infty(\mathbb{R}) \setminus \mathscr{D}(\mathbb{R})$ such that $f_n \overset{\tau_{C^\infty}}{\to} f$. Deduce that $(\mathscr{D}(\mathbb{R}), \tau_{C^\infty})$ is not sequentially complete.
(*Hint: Pick* $\psi \in \mathscr{D}(\mathbb{R})$ *satisfying* $\operatorname{supp} \psi = [0, 1]$ *and show that* $f_n(x) := \sum_{i=1}^n \frac{\psi(x-i)}{i^2}$ *is cauchy in* $C^\infty(\mathbb{R})$ *and let* f *be the limit of* (f_n) *in* $C^\infty(\mathbb{R})$)
c) Prove that $\mathscr{D}(\Omega)$ is sequentially complete.
d) Let $K \subset \Omega$ be compact with nonempty interior, $x \in \operatorname{Int} K$, $N \in \mathbb{N}$, $\varepsilon > 0$ and $M > 0$. Find $\varphi \in \mathscr{D}(K)$, $\varphi \geq 0$ such that $\|\varphi\|_N < \varepsilon$ and $D^{(\alpha)}\varphi(x) = 0$ whenever $|\alpha| \leq N$, but there is $\beta \in \mathbb{N}_0^d$, $|\beta| = N + 1$ with $|D^{(\beta)}\varphi(x)| > M$.
(*Hint: show that it suffices to handle the case when* $x = 0$ *and dimension* $d = 1$. *In this special case use* $\varphi(t) = t^{N+1}\phi(t)$ *for a suitable function* ϕ)

3. Bonus exercise (not intended for exams): Consider the set

$$V := \{f \in \mathscr{D}(\mathbb{R}) : \ |f(k) f^{(k)}(0)| < 1 \text{ for every } k \in \mathbb{N}\}.$$

Prove that

(i) If $f \in V$ and $W \subset \mathscr{D}(\mathbb{R})$ is an absolutely convex set satisfying $W \cap \mathscr{D}(K) \in \tau_K(0)$ for every compact $K \subset \mathbb{R}$, then $(f + W) \setminus V \neq \emptyset$. In particular, the set $\mathscr{D}(\mathbb{R}) \setminus V$ is dense in $\mathscr{D}(\mathbb{R})$.
(*Hint: By the assumption there are* $N(n) \in \mathbb{N}$ *and* $\varepsilon(n) > 0$ *such that*

$$U_n := U_{\|\cdot\|_{N(n)}, \varepsilon(n)} = \{f \in \mathscr{D}([-n, n]) : \ \|f\|_{N(n)} < \varepsilon(n)\} \subset W \cap \mathscr{D}([-n, n]]).$$

Put $N := N(1)$ *and find* $g \in U_{N+1}$ *satisfying* $|f(N + 1) + \frac{1}{2}g(N + 1)| > 0$ *and* $|g(N + 1)| > 0$. *Observe that by 2.d for any* $M > 0$ *there exists* $\varphi \in U_1$ *satisfying* $|\varphi^{N+1}(0)| > M$. *Use this observation to show that if* M *is big enough, we obtain* $f + \frac{\varphi + g}{2} \in (f + W) \setminus V$.)

(ii) $V \cap \mathscr{D}(K) \in \tau_K(0)$ for every compact $K \subset \mathbb{R}$, but V is not a neighborhood of zero in $\mathscr{D}(\mathbb{R})$.

(iii) The set $\mathscr{D}(\mathbb{R}) \setminus V$ is sequentially closed in $(\mathscr{D}(\mathbb{R}), \tau)$, that is, every convergent sequence of points from $\mathscr{D}(\mathbb{R}) \setminus V$ has the limit in the set $\mathscr{D}(\mathbb{R}) \setminus V$.

Deduce that there exists $f \in \overline{\mathscr{D}(\mathbb{R}) \setminus V}$, which is not a limit of a sequence of functions from $\mathscr{D}(\mathbb{R}) \setminus V$. In particular, $\mathscr{D}(\mathbb{R})$ is not metrizable.

Suitable for credit: exercises 2.a+b, 2.c, 2.d, 3.

1. Essential exercises:
a) Let $\Lambda_{\log|x|}$ be the regular distribution on $\mathbb{R}$ corresponding to the locally integrable function $\log|x|$. Prove that its derivative $(\Lambda_{\log|x|})'$ is the distribution $\Lambda_{\frac{1}{x}}$ on $\mathbb{R}$ given by the formula

$$\Lambda_{\frac{1}{x}}(\varphi) := \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(x)}{x}\,\mathrm{d}x, \qquad \varphi \in \mathscr{D}(\mathbb{R})$$

and moreover we have $x\Lambda_{\frac{1}{x}} = \Lambda_1$.

2. Further exercises:
a) Which of the following formulas define a distribution on $\mathbb{R}$ and which define a distribution on $(0,\infty)$? If the formula defines a distribution find out whether it is of finite order.

(i) $\Lambda(\varphi) = \sum_{n=1}^{\infty} n\varphi^{(n)}(n)$.

(ii) $\Lambda(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n}\varphi(\frac{1}{n})$.

(iii) $\Lambda(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n^2}\varphi^{(n)}(\frac{1}{n})$.

(*Hint: sometimes it helps to use 2.d from Exercises 6*)
b) Let $(a,b) \subset \mathbb{R}$ and $x_0 \in (a,b)$. Prove that $S \in \mathscr{D}((a,b))$ is a solution of the equation $(x-x_0)S = 0$ if and only if there is $c \in \mathbb{K}$ satisfying $S = c\Lambda_{\delta_{x_0}}$. Then deduce that $(x-x_0)^2S = 0$ if and only if $S \in \operatorname{span}\{\Lambda_{\delta_{x_0}}, (\Lambda_{\delta_{x_0}})'\}$.
(*Hint: For the nontrivial implication in the first part consider $Q : \mathscr{D}(\mathbb{R}) \to \mathscr{D}(\mathbb{R})$ given by the formula*

$$Q(\psi)(x) := \int_0^1 \psi'(x_0 + t(x-x_0))\,\mathrm{d}t.$$

Prove that Q is well-defined mapping satisfying $(x-x_0)Q(\psi) = \psi$ whenever $\psi(x_0) = 0$. Deduce that if $(x-x_0)S = 0$ then $\operatorname{Ker}\Lambda_{\delta_{x_0}} \subset \operatorname{Ker} S$. *For the second part, by the first part we have $(x-x_0)S = c\Lambda_{\delta_{x_0}}$, then notice that $(x-x_0)(\Lambda_{\delta_{x_0}})' = -\Lambda_{\delta_{x_0}}$, and finally apply the already proven part to $S + c(\Lambda_{\delta_{x_0}})'$.*)
c) Find all the solutions of the following equations for $S \in \mathscr{D}(\mathbb{R})^*$.

(i) $S' = \Lambda_{\delta_{x_0}}$ $(x_0 \in \mathbb{R})$.

(ii) $S'' = \Lambda_{\delta_{x_0}}$ $(x_0 \in \mathbb{R})$.

(iii) $(1+x)^2S'' = 0$.

(iv) $(x-1)S = \Lambda_1$.

(*Hint: find one "particular solution" and prove that any solution is a particular solution plus general solution of a homogeneous equation .. for the solution of a homogeneous equation use Exercise 2.b) above or Theorem 72*)

3. Bonus exercises (not intended for exams): a) Prove that given $f \in C^\infty(\mathbb{R})$, distribution $S \in \mathscr{D}(\mathbb{R})^*$ solves the equation $S' + fS = 0$ if and only if $S = c\Lambda_{e^{-F(x)}}$ for some constant $c \in \mathbb{K}$ and some function F satisfying $F' = f$.
(*Hint: prove that we have $(e^{F(x)}S)' = e^{F(x)}(S' + fS)$ so S is the solution of our equation iff $(e^{F(x)}S)' = 0$*)
b) Prove that for any $S \in \mathscr{D}(\mathbb{R})^*$ and $x_0 \in \mathbb{R}$ there exists $\Lambda \in \mathscr{D}(\mathbb{R})^*$ satisfying $(x-x_0)\Lambda = S$.
(*Hint: pick any $\phi \in \mathscr{D}(\mathbb{R})$ with $\phi(x_0) = 1$ and consider $Q : \mathscr{D}(\mathbb{R}) \to \mathscr{D}(\mathbb{R})$ given by the formula*

$$Q(\psi)(x) := \int_0^1 \psi'(x_0 + t(x-x_0)) - \psi(x_0)\phi(x_0 + t(x-x_0))\,\mathrm{d}t.$$

Prove that Q is well-defined sequentially continuous mapping satisfying $Q((x-x_0)\varphi) = \varphi$. Finally, put $\Lambda(\psi) := S(Q(\psi))$ for $\psi \in \mathscr{D}(\mathbb{R})$)

Suitable for credit: exercises 2.a, 2.b, 2.c

EXERCISES 8 (25.11.2022)

1. Essential exercises:
a) Prove that

$$\Lambda_1 * \left((\Lambda_{\delta_0})' * \Lambda_{\chi_{(0,\infty)}}\right) \neq \left(\Lambda_1 * (\Lambda_{\delta_0})'\right) * \Lambda_{\chi_{(0,\infty)}}.$$

(that is, prove that all the expressions are well-defined and that the inequality holds)
b) Prove that $\Lambda_{\chi_{(0,\infty)}} * \Lambda_{\chi_{(0,\infty)}} = \Lambda_{\mathrm{id}}$.

2. Further exercises:
a) Given $c > 0$, consider the function

$$f(t,x) := \begin{cases} \frac{1}{2c}, & |x| < ct, \\ 0, & \text{otherwise}, \end{cases} \qquad (t,x) \in \mathbb{R}^2.$$

Prove that

(i) Distribution Λ_f solves the equation $D^{(2,0)}\Lambda - c^2 D^{(0,2)}\Lambda = \Lambda_{\delta_{(0,0)}}$.

(ii) Given $\varphi \in \mathscr{D}(\mathbb{R}^2)$ satisfying $\operatorname{supp}\varphi \subset \mathbb{R} \times (t_0, \infty)$ for some $t_0 \in \mathbb{R}$, there exists $g \in C^\infty(\mathbb{R}^2)$ such that $\operatorname{supp} g \subset \mathbb{R} \times (t_0, \infty)$ and $\partial_t^2 g - c^2 \partial_x^2 g = \varphi$.

(iii) For every $(x_0, t_0) \in \mathbb{R}^2$ find a distribution Λ satisfying equation $D^{(2,0)}\Lambda - c^2 D^{(0,2)}\Lambda = \Lambda_{\delta_{(x_0,y_0)}}$.

(Note: Λ_f is fundamental solution of the "Wave equation")
b) Consider the function

$$f(t,x) := \begin{cases} \frac{1}{\sqrt{4\pi t}} \exp(-\frac{|x|^2}{4t}), & t > 0, \\ 0, & \text{otherwise}, \end{cases} \qquad (t,x) \in \mathbb{R}^2.$$

Prove that

(i) f is locally integrable on $\mathbb{R}^2$,

(ii) $(\partial_t f - \partial_x^2 f)(x,t) = 0$ whenever $t > 0$,

(iii) $\int_{\mathbb{R}} f(t,x)\,\mathrm{d}x = 1$ for every $t > 0$,
(*Hint: use the well-known value* $\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$),

(iv) Distribution Λ_f solves the equation $\partial_t \Lambda - \partial_x^2 \Lambda = \Lambda_{\delta_{(0,0)}}$.
(*Hint: First, using per partes and (i) show that for every* $\varphi \in \mathscr{D}(\mathbb{R})$ *we have*

$$(\partial_t\Lambda - \partial_x^2\Lambda)(\varphi) = \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \int_{\mathbb{R}} f(t,x)(\partial_t\varphi - \partial_x^2\varphi)(t,x)\,\mathrm{d}x\,\mathrm{d}t = \ldots = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} f(\varepsilon, x)\varphi(\varepsilon, x)\,\mathrm{d}x$$

and then using (ii) and the fact that φ *is a Lipchitz map prove that the limit above is equal to* $\varphi(0,0)$)

(Note: Λ_f is fundamental solution of the "Heat equation")

3. Bonus exercises (not intended for exams):
a) Let $f(x) = \|x\|^{-1}$, $x \in \mathbb{R}^3$. Prove that f is locally integrable on $\mathbb{R}^3$ and that the distribution Λ_f solves the equation $\triangle\Lambda_f = -4\pi\Lambda_{\delta_{(0,0,0)}}$.
(Note: $-\frac{1}{4\pi}\Lambda_f$ is fundamental solution of the "Laplace equation")

Suitable for credit: exercises 2.a, 2.b, 3.a

EXERCISES 9 (2.12.2022)

1. Essential exercises:
a) Prove that on $\mathbb{R}$ we have $\widehat{\Lambda_{\delta_0}} = \frac{1}{\sqrt{2\pi}}\Lambda_1$, $\widehat{\Lambda_1} = \sqrt{2\pi}\Lambda_{\delta_0}$ and $\widehat{\Lambda_{\delta_a}} = \frac{1}{\sqrt{2\pi}}\Lambda_{e^{-iax}}$ for every $a \in \mathbb{R}$.

b) Let Λ be a tempered distribution on $\mathbb{R}$. Prove that $\widehat{\widehat{\Lambda}}(\varphi) = \Lambda(\check{\varphi})$ for every $\varphi \in \mathcal{S}_1$.

c) Express on $\mathbb{R}$ the Fourier transform $\widehat{\Lambda_{\cos x}}$ as a linear combination of tempered distributions of the form Λ_{δ_a}, $a \in \mathbb{R}$. (*Hint: express cosinus as exponential and use (a) and then (b)*)

2. Further exercises:
a) Let $f \in L_1^{loc}(\mathbb{R})$, $f \geq 0$. Prove that if Λ_f is tempered distribution, then there are $C > 0$ and $N \in \mathbb{N}_0$ satisfying

$$\forall R \geq 1: \quad \int_{-R}^{R} f(x)\,\mathrm{d}x \leq C(1+R)^N.$$

Deduce that Λ_{e^x} is not a tempered distribution. On the other hand, prove that $\Lambda_{e^x \cos(e^x)}$ is tempered distribution. (*Hint: Pick $A > 0$ and $N \in \mathbb{N}_0$ satisfying $|\Lambda_f(\phi)| \leq A\nu_N(\phi)$, $\phi \in \mathscr{D}(\mathbb{R})$. Fix some $\psi \in \mathscr{D}([-2,2])$ satisfying $\psi|_{[-1,1]} \equiv 1$, then check that for every $R > 0$ we have*

$$0 \leq \int_{-R}^{R} f(x)\,\mathrm{d}x \leq \int_{-R}^{R} f(x)\psi(\tfrac{x}{R})\,\mathrm{d}x \leq A\nu_N(\psi(\frac{\cdot}{R})) \leq \ldots \leq C(1+R)^N.$$

For the "on the other hand" part note that we have $(\sin(e^x))' = e^x\cos(e^x)$ and that $\sin(e^x)$ is bounded function)

b) Which of the following formulas define a tempered distribution on $\mathbb{R}$?

(i) $\Lambda(\varphi) := \sum_{j=-\infty}^{\infty} j^2\varphi(j)$, $\varphi \in \mathscr{D}(\mathbb{R})$.

(ii) $\Lambda(\varphi) := \sum_{j=-\infty}^{\infty} e^j\varphi(j)$, $\varphi \in \mathscr{D}(\mathbb{R})$.

(iii) $\Lambda(\varphi) := \int_0^{10} \frac{\varphi(x)-\varphi(0)}{x}\,\mathrm{d}x + \int_{10}^{\infty} \frac{\varphi(x)}{x}\,\mathrm{d}x$, $\varphi \in \mathscr{D}(\mathbb{R})$.

(*Hint: for (ii) use similar strategy as in Exercise 2.a*)

c) Prove that for a tempered distribution Λ on $\mathbb{R}$ we have

$$\Lambda \in \operatorname{span}\{(\Lambda_{\delta_0})^{(n)} : n \in \mathbb{N}_0\} \Leftrightarrow \widehat{\Lambda} \in \{\Lambda_P : P \text{ is a polynomial}\}.$$

d) Let $d \in \mathbb{N}$ and $(a_\alpha)_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq N}$ be a finite sequence of complex numbers satisfying that the polynom $\sum_{|\alpha| \leq N} a_\alpha (ix)^\alpha$ does not have root in $\mathbb{R}^d$. Prove that then the only tempered distribution Λ satisfying $\sum_{|\alpha| \leq N} a_\alpha D^\alpha \Lambda = 0$ is $\Lambda = 0$.

3. Bonus exercise (not intended for exams): Let Λ be a tempered distribution satisfying the equation $\sum_{|\alpha| \leq N} a_\alpha D^\alpha \Lambda = 0$ (where $(a_\alpha)_{|\alpha| \leq N}$ is finite sequence in $\mathbb{K}$). Consider then the polynomial $P(x) = \sum_{|\alpha| \leq N} a_\alpha (ix)^\alpha$. Prove that the following holds.

(a) If polynomial P does not have root in $\mathbb{R}^d$, then $\Lambda = 0$.

(b) If polynomial P does not have root in $\mathbb{R}^d \setminus \{0\}$, then $\Lambda = \Lambda_Q$ for some polynomial Q.

(c) Aply the above to prove the following generalization of the Liouville theorem: Let $f \in H(\mathbb{C})$ be a holomorphic function satisfying for some $C > 0$ and $N \in \mathbb{N}_0$ that $|f(x)| \leq C(1+|x|)^N$, $x \in \mathcal{C}$. Then f is polynomial of degree at most N.

For the proof of (b) you may without proof use the following well-known result.

Theorem 8. *Let Λ be a distribution on $\mathbb{R}^d$ such that for any $\varphi \in \mathscr{D}(\mathbb{R}^d \setminus \{0\})$ we have $\Lambda(\varphi) = 0$. Then*

$$\Lambda \in \operatorname{span}\{D^\alpha \Lambda_{\delta_0} : \alpha \in \mathbb{N}_0^d\}.$$

Proof. viz. skripta od doc. Johanise a prof. Spurného (Věta 33 on page 136 here: `https://www2.karlin.mff.cuni.cz/~spurny/doc/ufa/funkcionalka.pdf`) □

Suitable for credit: exercises 2.a, 2.b, 2.c+d

EXERCISES 10 (9.12.2022)

1. Essential exercises:

a) Consider on an uncountable set I the σ-algebra $\mathcal{A} := \mathcal{P}(I)$ consisting of all the subsets of I. Prove that the mapping $I \ni i \mapsto e_i \in c_0(I)$ is borel $\mathcal{A}$-measurable, but not strongly $\mathcal{A}$-measurable.

b) Consider the σ-algebra $\mathcal{A}$ consisting of Lebesgue-measurable sets on $[0,1]$. Prove that the mapping $[0,1] \ni x \mapsto e_x \in \ell_2([0,1])$ is weakly $\mathcal{A}$-measurable, but not borel $\mathcal{A}$-measurable.

2. Further exercises:

a) Let $(\Omega, \mathcal{A}, \mu)$ be the interval $(0,\infty)$ with the Lebesgue measure, $\psi : (0,\infty) \to (0,\infty)$ a function and $X = L_p(0,\infty)$ for some $p \in (1,\infty]$. Consider the function $\phi : (0,\infty) \to X$ given by the formula $\phi(t) := \chi_{(0,\psi(t))}$, $t > 0$. Prove that

- If $p \in (1,\infty)$, then ϕ is strongly μ-measurable $\Leftrightarrow$ ϕ is weakly μ-measurable $\Leftrightarrow$ ψ is μ-measurable.
  (*Hint: since X is separable, strong and weak measurability coincide. Next, use without proof the well-known fact that simple functions are dense in L_q and deduce that functions of the form $\{\chi_{(0,T)} :\ T > 0\}$ are linearly dense in X^*, so to test weak measurability it suffices to consider functions of the form $\chi_{(0,T)} \in L_q = X^*$*)

- if $p = \infty$, then ϕ is strongly μ-measurable $\Leftrightarrow$ ψ is μ-measurable and there exists a countable set $C \subset (0,\infty)$ such that $\psi(t) \in C$ for a.e. $t \in (0,\infty)$.
  (*Hint: $\Rightarrow$ to prove measurability of ψ consider functions of the form $\chi_{(0,T)}$ similarly as above, to prove the existence of C note that for characteristic functions in X form a discrete set and use that the range of ϕ is a.e. contained in a separable set; $\Leftarrow$ prove that ϕ is borel μ-measurable and the range of ϕ is a.e. contained in a separable set*)

b) In this exercise we work with real Banach spaces, that is, $\mathbb{K} = \mathbb{R}$. Let $(\Omega, \mathcal{A}, \mu)$ be the interval $(0,1)$ with the Lebesgue measure, $\psi : (0,\infty) \to \mathbb{R}$ a function and $X = L_p(0,\infty)$ for some $p \in [1,\infty)$. Consider the function ϕ given by the formula $\phi(t)(u) := \psi(u)\chi_{(0,t)}(u)$, $t, u \in (0,1)$. Prove that $\phi(t) \in X$ for every $t \in (0,1)$ if and only if $\psi|_{(0,T)} \in L_p((0,T))$ for every $T > 0$. Assume now that $\phi(t) \in X$ for every $t \in (0,1)$ and prove the following.

- The mapping $\phi : (0,1) \to X$ is strongly μ-measurable. Moreover, it is weakly integrable iff $(1-u)\psi(u) \in L_p(0,1)$.
  (*Hint: you may use without the proof the fact that $f \in L_p$ if and only if for every $g \in L_q$ we have $fg \in L_1$, see Exercise 3.a below*)

- Assume $\phi : (0,1) \to X$ is weakly integrable. Prove that it is Pettis integrable and compute the value of the Pettis integral $(P)\int_E \phi \, d\mu$ for any measurable $E \subset (0,1)$.

3. Bonus exercises (not intended for exams):

a) Let $f : (0,1) \to [0,\infty)$ be a measurable function and $p \in (1,\infty)$. Prove that $f \in L_p(0,1)$ if and only if for every $g \in L_q(0,1)$, $g \geq 0$ we have $fg \in L_1(0,1)$.

b) Let $(X, \mathcal{A})$ be a measurable space such that the cardinality of X is greater than continuum. Prove that $\{(x,x) :\ x \in X\}$ is not in the σ-algebra $\mathcal{A} \otimes \mathcal{A}$ on $X \times X$ generated by sets $\{A \times B :\ A, B \in \mathcal{A}\}$.
(*Hint: pick any $U \in \mathcal{A} \otimes \mathcal{A}$. First, prove that there exists a sequence (A_n) in $\mathcal{A}$ such that $U \in \sigma\{A_n \times A_m :\ n, m \in \mathbb{N}\}$. Then for $\sigma \in 2^\omega$ put $B_\sigma := \bigcap_{\{n:\, \sigma(n)=1\}} A_n \cap \bigcap_{\{n:\, \sigma(n)=0\}} (X \setminus A_n)$ and prove that U is union of sets of the from $B_\sigma \times B_\tau$ for some $\sigma, \tau \in 2^\omega$. Deduce that any $\mathcal{A} \otimes \mathcal{A}$-measureable set is union of 2^ω sets of the form $A \times B$ for some $A, B \in \mathcal{A} \otimes \mathcal{A}$. Finally, use the assumption on the cardinality of X to prove that the set $\{(x,x) :\ x \in X\}$ cannot be written as a union of 2^ω sets of the form $A \times B$ for some $A, B \in \mathcal{A} \otimes \mathcal{A}$.*)

c) Consider the Banach space $X = \ell_2(I)$ where the cardinality of I is greater than continuum. Consider the σ-algebra $\mathcal{A}$ on X consisting of borel subsets of X and the measurable space $(X \times X, \mathcal{A} \otimes \mathcal{A})$. Let $f, g : X \times X \to X$ be defined as $f(x,y) = x$ and $g(x,y) = -y$. Prove that both f, g are $\mathcal{A} \otimes \mathcal{A}$-measurable, but $f + g$ is not $\mathcal{A} \otimes \mathcal{A}$-measurable.
(*Hint: use exercise 2a above*)

Suitable for credit: exercises 2.a, 2.b, 3.a, 3.b+c

EXERCISES 11 (16.12.2022)

1. Essential exercises:

a) Let $(\Omega, \mathcal{A}, \mu)$ be the set $\mathbb{N}$ with the counting measure. Consider the function $f : \mathbb{N} \to c_0$ given as $f(n) := \frac{1}{n} e_n$. Prove that f is Pettis integrable, but not Bochner integrable.

2. Further exercises:

a) Let $(\Omega, \mathcal{A}, \mu)$ be the interval $(0, \infty)$ with the Lebesgue measure, $\psi : (0, \infty) \to \mathbb{K}$ a measurable function and $X = L_p(0, \infty)$ for some $p \in [1, \infty)$. Consider the function $f : (0, \infty) \to X$ given by the formula $f(t) := \psi(t)\chi_{(0,t)}$, $t > 0$.

(i) Prove that f is strongly μ-measurable. (*Hint: since X is separable, strong and weak measurability coincide.*)

(ii) Prove that f is Bochner integrable if and only if $\int_0^\infty t^{1/p} |\psi(t)| \,\mathrm{d}t < \infty$. Moreover, if $p = 1$ and f is weakly integrable, then it is Bochner integrable. (*Hint: for the second part use that $x^* \circ f$ is integrable for $x^* = 1 \in L_\infty((0, \infty)) = X^*$*)

(iii) Prove that if $p > 1$ and $\int_0^\infty \Big(\int_u^\infty |\psi(t)| \,\mathrm{d}t \Big)^p \,\mathrm{d}u < \infty$, then f is weakly integrable and therefore also Pettis integrable.

(iv) If $p > 1$, find a function ψ such that the function f is Pettis integrable, but not Bochner integrable.
(*Hint: try to consider a function $\psi = \sum_{n=1}^\infty \varepsilon_n \chi_{[2^n, 2^{n+1})}$ a for a suitable sequence of positive numbers (ε_n).*)

b) For $f \in L_1(\mu; X)$ put

$$\|f\|_{Pettis} := \sup_{x^* \in B_{X^*}} \int_0^1 |x^* \circ f| \,\mathrm{d}t.$$

Let $(\Omega, \mathcal{A}, \mu)$ be the interval $[0, 1]$ with the Lebesgue measure, $X = \ell_2$ and consider functions $f_n : [0, 1] \to \ell_2$ given by

$$f_n(t) := \sum_{k=1}^{2^n} e_k \chi_{[\frac{k-1}{2^n}, \frac{k}{2^n})]}(t), \quad t \in [0, 1].$$

(i) Prove that $\|f_n\|_{L_1(\mu;X)} = 1$, $n \in \mathbb{N}$ but $\|f_n\|_{Pettis} \to 0$.

(ii) Find a sequence $\tilde{f}_n$ in $L_1(\mu; X)$ satisfying $\|\tilde{f}_n\|_{L_1(\mu;X)} \to \infty$, but $\|\tilde{f}_n\|_{Pettis} \to 0$. (*Hint: try to put $\tilde{f}_n = \alpha_n f_n$ for some sequence (α_n).*)

(iii) For $n \in \mathbb{N}$ consider functions $g_n : [0, 1] \to X$ defined as $g_n(t) := 2^n f_n(2^n t - 1)\chi_{[\frac{1}{2^n}, \frac{1}{2^{n-1}})}(t)$ and function $f : [0, 1] \to X$ defined as $g(t) := \sum_{n=1}^\infty g_n(t)\chi_{[\frac{1}{2^n}, \frac{1}{2^{n-1}})}(t)$. Prove that g is not Bochner integrable, but it is Pettis integrable.

(*Hint: first, show that for each $N \in \mathbb{N}$ we have $\int \|f\| \geq \sum_{n=1}^N \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} \|g_n(t)\| = \ldots = N \to \infty$. Then, note that since X is reflexive it suffices to show weak integrability of f, for this purpose compute first the value of $\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} |h(f(t))| \,\mathrm{d}t$ for every $h \in \ell_2$.*)

c) Let $(\Omega, \mathcal{A}, \mu)$ be the interval $[0, 1]$ with the Lebesgue measure, $X = c_0$ and consider the function $F : \mathcal{A} \to X$ given as

$$F(E) := \Big(\int_E \sin(2^n \pi t) \,\mathrm{d}t \Big)_{n=1}^\infty, \quad E \in \mathcal{A}.$$

Prove that $F(E) \in c_0$ and $\|F(E)\| \leq \mu(E)$ for every $E \in \mathcal{A}$. Deduce that F is also σ-additive (that is, for pairwise disjoint sequence (E_n) from $\mathcal{A}$ we have $F(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty F(E_n)$). On the other hand, prove that there does not exist $f \in L_1(\mu; X)$ satisfying $F(E) = \int_E f \,\mathrm{d}\mu$, $E \in \mathcal{A}$. Note: this witnesses that c_0 does not have RNP.
(*Hint: in order to prove $F(E) \in c_0$ use Bessel inequality and the well-known fact that $\{\sqrt{2}\sin(n\pi t) \colon n \in \mathbb{N}\}$ is orthonormal system in $L_2([0, 1])$; In order to prove the nonexistence of $f \in L_1(\mu; X)$ suppose it exists and deduce that then $e_n \circ f_n = \sin(2^n \pi t)$ for every $n \in \mathbb{N}$, prove that for $E_n := \{t \in [0, 1] \colon \sin(2^n \pi t) \geq \frac{1}{\sqrt{2}}\}$ we have $\mu(E_n) = \frac{1}{4}$, deduce that $\mu(\bigcap_{n \in \mathbb{N}} \bigcup_{k=n}^\infty E_k) \geq \limsup \mu(E_k) \geq \frac{1}{4}$ and from this deduce that $\mu(\{t \colon f(t) \notin c_0\}) > 0$, a contradiction.*)

Suitable for credit: exercises 2.a, 2.b, 2.c

EXERCISES 12 (6.1.2022)

1. Essential exercises:

a) Prove that $\operatorname{ext} B_{\ell_1} = \{te_n : n \in \mathbb{N}, t \in S_{\mathbb{K}}\}$.
b) Prove that $\operatorname{ext} B_{\ell_\infty} = \{f \in \ell_\infty : |f(n)| = 1 \text{ for every } n \in \mathbb{N}\}$.
c) Prove that $\operatorname{ext} B_{L_1([0,1])} = \emptyset$.

2. Further exercises:

a) Let H be a Hilbert space. Prove that $\operatorname{ext} B_H = S_H$.(*Hint: use the parallelogram law.*)
b) Prove that $\overline{\operatorname{conv}\operatorname{ext} B_X}^{\|\cdot\|} = B_X$ for $X = \ell_p$, where $p \in [1, \infty)$. (*Hint: for $p > 1$ use Krein-Milman tehorem together with the fact that B_X is weakly closed because X is reflexive. For $p = 1$ proceed directly.*)

Solutions are available at `https://www2.karlin.mff.cuni.cz/~cuth/fa-priklady.pdf`