EXERCISES 1 (14.2.2023)

1. Given a Hausdorff topological space T, prove that
a)

$$C_b(T) := \{f \in C(T) \colon f \text{ is bounded}\}$$

is a closed subalgebra of $\ell_\infty(T)$, so it is a commutative Banach algebra with a unit.
b) If T is locally compact and not compact, then consider

$$C_0(T) := \{f \in C(T) \colon \text{ for any } \varepsilon > 0 \text{ the set } \{t \in T \colon |f(t)| \geq \varepsilon\} \text{ is compact}\}.$$

First, consider the one point compactification $K := T \cup \{\infty\}$ of T.
(recall that topology τ on $T \cup \{\infty\}$ consist of open subsets of T and sets of the from $G = \{\infty\} \cup (L \setminus K_0)$, where $K_0 \subset L$ is a compact set; it is a known fact from the course on general topology that then $(T \cup \{\infty\}, \tau)$ is a compact Hausdorff space and T is dense subset of $T \cup \{\infty\}$)
Prove that the mapping

$$C(K) \supset \{f \in C(K) \colon f(\infty) = 0\} \ni f \mapsto f|_T \in C_0(T)$$

is a surjective linear isometry. Deduce that $C_0(T)$ is a Banach space and that $C_0(T)$ is a closed subalgebra of $C_b(T)$, so it is a commutative Banach algebra. Finally, prove that $C_0(T)$ does not have a unit. (*Hint: prove that if the unit exists, then is has to be constant one function*)

2. Let X be a Banach space with $\dim X > 1$.
a) $\mathcal{L}(X)$ is a Banach algebra (where multiplication is given by composition of operators) with a unit. Prove that $\mathcal{L}(X)$ is not commutative. (*Hint: if $y, z \in X$ are linearly independent, then the operators which do no commute may be taken of the from $x \mapsto x^*(x)y$ and $x \mapsto x^*(x)z$ for some $x^* \in X^*$*)
b) The space of compact operators $\mathcal{K}(X) \subset \mathcal{L}(X)$ is a closed subalgebra. Prove that $\mathcal{K}(X)$ is a Banach algebra, which is not commutative and that it does not have a unit if $\dim X = +\infty$. (*Hint: prove that if the unit exists, then is has to be the identity operator*)

3. a) Prove that $L_1(\mathbb{R}^d)$ with multiplication given by convolution (that is, $f * g(x) := \int f(y)g(x-y)\,\mathrm{d}y$) is a commutative Banach algebra without a unit. (*Hint: in order to see that it does not have a unit, pick $g = \frac{\chi_{B(0,1)}}{\|\chi_{B(0,1)}\|_1} \in S_{L_1}$ put $g_n(x) := n^d g(nx)$ and try to use the fact that if $e \in L_1(\mathbb{R}^d)$ was a unit, we would have $g_n = e * g_n \to e$*)
b) Let G be a commutative group. Prove that $\ell_1(G)$ with multiplication $*$ given by

$$(x * y)(g) := \sum_{h \in G} x(h)y(g-h), \quad x, y \in \ell_1(G)$$

is a commutative Banach algebra with a unit.

EXERCISES 2 (21.2.2023)

1. Let us consider operators $S, T \in \mathcal{L}(\ell_2)$ given by

$$T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots), \quad S(x_1, x_2, \ldots) = (x_2, x_3, \ldots), \qquad x \in \ell_2.$$

a) Prove that T does not have right inverse, but it has left inverse (and describe all the left inverses of T).
b) Prove that S does not have left inverse, but it has right inverse (and describe all the right inverses of S).

2. Consider the commutative group $G = (\mathbb{Z}_n, +)$, where $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and addition is modulo n.
a) Find explicit formula for an isomorphism from the Banach algebra $\ell_1(G)$ into the Banach algebra M_n.
(*Hint: use Theorem 8 from the lecture*).
b) For $n = 2$ and $n = 3$ give an explicit characterization of invertible elements in $\ell_1(G)$.
(*Hint: use the representation by matrices from the previous item*)

EXERCISES 3 (28.2.2023)

1. a) Prove that given a Hausdorff compact space K and $f \in C(K)$, we have $\sigma(f) = \operatorname{Rng} f$.
b) Prove that given a Hausdorff locally compact space T which is not compact and $f \in C_0(T)$, we have $\sigma(f) = \operatorname{Rng} f \cup \{0\}$.
c) Find two examples of Hausdorff locally compact spaces T_1, T_2 which are not compact such that: for every $f \in C_0(T_1)$ we have $\sigma(f) = \operatorname{Rng} f$; there exists $f \in C_0(T_2)$ such that $\sigma(f) \neq \operatorname{Rng} f$. (*Hint: T_1 may be any non σ-compact space; T_2 may be e.g. the real line*)

2. Let us consider the commutative Banach algebra $A = (\ell_1(\mathbb{Z}), *)$ and pick any $n \in \mathbb{Z} \setminus \{0\}$. Prove that $\sigma(e_n) = \{\lambda \in \mathbb{K} \colon |\lambda| = 1\}$ and that

$$R_{e_n}(\lambda) = \begin{cases} \sum_{k=0}^{\infty} \frac{e_{kn}}{\lambda^{k+1}}, & |\lambda| > 1, \\ \sum_{k=1}^{\infty} -\lambda^{k-1} e_{-kn}, & |\lambda| < 1. \end{cases}$$

EXERCISES 4 (7.3.2023)

In this series of exercises, given a Banach algebra A, $x \in A$ and a function f holomorphic on a neighborhood of $\sigma(x)$, we denote $f(x) := \phi(x)f$ (that is, the value of f under the holomorphic calculus corresponding to the element x).

1. Consider the Banach algebra $A = M_n$, $n \geq 2$. Pick some $z \in \mathcal{C}$ and consider the matrix

$$J = \begin{pmatrix} z & 1 & 0 & \cdots & 0 & 0 \\ 0 & z & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z & 1 \\ 0 & 0 & 0 & \cdots & 0 & z \end{pmatrix}$$

a) Prove that $\sigma(J) = \{z\}$.

b) Prove that for $\lambda \in \rho(J)$ we have

$$(\lambda I - J)^{-1} = \begin{pmatrix} \frac{1}{\lambda - z} & \frac{1}{(\lambda-z)^2} & \cdots \frac{1}{(\lambda-z)^n} & \\ 0 & \frac{1}{\lambda-z} & \cdots & \frac{1}{(\lambda-z)^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda-z} \end{pmatrix}$$

c) Let f be a holomorphic function on a neighborhood of z. Prove that

$$f(J) = \begin{pmatrix} f(z) & f'(z) & \frac{f''(z)}{2} & \cdots \frac{f^{(n-1)}}{(n-1)!} & \\ 0 & f(z) & f'(z) \cdots & \frac{f^{(n-2)}}{(n-2)!} & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f(z) \end{pmatrix}$$

d) From the above deduce that the value of $f(J)$ is not given just by $f|_{\sigma(J)}$.

2. (not suitable for a credit, but interesting) a) Let f be a holomorphic function on $\mathbb{C}$ and let $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$, $\lambda \in \mathbb{C}$ be its Taylor expansion. Prove that for every Banach algebra A and every $x \in A$ we have $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

b) Consider the Banach algebra $A = C(K)$ and $g \in A$. Prove that whenever F is a holomorphic function on a neighborhood of $\sigma(g) = \operatorname{Rng} g$, then $F(g) = F \circ g$.

3. Consider the Banach algebra $A = \mathcal{L}(X)$ (X is infinite-dimensional Banach space) and for $x_0 \in X \setminus \{0\}$ and $x^* \in X^*$ with $x^*(x_0) \neq 0$ consider $T \in A$ given by the formula $Tx := x^*(x)x_0$, $x \in X$.

a) Prove that $\sigma(T) = \{0, x^*(x_0)\}$ and for $\lambda \notin \{0, x^*(x_0)\}$ find a formula for $R_T(\lambda)$.

(*Hint: the solution is* $R_T(\lambda) = \frac{1}{\lambda} I + \frac{1}{\lambda(\lambda - x^*(x_0))} T$).

b) Given a function f holomorphic on a neighborhood of $\sigma(T)$ compute the value of $f(T)$.

(*Solution:* $f(T) = I \cdot f(0) + T \cdot \frac{f(x^*(x_0)) - f(0)}{x^*(x_0)}$*; Hint: first using the formula for* $R_T(\lambda)$ *observe that it suffices to compute the curve integrals of functions* $\frac{f(\lambda)}{\lambda}$ *and* $\frac{f(\lambda)}{\lambda(\lambda - x^*(x_0))}$*, when computing the integral of the second function observe that decomposition using partial fractions we have* $\frac{f(\lambda)}{\lambda(\lambda - x^*(x_0))} = \frac{1}{x^*(x_0)}\left(\frac{f(\lambda)}{\lambda - x^*(x_0)} - \frac{f(\lambda)}{\lambda}\right)$)

EXERCISES 5 (14.3.2023)

1. Let K be a compact Hausdorff space and for closed $F \subset K$ denote $I(F) := \{f \in C(K) \colon f|_F \equiv 0\}$. Prove that all the closed ideals of $C(K)$ are $\{I(F) \colon F \subset K \text{ closed}\}$.
(*Hint: each $I(F)$ is a closed ideal; if $I \subset C(K)$ is an ideal, put $F := \bigcap_{f \in I} f^{-1}(0)$ and prove that $I(F) = \overline{I}$.*)

2. a) Let $(G, +)$ be a commutative group and $A = \ell_1(G)$. Prove that $\varphi \in \ell_\infty(G) = A^*$ belongs to $\Delta(A)$ if and only if $\varphi : G \to \mathbb{T} := \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}$ is a group homomorphism.
(*Hint: Note that $e_{g+h} = e_g * e_h$.*)
b) For $A = \ell_1(\mathbb{Z})$ use a) to describe $\Delta(A)$ and explain how to understand the equality $\Delta(A) = \mathbb{T}$.
c) Consider the commutative group $G = (\mathbb{Z}_n, +)$, where $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and addition is modulo n. For $A = \ell_1(\mathbb{Z}_n)$ use a) to describe $\Delta(A)$ and prove that it consists of exactly n elements.

EXERCISES 6 (21.3.2023)

1. Consider the commutative Banach algebra $A = \ell_1(\mathbb{Z})$.
a) Using the identification $\Delta(A) = \mathbb{T}$ from one Exercise 5.2b) above, describe the Gelfand transform of A and (using it) express the spectrum of a general element of A.
b) Is the Gelfand transform one-to-one? If yes, what is its inverse? (*Hint: use knowledge of Fourier series*)
c) Consider the mapping $* : A \to A$ given by $x^*(n) := \overline{x(-n)}$, $n \in \mathbb{Z}$ for every $x \in A$. Prove that $*$ is an involution on A and that for $x = e_0 - e_1 - e_2$ we have $\|x^* x\| = 5$ and $\|x\|^2 = 9$ (therefore, $(A, *)$ is not a C^*-algebra).

2. Consider the commutative Banach algebra $A = L_1(\mathbb{R})$ and representation of its dual $A^* = L_\infty(\mathbb{R})$.
a) For $x \in \mathbb{R}$, consider the function $\phi_x \in L_\infty$ defined as $\phi_x(t) := e^{itx}$, $t \in \mathbb{R}$. Prove that $\{\phi_x \colon x \in \mathbb{R}\} \subset \Delta(A)$.
b) Let us mention the (nontrivial) known fact that

$$\Delta(A) \subset \{f \in C(\mathbb{R}) \colon |f(t)| = 1 \text{ and } f(t+s) = f(t)f(s) \text{ for every } t, s \in \mathbb{R}\}$$

(proof is e.g. on page 288 here: `https://www2.karlin.mff.cuni.cz/~spurny/doc/ufa/funkcionalka.pdf`).
Using the above mentioned fact, prove that $\{\phi_x \colon x \in \mathbb{R}\} = \Delta(A)$.
(*Hint: pick $f \in C(\mathbb{R})$ satisfying $f(t+s) = f(t)f(s)$ for $t, s \in \mathbb{R}$, prove equality $f(t) \int_0^{t_0} f(s)\,\mathrm{d}s = \int_t^{t+t_0} f(s)\,\mathrm{d}s$ for ever y t and deduce that f is differentiable, then observe that f satisfies differential equation $f'(t) = f'(0)f(t)$ for every t*)
c) Using the identification of $\Delta(A)$ with $\mathbb{R}$ from part b), show that the Gelfand transform on $L_1(\mathbb{R})$ and the Fourier transform on $L_1(\mathbb{R})$ are up to a constant identical.
d) Consider the mapping $* : A \to A$ given by $f^*(x) := \overline{f(x)}$, $x \in \mathbb{R}$ for every $f \in A$. Prove that $*$ is an involution on A and that for $f = i(\chi_{(0,1)} - \chi_{(-1,0)})$ we have $\|f * f^*\| = \frac{8}{3}$ and $\|f\|^2 = 4$ (therefore, $(A, *)$ is not a C^*-algebra).

EXERCISES 7 (28.3.2023)

1. Let H be a Hilbert space and $A \subset \mathcal{L}(H)$ be a closed $*$-subalgebra. Consider the set $M_n(A)$ consisting of $n \times n$ matrices with entries belonging to A.
a) Define natural algebraic operations on $M_n(A)$ and a norm on $M_n(A)$ in such a way that $M_n(A)$ is isometric to a $*$-subalgebra of $\mathcal{L}(H^n) = \mathcal{L}(H \oplus_2 \ldots \oplus_2 H)$. Prove that then $M_n(A)$ is a C^*-algebra.
b) For a Hausdorff compact space K consider the C^*-algebra $C(K, A) := \{f : K \to A\colon f \text{ is continuous}\}$ (on $C(K, A)$ we consider the supremum norm and the algebraic operations are defined pointwise). Prove that if $A = M_n(\mathbb{C})$, then the C^*-algebras $C(K, A)$ and $M_n(C(K))$ are isometrically $*$-isomorphic.

2. a) Let A be a C^*-algebra with a unit. Let $a, b \in A$ be normal elements which are unitarily equivalent (that is, there exists $u \in A$ with $u^* = u^{-1}$ and $u^*au = b$). Prove that the C^*-algebras $\overline{\text{alg}}\{e, a, a^*\}$ and $\overline{\text{alg}}\{e, b, b^*\}$ are isometrically $*$-isomorphic.
(*Hint: prove that* $\sigma(a) = \sigma(b)$ *and use continuous calculus*)
b) Let A be a C^*-algebra with a unit, $a \in A$ be self-adjoint with $\sigma(a) \subset [0, \varepsilon] \cup [1 - \varepsilon, 1]$ for some $\varepsilon \in (0, \frac{1}{4})$. Prove that there exists a projection $p \in A$ (that is, some $p \in A$ satisfying $p = p^* = p^2$) such that $\|p - a\| \leq \varepsilon$.
(*Hint: using the continuous calculus, define* $p = g(a)$ *for a suitable* $g \in C(\sigma(a))$ *with* $\text{Rng}(g) \subset \{0, 1\}$)
c) Let $i : (0, 1) \to \mathbb{R}$ be the inclusion map. Then i is continuous. Show that $\varphi : C_0(\mathbb{R}) \to C_0((0, 1))$ defined by $\varphi(f) := f \circ i$ is not a $*$-homomorphism. What goes wrong?

EXERCISES 8 (4.4.2023)

1. Consider the operator $T : L_2([0, 1]) \to L_2([0, 1])$ given by the formula $Tf(x) := \int_0^x f(t)\,\mathrm{d}t$, $x \in [0, 1]$.
a) Prove that T is a compact operator. (*Hint: consider operators* $A : L_2([0, 1]) \to C([0, 1])$ *and* $B : C([0, 1]) \to L_2([0, 1])$*, where* A *is given by the same formula as* T *and* B *is the "identity"; prove that both* A *and* B *are continuous linear and using Arzela-Ascoli theorem show that* A *is compact; then use the identity* $T = B \circ A$*.*)
b) Prove that if $T^*Tf = \lambda f$ for some $\lambda > 0$ and $f \in L_2([0, 1])$, then $f \in C^2([0, 1])$ and $\lambda f'' + f = 0$.

2. Consider the operator $T : L_2([0, 1]) \to L_2([0, 1])$ given by the formula $Tf(x) := \int_0^x f(t)\,\mathrm{d}t$, $x \in [0, 1]$. In this exercise you may use both **1.**a and **1.**b from the exercise above. Using the proof of the Schmidt theorem, find positive numbers $(\lambda_n)_{n\in\mathbb{N}}$ and orthogonal systems of functions $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $L_2([0, 1])$ such that

$$Tf = \sum_{n=0}^{\infty} \lambda_n \Big(\int_0^1 f(t) f_n(t)\,\mathrm{d}t \Big) g_n.$$

(Recall: in what sense is the sum above convergent?)

EXERCISES 9 (11.4.2023)

1. Let $H = \ell_2$. For $z \in \ell_\infty$ consider the operator $M_z \in \mathcal{L}(\ell_2)$ defined as $M_z(x) = (z_n x_n)_{n=1}^\infty$ (recall that $\|M_z\| = \|z\|_\infty$ and $\sigma(M_z) = \overline{\{z_n : n \in \mathbb{N}\}}$, see e.g. Příklad 8 here: `https://www2.karlin.mff.cuni.cz/~spurny/doc/fa2/fa-priklady.pdf`). Fix some $z \in \ell_\infty$.

a) Prove that M_z is normal operator.

Further, by $\phi : \mathrm{Bor}_b(\sigma(M_z)) \to \mathcal{L}(H)$ denote the borel calculus from Theorem 89 and by $\mu_{x,y}$ the measures from Definition 86. Prove that the following holds.

b) For $f \in C(\sigma(M_z))$ we have $\phi(f) = M_{f\circ z}$, where $f \circ z = (f(z_n))_{n=1}^\infty$.

c) For every $n, m \in \mathbb{N}$ we have $\mu_{e_n,e_n} = \delta_{z_n}$ and $\mu_{e_n,e_m} = 0$ if $n \neq m$.

d) For $x, y \in H$ we have $\mu_{x,y} = \sum_{n=1}^\infty x_n \overline{y_n} \delta_{z_n}$.

(*Hint: for finitely supported vectors use the already proven part and Remark 87; for the general case consider* $a_N = \sum_{i=1}^N x_i e_i$, $b_N = \sum_{i=1}^N y_i e_i$, *note that for any* $T \in \mathcal{L}(H)$ *we have* $\langle Ta_N, b_N\rangle \to \langle Tx, y\rangle$ *and apply it for* $T = \phi(f)$)

e) For $g \in \mathrm{Bor}_b(\sigma(M_z))$ we have $\phi(g) = M_{g\circ z}$.

f) Every $A \subset \sigma(M_z)$ is $\mu_{x,y}$-measurable and if A is moreover Borel, then $\phi(\chi_A)x = \sum_{n, z_n \in A} x_n e_n$ for every $x \in H$.

(*Hint: use the well-known fact that given* $\mu \in M(\sigma(M_z))$ *with* $\mu \geq 0$, *a set* $A \subset \sigma(M_z)$ *is* μ*-measurable if and only if there are Borel sets* $B \subset A \subset C$ *such that* $\mu(C \setminus B) = 0$.)

2. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space with the property that given $A \subset \Omega$ with $\mu(A) > 0$, there exists $B \subset A$ with $\mu(B) \in (0, \infty)$. Let $H = L_2(\mu)$ and $g \in L_\infty(\mu)$ and consider the operator $M_g \in \mathcal{L}(H)$ given as $M_g(f) = g \cdot f$.

a) Prove that $\|M_g\| = \|g\|_\infty$ and that M_g is normal operator.

b) Prove that

$$\sigma(M_g) = \operatorname{essRng} g := \{\lambda \in \mathbb{K} : \mu\big(g^{-1}(U(\lambda, \varepsilon))\big) > 0 \text{ for every } \varepsilon > 0\}.$$

c) If $\phi : \mathrm{Bor}_b(\sigma(M_g)) \to \mathcal{L}(H)$ is the borel calculus from Theorem 89, then for any $f \in \mathrm{Bor}_b(\sigma(M_g))$ we have $\phi(f) = M_{f\circ g}$.

d) Given a borel set $A \subset (\sigma(M_g))$, we have $\phi(\chi_A) = M_{\chi_{g^{-1}(A)}}$.

(*Hint: for a sketch of the solution see page 229 here*:
`https://www2.karlin.mff.cuni.cz/~spurny/doc/ufa/funkcionalka.pdf`)

EXERCISES 10 (18.4.2023)

1. Let H be a Hilbert space and $T \in \mathcal{L}(H) \setminus \{0\}$ compact normal operator. Let $\{\lambda_n\}_{n=1}^M$, $M \in \mathbb{N} \cup \{\infty\}$ be one-to-one sequence of all the eigenvalues of the operator T and P_n orthogonal projection onto $\mathrm{Ker}(\lambda_n I - T)$. Using Corollary 97 prove that

- $T = \sum_{n=1}^M \lambda_n P_n$, where the series converges in the SOT topology on the space $\mathcal{L}(H)$;
- the mapping $\sigma(T) \supset A \mapsto E(A) := \sum_{\{n:\lambda_n \in A\}} P_n$ is spectral measure satisfying that $T = \int \mathrm{id}\, \mathrm{d}E$;
- if $f : \sigma(T) \to \mathbb{C}$ is a bounded function and E is as above, then $\int f \, \mathrm{d}E = \sum_{n=1}^M f(\lambda_n) P_n$, where the series converges in the SOT topology on the space $\mathcal{L}(H)$.

2. a) Let H, K be Hilbert spaces, $T \in \mathcal{L}(H) \setminus \{0\}$ normal operator and $U : H \to K$ unitary operator (that is, surjective isometry). Let E be the spectral measure satisfying $T = \int \mathrm{id}\, \mathrm{d}E$. Consider now the operator $T_U := U^*TU \in \mathcal{L}(K)$. Prove that T_U is normal operator and if E_U is defined as $E_U(A) := U^*E(A)U$, then E_U is the spectral measure satisfying $T_U = \int \mathrm{id}\, \mathrm{d}E_U$.
b) Consider now the Hilbert space $H = \ell_2(\mathbb{Z})$ and the unique operator $T \in \mathcal{L}(H)$ satisfying $T(e_k) = e_{k-1}$, $k \in \mathbb{Z}$. Further, consider the Hilbert space $K = L_2(\mathbb{T}, \mu)$, where μ is the normalized Lebesgue measure on the circle (that is, $\int_{\mathbb{T}} f(t)\, \mathrm{d}\mu = \frac{1}{2\pi} \int f(e^{ix})\, \mathrm{d}x$) and the unique operator $U : \ell_2 \to L_2(\mathbb{T}, \mu)$ given by $Ue_k = t^k$. Prove that then T is normal, U is unitary and that for the function $g \in L_\infty(\mu)$ satisfying $g(t) = t^{-1}$, $t \in \mathbb{T}$ we have $T = U^*M_gU$, where M_g is the operator from Exercises 9.2. As a corollary, find spectral decomposition of the operator T (that is, a formula for the spectral measure E such that $T = \int \mathrm{id}\, \mathrm{d}E$).
(*Hint: for a sketch of the solution see page 113 here*:
`https://www2.karlin.mff.cuni.cz/~spurny/doc/fa2/fa-priklady.pdf`)

EXERCISES 11 (25.4.2023)

1. a) Let $X = \ell_p$, $p \in [1, \infty)$ and $z = (z_n)$ be a sequence of (real or complex) numbers. Let

$$D(M_z) := \{x \in X \colon (x_n z_n) \in X\}.$$

Prove that $D(M_z) \subset X$ is dense subspace. Consider now the operator $M_z : D(M_z) \to X$ defined as $M_z(x) := (x_n z_n)$, $x \in D(M_z)$. Prove that M_z is densely defined, closed and that it is bounded (hence everywhere defined) if and only if the sequence z is bounded.

b) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with μ semifinite (that is, if $A \in \mathcal{A}$ is such that $\mu(A) > 0$, then there exists $B \subset A$ with $\mu(B) \in (0, \infty)$). Let $X = L_p(\mu)$, $p \in [1, \infty)$ and $g : \Omega \to \mathbb{K}$ be a measurable function. Consider

$$D(M_g) := \{f \in X \colon gf \in X\}.$$

Prove that $D(M_g)$ is dense subspace of X. Consider now the operator $M_g : D(M_g) \to X$ defined as $M_g(f) = gf$, $f \in D(M_g)$. Prove that M_g is densely defined, closed and that it is bounded (hence everywhere defined) if and only if the function g is essentially bounded.

2. a) Let $X = \ell_p$, $p \in (1, \infty)$ and

$$Y := \{x \in X \colon (nx_n) \in X \& \sum_{n=1}^{\infty} x_n = 0\}.$$

Prove that Y is a dense subspace. Consider now the operator $T : Y \to X$ defined as $Tx := (nx_n)$, $x \in Y$. Prove that T is densely defined and closed.

(*Hint: using Hölder inequality prove that if* $(nx_n) \in X$ *implies* $x \in \ell_1$*, this will prove that* Y *is well-defined and a linear subspace; in order to prove that* Y *is dense prove that* $c_{00} \subset \overline{Y}$ *and in order to prove that* T *is closed use definitions and Hölder inequality*)

b) Let $X = L_p((1, \infty))$, $p \in (1, \infty)$ and

$$Y := \{f \in X \colon (t \mapsto tf(t)) \in X \& \int_1^{\infty} f = 0\}.$$

Prove that Y is a dense subspace. Consider now the operator $T : Y \to X$ defined as $Tf(t) := tf(t)$, $f \in Y$, $t \in (1, \infty)$. Prove that T is densely defined and closed.

(*Hint: using Hölder inequality prove that if* $(t \mapsto tf(t)) \in X$ *implies* $x \in L_1$*, this will prove that* Y *is well-defined and a linear subspace; in order to prove that* Y *is dense prove that* $\chi_A \in \overline{Y}$ *for* $A \subset (1, \infty)$ *bounded and in order to prove that* T *is closed use definitions and Hölder inequality*)

3. Let $X = L_p((0, 1))$, $p \in [1, \infty)$ and

$$Y := \{f \in AC([0, 1]) \colon f' \in X\} \subset X.$$

Define the operators T_j, $j = 1, \ldots, 6$ by the same formula $T_j(f) := f'$ with domains

$$D(T_1) := Y, \quad D(T_2) := \{f \in Y \colon f(0) = 0\},$$
$$D(T_3) := \{f \in Y \colon f(1) = 0\}, \quad D(T_4) := \{f \in Y \colon f(0) = f(1) = 0\},$$
$$D(T_5) := \{f \in Y \colon f(0) = f(1)\}, \quad D(T_6) := \{f \in Y \colon f(0) = -f(1)\}.$$

Show that all those operators are densely defined and closed.

(*To prove density use that test functions are dense in* L_p*; to prove they are closed pick* $(f_n, f_n') \to (f, g)$ *and show e.g. that* $h_n(x) := f_n(x) - f_n(0) = \int_0^x f_n'$ *is a cauchy sequence in* $C([0, 1])$*, further show that* $f_n(0)$ *has a convergent subsequence and deduce that* $f(x) = \int_0^x g + const.$)

EXERCISES 12 (2.5.2023)

In the following exercises, given an operator T on a Banach space X, we put $\sigma_p(T) := \{\lambda \in \mathbb{K} \colon \lambda I - T \text{ is not one-to-one}\}$.
1. Let M_z and M_g be the operators from Exercises 11.1. Find $\sigma_p(M_z)$, $\sigma_p(M_g)$ and show that

$$\sigma(M_z) = \overline{\{z_n \colon n \in \mathbb{N}\}}$$

and

$$\sigma(M_g) = \operatorname{essRng} g := \{\lambda \in \mathbb{K} \colon \mu\big(g^{-1}(U(\lambda, \varepsilon))\big) > 0 \text{ for every } \varepsilon > 0\}.$$

2. Find $\sigma_p(T)$ and $\sigma(T)$ for the operators T from Exercises 11.2.
3. Find $\sigma_p(T)$ and $\sigma(T)$ for the operators $T \in \{T_i \colon i = 1, \ldots, 6\}$ from Exercises 11.3.
(*Hint: note that given $g \in X$, the function $f(t) := -e^{\lambda t} \int_0^t g(s) e^{-\lambda s}\,\mathrm{d}s$ solves the equation $\lambda f - f' = g$*)

EXERCISES 13 (9.5.2023)

1. Consider the operators T_i, $i \in \{1, \ldots, 6\}$ from Exercises 11.2 in the space $L_2((0,1))$. Prove that
a) we have $T_1^\star = -T_4$, $T_2^\star = -T_3$, $T_3^\star = -T_2$, $T_4^\star = -T_1$, $T_5^\star = -T_5$, $T_6^\star = -T_6$.
(*Hint: To prove the inclusions '$\supset$' use integration by parts. To prove '$\subset$' proceed as follows: Let $g \in D(T_i^\star)$. Then there is $h \in L_2((0,1))$ such that $\langle T_i f, g\rangle = \langle f, h\rangle$ for any $f \in D(T_i)$. Set $H(t) := \int_0^t h(s)\,\mathrm{d}s$, $t \in [0,1]$. Apply integration by parts. Note that $\mathcal{D}((0,1)) \subset D(T_i)$ and deduce that the distributive derivative of $g + H$ on $(0,1)$ is zero, thus $g + H$ is almost everywhere equal to a constant. So, $g \in AC([0,1])$ and $H = g(0) - g$. Plug this to the computation and conclude.*)
b) iT_5 and iT_6 are self-adjoint and iT_4 is symmetric.

2. a) Consider the operator T from Exercises 11.2a in the space ℓ_2. Prove that

$$D(T^\star) = \{x \in \ell_2 \colon \lim n x_n \text{ exists and is finite, and } (k x_k - \lim n x_n)_{k=1}^\infty \in \ell_2\}$$

and $T^\star x = (k x_k - \lim n x_n)_{k=1}^\infty$, $x \in D(T^\star)$.
(*Hint: "$\supset$" is easy using definitions; for "$\subset$" proceed as follows: Let $y \in D(T^\star)$, then there exists $z \in \ell_2$ such that $\langle Tx, y\rangle = \langle x, z\rangle$ for $x \in D(T)$, apply it for $x = e_1 - e_n$, $n \geq 2$ and deduce that $n y_n = y_1 - z_1 + z_n$. Using that $z_n \to 0$ compute $\lim n y_n$ and conclude.*)
b) Consider the operator T from Exercises 11.2b in the space $X = L_2((1,\infty))$. Set

$$D(\psi) = \{f \in X \colon \exists C \in \mathbb{C} \text{ such that the function } t \mapsto C + t f(t) \text{ is in } X\}.$$

Show that $D(\psi)$ is a dense linear subspace of X, that for $f \in D(\psi)$ the constant C from the definition is uniquely determined and the mapping $\psi : D(\psi) \to \mathbb{C}$ mapping f to the respective C (that is, such that $(t \mapsto \psi(f) + t f(t)) \in X$) is a linear functional defined on $D(\psi)$.

Show that $D(T^\star) = D(\psi)$ and $T^\star f(t) = \psi(f) + t f(t)$, $t \in (1,\infty)$ for $f \in D(T)$.
(*Hint: "$\supset$" is easy using definitions; for "$\subset$" proceed as follows: Let $g \in D(T^\star)$, then there exists $h \in X$ such that $\langle Tf, g\rangle = \langle f, h\rangle$ for $f \in D(T)$, apply it for $f = (r-1)\chi_{(1,2)} - \chi_{(r,1)}$. Differentiate the resulting equality with respect to r and deduce that $h(r) = \int_1^2 (h(t) - t g(t))\,\mathrm{d}t + r g(r)$ almost everywhere. Then complete the argument.*)

EXERCISES 14 (16.5.2023)

1. Let $(\Omega, \mathcal{A}, \mu)$ be as in Exercises 11.1b and for $g : \Omega \to \mathbb{C}$ measurable let M_g be the operator as in Exercises 11.1b.
a) Prove that for every $g : \Omega \to \mathbb{C}$ measurable we have $(M_g)^\star = M_{\overline{g}}$.
b) For a measurable $g : \Omega \to \mathbb{R}$ find $h : \Omega \to \mathbb{C}$ measurable such that M_h is the Cayley transformation of M_g.
(Hint: you may use without proof the fact used in the solution of Exercises 12.1 that if $\lambda \notin \operatorname{essRng} g$ then $(\lambda I - M_g)^{-1} = M_{\frac{1}{\lambda - g}}$.)
c) Find a characterization of measurable functions $g : \Omega \to \mathbb{C}$ such that $M_g \in \mathcal{L}(\mu)$ is a unitary operator and $I - M_g$ is one-to-one. Further, for those functions g find $h : \Omega \to \mathbb{C}$ measurable such that Cayley transformation of M_h is M_g.
(Hint: you may use without proof the fact used in the solution of Exercises 12.1 that $\lambda I - M_g$ is one-to-one if and only if $\mu(g^{-1}(\lambda)) = 0$.)

2. a) Compute the Cayley transform of the self-adjoint operators iT_5 and iT_6 from Exercises 13.1b.
Hint: you may without proof use the fact used in the solution of Exercises 12.3 that $\sigma(T_5) = \sigma_p(T_5) = \{2\pi i n \colon n \in \mathbb{Z}\}$, $\sigma(T_6) = \sigma_p(T_6) = \{-i\frac{\pi}{2} + 2\pi i n \colon n \in \mathbb{Z}\}$, for $\lambda \notin \sigma(T_5)$ we have

$$(\lambda I - T_5)^{-1} g(t) = -e^{-\lambda t}\Big(\int_0^t g(s) e^{-\lambda s}\, \mathrm{d}s + \tfrac{e^\lambda}{1-e^\lambda}\int_0^1 g(s) e^{-\lambda s}\, \mathrm{d}s\Big)$$

and for $\lambda \notin \sigma(T_6)$ we have

$$(\lambda I - T_6)^{-1} g(t) = -e^{-\lambda t}\Big(\int_0^t g(s) e^{-\lambda s}\, \mathrm{d}s - \tfrac{e^\lambda}{1+e^\lambda}\int_0^1 g(s) e^{-\lambda s}\, \mathrm{d}s\Big).$$

b) Compute the Cayley transform of the symmetric operator iT_4 from Exercises 13.1b.
Hint: you may without proof use the fact used in the solution of Exercises 12.3 that $\sigma(T_4) = \emptyset$ and for $\lambda I - T_4$ and $g \in L_2((0,1))$ the equation $(\lambda I - T)f = g$, $f \in D(T_4)$ has a solution if and only if $g \in \{e^t\}^\perp$ and in this case the solution is

$$f(t) := -e^{\lambda t}\int_0^t g(s) e^{-\lambda s}\, \mathrm{d}s.$$

FURTHER EXERCISE

Not covered during semester, may be used during oral Exam.

1. Let $(\Omega, \mathcal{A}, \mu)$, $g : \Omega \to \mathbb{C}$ and M_g be as in Exercises 11.1b. For a Borel set $A \subset \mathbb{C}$ and $f \in L_2(\mu)$ put $E(A)f := M_{\chi_{g^{-1}(A)}} f$. Prove that E is spectral measure for $(\mathbb{C}, \operatorname{Bor}(\mathbb{C}), L_2(\mu))$ satisfying $M_g = \int \mathrm{id}\, \mathrm{d}E$.
(Hint: see Příklad 69 here `https://www2.karlin.mff.cuni.cz/~spurny/doc/ufa/funkcionalka.pdf`)