

# 1. Uniform Spaces

## 1.1. Relation to Topological and Metric Spaces

**Remark.** For a set  $X$ , denote the diagonal by  $\Delta(X) = \{(x, y) \in X^2 : x = y\}$ . The inverse relation to a relation  $E$  is the relation  $\{(x, y) : (y, x) \in E\}$ . For binary relations  $C, D$  on a set  $X$  we define  $C \circ D = \{(x, z) \in X \times X : (x, y) \in C, (y, z) \in D \text{ for some } y \in X\}$ .

**Definition.** A pair  $(X, \mathcal{D})$  is called a *uniform space* (US) if  $X$  is a set and  $\mathcal{D} \subseteq \mathcal{P}(X \times X)$  is nonempty and satisfies:

- (Ua)  $\forall D \in \mathcal{D} : \Delta(X) \subseteq D$ , (Ud)  $\forall D \in \mathcal{D} : D^{-1} \in \mathcal{D}$ ,  
 (Ub)  $\forall C, D \in \mathcal{D} : C \cap D \in \mathcal{D}$ ,  
 (Uc) if  $D \in \mathcal{D}$  and  $D \subseteq E$ , then  $E \in \mathcal{D}$ , (Ue)  $\forall D \in \mathcal{D} \exists C \in \mathcal{D} : C \circ C \subseteq D$ .

The system  $\mathcal{D}$  is called a *uniformity*. Elements of  $\mathcal{D}$  are called *neighborhoods of the diagonal*. A uniformity  $\mathcal{D}$  is called separated if, in addition,

- (Uf)  $\bigcap \mathcal{D} = \Delta(X)$ . (Equivalently, for every  $x, y \in X, x \neq y$ , there exists  $D \in \mathcal{D}$  such that  $(x, y) \notin D$ .)

If  $\mathcal{D}$  is separated, then we say that the uniform space  $(X, \mathcal{D})$  is  $T_1$ .

A system  $\mathcal{B} \subseteq \mathcal{P}(X^2)$  is called a *base of a uniformity* (respectively, a *base of the uniformity*  $\mathcal{D}$ ) if closing  $\mathcal{B}$  under supersets yields a uniformity (respectively, the uniformity  $\mathcal{D}$ ). A system  $\mathcal{S} \subseteq \mathcal{P}(X^2)$  is called a *subbase of a uniformity* (respectively, a *subbase of the uniformity*  $\mathcal{D}$ ) if closing it under finite intersections yields a base of a uniformity (respectively, a base of the uniformity  $\mathcal{D}$ ).

If  $(X, \mathcal{D})$  and  $(Y, \mathcal{E})$  are uniform spaces, then we say that a mapping  $f : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})$  is *uniformly continuous* if  $(f \times f)^{-1}(E) \in \mathcal{D}$  for every  $E \in \mathcal{E}$ . A mapping  $f$  is called a *uniform homeomorphism* if  $f$  is a bijection and both  $f$  and  $f^{-1}$  are uniformly continuous.

**Lemma 1.** A nonempty system  $\mathcal{B} \subseteq \mathcal{P}(X^2)$  forms a base of some uniformity on  $X$  if and only if the following conditions hold:

- (a)  $\forall C \in \mathcal{B} : \Delta(X) \subseteq C$ , (c)  $\forall C \in \mathcal{B} \exists D \in \mathcal{B} : D \subseteq C^{-1}$ ,  
 (b)  $\forall C, D \in \mathcal{B} \exists E \in \mathcal{B} : E \subseteq C \cap D$ , (d)  $\forall D \in \mathcal{B} \exists C \in \mathcal{B} : C \circ C \subseteq D$ .

Moreover, if  $\mathcal{B}$  is a base of a uniformity, then it is a base of a separated uniformity if and only if  $\bigcap \mathcal{B} = \Delta(X)$ .

*Důkaz.* The proof is an easy exercise, it can be used as an additional exercise during the exams.  $\square$

**Examples.** • The *discrete uniformity* on a set  $X$  consists of all supersets of  $\Delta(X)$ .

- If  $(X, \rho)$  is a pseudometric space, define  $E_\rho(r) := \{(x, y) \in X \times X : \rho(x, y) < r\}$  for  $r > 0$ . Then  $\{E_\rho(r) : r > 0\}$  is a base of a uniformity on  $X$ , which is a base of a separated uniformity if, in addition,  $\rho$  is a metric. This uniformity is denoted by  $\mathcal{D}_\rho$ . We say that a uniformity  $\mathcal{D}$  is *metrizable* if there exists a metric  $\rho$  such that  $\mathcal{D} = \mathcal{D}_\rho$ .

It is easy to verify that if  $(X, \rho)$  and  $(Y, \sigma)$  are metric spaces, then a mapping  $f : (X, \rho) \rightarrow (Y, \sigma)$  is uniformly continuous if and only if it is uniformly continuous as a mapping between the uniform spaces  $(X, \mathcal{D}_\rho)$  and  $(Y, \mathcal{D}_\sigma)$ .

**Definition.** If  $R$  is a system of pseudometrics on a set  $X$ , then the *uniformity generated by  $R$*  (denoted by  $\mathcal{D}_R$ ) is the uniformity whose subbase is  $\{E_\rho(r) : r > 0, \rho \in R\}$ .

**Remark.** It is easy to see that if  $R$  is a system of pseudometrics on a set  $X$ , then  $\mathcal{D}_R$  is separated if and only if  $R$  separates points of  $X$  (i.e.  $\forall x \neq y \exists \rho \in R : \rho(x, y) > 0$ ). Furthermore, if  $S$  is a system of pseudometrics on a set  $Y$ , then a mapping  $f : (X, \mathcal{D}_R) \rightarrow (Y, \mathcal{D}_S)$  is uniformly continuous if and only if

$$\forall \rho \in S \forall \varepsilon > 0 \exists \sigma \in R \exists \delta > 0 \forall x, y \in X : \sigma(x, y) < \delta \implies \rho(f(x), f(y)) < \varepsilon.$$

**Notation.** For  $E \subset X \times X$  and  $x \in X$  we denote  $E[x] := \{y \in X : (x, y) \in E\}$ .

**Proposition 2.** *If  $(X, \mathcal{D})$  is a uniform space, then*

$$\tau_{\mathcal{D}} = \{A \subseteq X : \forall x \in A \exists D \in \mathcal{D} : D[x] \subseteq A\}$$

*is a topology on  $X$ . Moreover, the following hold.*

- (a) *If  $\mathcal{B}$  is a base of the uniformity  $\mathcal{D}$ , then  $\mathcal{B}(x) := \{D[x] : D \in \mathcal{B}\}$ ,  $x \in X$ , are neighborhood bases at points in  $(X, \tau_{\mathcal{D}})$ .*
- (b)  *$\mathcal{D}$  is separated if and only if  $(X, \tau_{\mathcal{D}})$  is  $T_1$ .*
- (c) *If  $(Y, \mathcal{E})$  is a uniform space and  $f : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})$  is uniformly continuous, then  $f : (X, \tau_{\mathcal{D}}) \rightarrow (Y, \tau_{\mathcal{E}})$  is continuous.*
- (d) *If  $\mathcal{D}$  is generated by a system of pseudometrics  $R$ , then for every net  $(x_i)_{i \in I}$  and every  $x \in X$  we have that  $x_i \xrightarrow{\tau_{\mathcal{D}}} x$  if and only if  $\rho(x_i, x) \rightarrow 0$  for every  $\rho \in R$ .*

*Důkaz.* The proof was presented, it can be examined. □

**Definition.** A topological space  $(X, \tau)$  is called *uniformizable* if there exists a uniformity  $\mathcal{D}$  such that  $\tau = \tau_{\mathcal{D}}$ .

**Lemma 3** (On a pseudometric). *Let  $(X, \mathcal{D})$  be a uniform space and let  $\{D_n : n \in \mathbb{N} \cup \{0\}\} \subset \mathcal{D}$  satisfy*

- (i)  $D_0 = X \times X$ ,
- (ii)  $\forall n \in \mathbb{N} : D_n = (D_n)^{-1}$ ,
- (iii)  $\forall n \in \mathbb{N} : D_{n+1} \circ D_{n+1} \circ D_{n+1} \subseteq D_n$ .

*Then there exists a pseudometric  $\rho$  on  $X$  satisfying*

- (a)  $\forall n \geq 1 : \{(x, y) : d(x, y) < 2^{-n-1}\} \subseteq D_n \subseteq \{(x, y) : d(x, y) \leq 2^{-n}\}$ ,
- (b)  $\mathcal{D}_{\rho} \subset \mathcal{D}$  and  $\rho \leq 1$ .

*Důkaz.* The proof was presented, it can be examined. □

**Corollary 4.** *Every uniformity is generated by some system of pseudometrics. Every  $T_1$  uniformity is generated by a system of pseudometrics that separates points.*

*Důkaz.* The proof was presented, it can be examined. □

**Theorem 5.** *A  $T_1$  uniform space is metrizable if and only if it has a countable base.*

*Důkaz.* The proof was presented, it can be examined. □

**Remark.** A countable base of  $\mathcal{D}$  is something different from a countable base of  $\tau_{\mathcal{D}}$ . An example is the discrete uniformity whose base consists of a single element  $\Delta(X)$ , but  $\tau_{\mathcal{D}}$  is the discrete topology and hence the weight of  $(X, \tau_{\mathcal{D}})$  equals  $|X|$ .

**Theorem 6.** *A  $T_1$  topological space is uniformizable if and only if it is  $T_{3\frac{1}{2}}$ .*

*Důkaz.* The proof was presented, it can be examined. □

## 1.2. Subspace, Sum and Product

**Definition.** • Let  $(X, \mathcal{D})$  be a uniform space and  $A \subset X$ . Define  $\mathcal{D}|_A := \{D \cap (A \times A) : D \in \mathcal{D}\}$ . Then  $(A, \mathcal{D}|_A)$  is a *subspace* of  $(X, \mathcal{D})$ .

- Let  $(X_i, \mathcal{D}_i)$  be uniform spaces. The *product* of uniformities is the uniformity  $\mathcal{D}_{\Pi_I X_i}$  on  $\Pi_I X_i$  whose subbase is  $\{(\pi_i \times \pi_i)^{-1}(D) : i \in I, D \in \mathcal{D}_i\}$ . Then  $(\Pi_I X_i, \mathcal{D}_{\Pi_I X_i})$  is the *product of uniform spaces*.
- Let  $(X_i, \mathcal{D}_i)$  be uniform spaces. On  $\biguplus_I X_i := \bigcup_{i \in I} (\{i\} \times X_i)$  we define the *sum* of uniformities as the uniformity  $\biguplus_I \mathcal{D}_i := \{\bigcup_{i \in I} (\{i\} \times D_i) : D_i \in \mathcal{D}_i \text{ for each } i \in I\}$ . Then  $(\biguplus_I X_i, \biguplus_I \mathcal{D}_i)$  is the *sum of uniform spaces*.

**Remark.** It is easy to verify that the sum/product/subspace of uniform spaces is a well-defined uniform space. The topology generated by the uniformity  $\mathcal{D}|_A$  is the subspace topology, and the topology generated by the uniformity  $\mathcal{D}_{\Pi_I X_i}$  is the product topology.

**Proposition 7.** Let  $(Z, \mathcal{D})$  and  $(X_i, \mathcal{D}_i)$  for  $i \in I$  be uniform spaces.

- A mapping  $f : Z \rightarrow \Pi_I X_i$  is uniformly continuous if and only if the mappings  $\pi_i \circ f$  are uniformly continuous for each  $i \in I$ .
- Let  $f_i : X_i \rightarrow (Y_i, \mathcal{E}_i)$ ,  $i \in I$ , be uniformly continuous mappings. Then the mapping  $\Pi_I f_i : \Pi_I X_i \rightarrow \Pi_I Y_i$  is also uniformly continuous.
- Let  $f_i : Z \rightarrow X_i$ ,  $i \in I$ , be uniformly continuous mappings. Then the mapping  $\Delta_I f_i : Z \rightarrow \Pi_I X_i$  is also uniformly continuous.
- If  $f, g : Z \rightarrow \mathbb{R}$  are uniformly continuous mappings, then the mappings  $f+g$ ,  $f-g$ ,  $\max\{f, g\}$ ,  $\min\{f, g\}$ , and  $|f|$  are also uniformly continuous. Moreover, if  $f, g$  are bounded functions, then the mapping  $f \cdot g$  is also uniformly continuous.

*Důkaz.* The proof was presented, it can be examined. □

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## 1.3. Completeness and Total Boundedness

**Definition.** • A net  $(x_i)_{i \in I}$  in a uniform space  $(X, \mathcal{D})$  is called *Cauchy* if for every  $D \in \mathcal{D}$  there exists  $i_0 \in I$  such that for all  $i, j \geq i_0$  we have  $(x_i, x_j) \in D$ .

- A uniform space  $(X, \mathcal{D})$  is called *complete* if every Cauchy net converges in  $(X, \tau_{\mathcal{D}})$ .
- A uniform space  $(X, \mathcal{D})$  is called *totally bounded* if for every  $E \in \mathcal{D}$  there exists a finite set  $K \subseteq X$  such that  $E[K] = X$ . (Where  $E[K] := \bigcup_{x \in K} E[x]$ .)

**Remark.** If the uniformity  $\mathcal{D}$  is generated by a system of pseudometrics  $R$ , then a net  $(x_i)$  is Cauchy in  $(X, \mathcal{D})$  if and only if

$$\forall \rho \in R \forall \varepsilon \exists i_0 \forall i, j \geq i_0 : \rho(x_i, x_j) < \varepsilon.$$

**Remark.** It is not difficult to see that in a complete metric space Cauchy nets are convergent. It then easily follows that a metric space  $(X, \rho)$  is complete (resp. totally bounded) if and only if the uniform space  $(X, \mathcal{D}_\rho)$  is complete (resp. totally bounded).

**Remark.** Let  $(X, \mathcal{D})$  be a uniform space. It is easy to verify that convergent nets in  $(X, \tau_{\mathcal{D}})$  are Cauchy in  $(X, \mathcal{D})$  and that uniformly continuous mappings send Cauchy nets to Cauchy nets.

**Proposition 8.** Let  $(X, \mathcal{D})$  be a  $T_1$  uniform space. Then

$$X \text{ is totally bounded} \Leftrightarrow \text{every net in } X \text{ has a Cauchy subnet.}$$

*Důkaz.* The proof was presented, it can be examined. □

**Proposition 9.** (i) *If a subspace of a complete  $T_1$  uniform space is complete, then it is closed.*

(ii) *A subspace of a totally bounded uniform space (resp. a closed subspace of a complete uniform space) is totally bounded (resp. complete).*

(iii) *The product of totally bounded (resp. complete) uniform spaces is totally bounded (resp. complete).*

*Důkaz.* The proof was presented, it can be examined. □

**Theorem 10.** *Let  $X$  be a  $T_1$  uniform space. Then  $(X, \tau_{\mathcal{D}})$  is compact if and only if  $(X, \mathcal{D})$  is complete and totally bounded.*

*Důkaz.* The proof was presented, it can be examined. □

**Proposition 11.** *Let  $X$  and  $Y$  be  $T_1$  uniform spaces, let  $Y$  be complete, let  $A \subset X$ , and let  $f : A \rightarrow Y$  be uniformly continuous. Then there exists a uniformly continuous mapping  $F : \overline{A} \rightarrow Y$  such that  $F|_A = f$ .*

*Důkaz.* The proof was presented, it can be examined. □

**Definition.** Let  $(X, \mathcal{D})$  be a  $T_1$  uniform space. Its *completion* is a pair  $(e, Y)$ , where  $Y$  is a complete  $T_1$  uniform space and  $e : X \rightarrow Y$  is a uniform embedding onto a dense subset (i.e.  $e(\overline{X}) = Y$  and  $e : X \rightarrow e(X)$  is a uniform homeomorphism).

**Theorem 12.** *Every  $T_1$  uniform space has a completion. Moreover, if  $(e, Y)$  and  $(e', Y')$  are completions of a  $T_1$  uniform space  $X$ , then there exists a uniform homeomorphism  $F : Y \rightarrow Y'$  such that  $F \circ e = e'$ .*

*Důkaz.* The proof was omitted, it will not be examined. □

## 1.4. Uniformity on Compact Spaces

**Theorem 13.** *Let  $(X, \tau)$  be a compact Hausdorff space. Then there exists exactly one uniformity on  $X$  that generates the topology  $\tau$ ; a base of this unique uniformity is formed by the open neighborhoods of the diagonal  $\Delta(X)$ .*

*Důkaz.* The proof was presented, it can be examined. □

**Proposition 14.** *Let  $(X, \mathcal{D})$  and  $(Y, \mathcal{E})$  be  $T_1$  uniform spaces and let  $(X, \tau_{\mathcal{D}})$  be compact. Then every continuous mapping  $f : X \rightarrow Y$  is uniformly continuous.*

*Důkaz.* The proof was presented, it can be examined. □

## 2. Topological Groups

**Definition.** A triple  $(G, \cdot, \tau)$  is called a *topological group* (TG) if  $(G, \cdot)$  is a group,  $(G, \tau)$  is a topological space, and the multiplication operation  $\cdot : G \times G \rightarrow G$  (where  $G \times G$  is equipped with the product topology) and the inversion operation  $^{-1} : G \rightarrow G$  are continuous.

**Examples.** Examples of topological groups include the following.

- (a) Any group with the discrete topology.
- (b) Any normed linear space with addition as the group operation and the topology induced by the norm. More generally, every topological vector space is a commutative topological group.
- (c)  $GL(n, \mathbb{R})$  (the group of all real invertible  $n \times n$  matrices). The group operation is given by matrix multiplication, and the topology is given by coordinatewise convergence (i.e. the topology inherited from the product topology on  $\mathbb{R}^{n \times n}$ ).

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- (d)  $\text{Iso}(X)$ , where  $(X, \rho)$  is a metric space. The symbol  $\text{Iso}(X)$  denotes the set of all surjective isometries  $f : X \rightarrow X$  with the group operation given by composition and the topology of pointwise convergence (i.e. the product topology inherited from  $X^X$ ).

*Důkaz.* The proof was presented, it can be examined. □

- (e)  $\text{Iso}(V)$ , where  $(V, \|\cdot\|)$  is a normed linear space. Here  $\text{Iso}(V)$  denotes the set of all surjective linear isometries  $f : V \rightarrow V$  with the group operation given by composition and the topology of pointwise convergence (i.e. the product topology inherited from  $V^V$ ).
- (f)  $H(K)$ , where  $K$  is a compact Hausdorff space. The symbol  $H(K)$  denotes the set of all surjective homeomorphisms  $f : K \rightarrow K$  with the group operation given by composition and the *compact-open* topology, i.e. the topology whose subbase is formed by the sets  $E[L; U] := \{f \in H(K) : f(L) \subset U\}$ , where  $L \subset K$  is compact and  $U \subset K$  is open.  
(A proof that this is indeed a topological group is left as an exercise.)

The neutral element of a topological group  $G$  is denoted by  $e_G$  (or simply  $e$  if the group  $G$  is clear from the context). Recall that a subgroup  $N \subset G$  is a *normal subgroup* (denoted  $N \triangleleft G$ ) if  $gNg^{-1} = N$  for every  $g \in G$  (not to be confused with the unrelated notion of a normal topological space). Furthermore, recall that for  $N \triangleleft G$  we can define the *quotient group*  $G/N$ , i.e. the group whose elements are cosets of the form  $xN := \{xn : n \in N\}$ , with the group operation defined naturally by  $(xN)(yN) = (xy)N$  and  $(xN)^{-1} = x^{-1}N$ .

For each  $g \in G$  we define the left translation  $L_g : G \rightarrow G$  and the right translation  $R_g : G \rightarrow G$  by  $L_g(h) = gh$  and  $R_g(h) = hg$  for  $h \in G$ . A topological space  $X$  is called *homogeneous* if for every  $x, y \in X$  there exists a homeomorphism  $f : X \rightarrow X$  such that  $f(x) = y$ .

**Lemma 15.** *Let  $G$  be a topological group. Then the following hold.*

- (a) *The inversion mapping  $^{-1} : G \rightarrow G$  is a homeomorphism.*
- (b) *For every  $g \in G$ , the left and right translations  $L_g$  and  $R_g$  are homeomorphisms of  $G$ .*
- (c)  *$G$  is a homogeneous space.*

(d)  $\forall U \in \mathcal{U}(e) \exists V \in \mathcal{U}(e) : V \cdot V^{-1} \subset U$ .

(e) If one of the sets  $A, B \subset G$  is open, then  $A \cdot B$  is open.

(f) If  $H \subset G$  is a (normal) subgroup, then  $\overline{H}$  is also a (normal) subgroup.

(g) If  $H \subset G$  is a subgroup and  $\text{Int}(H) \neq \emptyset$ , then  $H$  is clopen.

(h) The product of topological groups equipped with the product topology is a topological group.

(i) A homomorphism of topological groups  $f : G \rightarrow H$  is continuous if and only if it is continuous at the neutral element  $e_G$ .

*Důkaz.* The proof was presented, it can be examined. □

## 2.1. Uniformities on Topological Groups

**Definition.** Let  $G$  be a topological group. Then

- the *right uniformity* on  $G$  is the uniformity  $\mathcal{D}_R$  whose base is given by the system  $\{R_U : U \in \mathcal{U}(e)\}$ , where  $R_U := \{(x, y) : xy^{-1} \in U\}$  for  $U \in \mathcal{U}(e)$ ,
- the *left uniformity* on  $G$  is the uniformity  $\mathcal{D}_L$  whose base is given by the system  $\{L_U : U \in \mathcal{U}(e)\}$ , where  $L_U := \{(x, y) : x^{-1}y \in U\}$  for  $U \in \mathcal{U}(e)$ .

**Lemma 16.** Let  $(G, \cdot, \tau)$  be a topological group. Then  $\tau = \tau_{\mathcal{D}_R} = \tau_{\mathcal{D}_L}$  (i.e. the topology generated by the right uniformity is the group topology). Moreover, the mappings  $R_g : (G, \mathcal{D}_R) \rightarrow (G, \mathcal{D}_R)$  and  $L_g : (G, \mathcal{D}_L) \rightarrow (G, \mathcal{D}_L)$  are uniform homeomorphisms for every  $g \in G$ .

*Důkaz.* The proof was presented, it can be examined. □

**Theorem 17.** Every  $T_1$  topological group is  $T_{3\frac{1}{2}}$ . Moreover, a  $T_1$  topological group is metrizable if and only if it has a countable character.

*Důkaz.* The proof was presented, it can be examined. □

**Lemma 18** (On right/left invariant pseudometrics). Let  $(G, \cdot, \tau)$  be a topological group and let  $\{U_n : n \in \mathbb{N} \cup \{0\}\} \subset \mathcal{U}(e)$  satisfy

- (i)  $U_0 = G$ ,
- (ii)  $\forall n \in \mathbb{N} : U_n = (U_n)^{-1}$ ,
- (iii)  $\forall n \in \mathbb{N} : U_{n+1} \cdot U_{n+1} \cdot U_{n+1} \subseteq U_n$ .

Then there exists a right-invariant (respectively, left-invariant) pseudometric  $\rho$  on  $G$  satisfying  $\rho \leq 1$ ,

(a)  $\forall n \geq 1 : B_\rho(e, 2^{-n-1}) \subseteq U_n \subseteq \overline{B}_\rho(e, 2^{-n})$ ,

(b)  $\mathcal{D}_\rho \subset \mathcal{D}_R$  (respectively,  $\mathcal{D}_\rho \subset \mathcal{D}_L$ ).

Moreover, if the left and right uniformities on  $G$  coincide, then there exists a bi-invariant pseudometric  $\rho$  with the properties above.

*Důkaz.* The proof was presented, it can be examined. □

**Theorem 19** (Birkhoff–Kakutani). Every metrizable topological group is metrizable by a right-invariant (respectively, left-invariant) metric.

*Důkaz.* The proof was presented, it can be examined. □

**Definition.** We say that a topological group is SIN (Small Invariant Neighborhoods) if there exists a base  $\mathcal{B}$  of neighborhoods of  $e$  such that  $gUg^{-1} \subset U$  for every  $U \in \mathcal{B}$  and every  $g \in G$ .

**Proposition 20.** A topological group is SIN if and only if its left and right uniformities coincide.

*Důkaz.* The proof was presented, it can be examined. □

**Theorem 21.** *A metrizable group is SIN if and only if it is metrizable by a bi-invariant metric.*

*Důkaz.* The proof was presented, it can be examined. □

**Examples.** Typical examples of SIN topological groups are compact groups, discrete groups, and commutative groups. An example of a metrizable topological group that is not SIN is  $GL(n, \mathbb{R})$ .

*Důkaz.* The proof was presented, it can be examined. □

## 2.2. Quotients of Topological Groups

**Theorem 22.** *Let  $G$  be a topological group and let  $N \triangleleft G$ . Consider on  $G/H$  the quotient topology induced by the mapping  $\pi : G \rightarrow G/H$ . Then  $G/H$  is a topological group and  $\pi$  is an open homomorphism. Moreover,  $G/H$  is  $T_1$  if and only if  $H$  is closed (regardless of whether  $G$  itself is  $T_1$ ).*

*Důkaz.* The proof was presented, it can be examined. □

**Theorem 23.** *Let  $G$  be a  $T_1$  topological group and let  $H \subset G$  be a locally compact subgroup. Then  $H$  is closed.*

*Důkaz.* The proof was presented, it can be examined. □

## 2.3. Representations of Topological Groups

**Definition.** Let  $G$  be a  $T_1$  topological group. We say that a function  $f : G \rightarrow \mathbb{R}$  is right uniformly continuous if  $f$  is uniformly continuous as a function from  $(G, \mathcal{D}_R)$  to  $\mathbb{R}$ . The set of all bounded right uniformly continuous functions  $f : G \rightarrow \mathbb{R}$  is denoted by  $RUC(G)$ .

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**Lemma 24.** *Let  $G$  be a  $T_1$  topological group. Then a function  $f : G \rightarrow \mathbb{R}$  is right uniformly continuous if and only if*

$$\forall \varepsilon > 0 \exists U \in \mathcal{U}(e) \forall u \in U \forall x \in G : |f(ux) - f(x)| < \varepsilon.$$

*If  $RUC(G)$  is equipped with the norm  $\|f\|_\infty := \sup_{x \in G} |f(x)|$ , then  $(RUC(G), \|\cdot\|_\infty)$  is a Banach space. Moreover,  $RUC(G)$  separates points and closed sets in  $G$ .*

*Důkaz.* The proof was presented, it can be examined. □

**Theorem 25 (Teleman).** *Let  $G$  be a  $T_1$  topological group. Then there exists a Banach space  $V$  such that  $G$  embeds as a topological group into  $\text{Iso}(V)$ .*

*Důkaz.* The proof was presented, it can be examined. □

**the end of 5. lecture (19.3.2026)**

### 3. Paracompact Spaces

**Definition.** If  $X$  is a set and  $\mathcal{U}$  is a cover of  $X$ , then a system  $\mathcal{V}$  is called a *refinement* of  $\mathcal{U}$  (denoted  $\mathcal{V} < \mathcal{U}$ ) if  $\mathcal{V}$  is a cover of  $X$  and for every  $V \in \mathcal{V}$  there exists  $U \in \mathcal{U}$  such that  $V \subseteq U$ .

Furthermore, let  $X$  be a topological space and let  $\mathcal{S} \subseteq \mathcal{P}(X)$ . The system  $\mathcal{S}$  is called

- *locally finite* if every point of  $X$  has a neighborhood intersecting only finitely many sets from  $\mathcal{S}$ ,
- *discrete* if every point of  $X$  has a neighborhood intersecting at most one set from  $\mathcal{S}$ ,
- *$\sigma$ -locally finite* (respectively,  *$\sigma$ -discrete*) if it is a countable union of locally finite (respectively, discrete) systems.

**Remarks.** Every ( $\sigma$ -)discrete system is ( $\sigma$ -)locally finite. The system  $\{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$  is  $\sigma$ -discrete but not locally finite.

**Fact 26** (Closure of a locally finite family). *If  $\mathcal{A}$  is a locally finite family in a topological space  $X$ , then  $\{\overline{A} : A \in \mathcal{A}\}$  is locally finite and  $\overline{\bigcup \mathcal{A}} = \bigcup \{\overline{A} : A \in \mathcal{A}\}$ .*

*Důkaz.* The proof was presented, it can be examined. □

**Definition.** A Hausdorff topological space  $X$  is called *paracompact* if every open cover of  $X$  has a locally finite open refinement.

**Examples.** All compact spaces and all discrete spaces are paracompact. (Later we will prove that every metric space is also paracompact.)

**Theorem 27** (Characterization of Paracompactness). *For a  $T_3$  topological space  $X$ , the following conditions are equivalent.*

- (a)  $X$  is paracompact.
- (b) Every open cover of  $X$  is refined by a  $\sigma$ -locally finite open cover.
- (c) Every open cover of  $X$  is refined by a locally finite cover.
- (d) Every open cover of  $X$  is refined by a locally finite closed cover.

*Důkaz.* The proof was presented, it can be examined. □

**Corollary 28.** *Every Lindelöf  $T_3$  space is paracompact.*

*Důkaz.* The proof was presented, it can be examined. □

**Definition.** For a system  $\mathcal{S}$  of subsets of a set  $X$  and  $x \in X$ , define  $st_{\mathcal{S}}(x) = \bigcup \{S \in \mathcal{S} : x \in S\}$ . We say that a cover  $\mathcal{V}$  *star-refines* a cover  $\mathcal{U}$  (denoted  $\mathcal{V} <_{st} \mathcal{U}$ ) if  $\{st_{\mathcal{V}}(x) : x \in X\}$  refines  $\mathcal{U}$ .

**Theorem 29** (Characterization of Paracompactness II). *For a Hausdorff topological space  $(X, \tau)$ , the following conditions are equivalent.*

- (a) Every open cover  $\mathcal{U}$  of  $X$  has an open star-refinement that is a cover.
- (b) There exists a uniformity  $\mathcal{D}$  on  $X$  generating the topology of  $X$  (i.e.  $\tau_{\mathcal{D}} = \tau$ ) such that for every open cover  $\mathcal{U}$  of  $X$  there exists  $D \in \mathcal{D}$  with  $\{D[x] : x \in X\} < \mathcal{U}$ .
- (c)  $X$  is  $T_{3\frac{1}{2}}$  and for every open cover  $\mathcal{U}$  of  $X$  there exists a continuous pseudometric  $\rho$  on  $X$  such that  $\{B_{\rho}(x, 1) : x \in X\} < \mathcal{U}$ .

(d)  $X$  is  $T_{3\frac{1}{2}}$  and every open cover of  $X$  is refined by a  $\sigma$ -discrete open cover.

(e)  $X$  is paracompact.

*Důkaz.* The proof was presented, it can be examined. □

### the end of 6. lecture (26.3. 2026)

**Definition.** A Hausdorff topological space  $X$  is called *collectively normal* if for every discrete family  $\mathcal{F}$  of closed sets there exists a family of pairwise disjoint open sets  $\{U(F) : F \in \mathcal{F}\}$  such that  $F \subset U(F)$  for every  $F \in \mathcal{F}$ .

**Remark.** If the family of closed sets is finite, then it is discrete if and only if it is disjoint. In particular, every collectively normal space is normal.

**Proposition 30.** *Every paracompact topological space is collectively normal, and hence normal.*

*Důkaz.* The proof was presented, it can be examined. □

**Theorem 31** (Stone). *Every metrizable topological space is paracompact.*

*Důkaz.* The proof was presented, it can be examined. □

**Definition.** Let  $\mathcal{G}$  be an open cover of a space  $X$ . A family of continuous functions  $\{f_i : X \rightarrow [0, 1] : i \in I\}$  is called a *locally finite partition of unity subordinate to  $\mathcal{G}$*  if the family  $\{\{f_i \neq 0\} : i \in I\}$  is locally finite, refines  $\mathcal{G}$ , and

$$\sum_{i \in I} f_i(x) = 1$$

for every  $x \in X$ .

**Theorem 32** (Partition of Unity). *In a paracompact topological space, for every open cover there exists a locally finite partition of unity subordinate to this cover.*

*Důkaz.* The proof was presented, it can be examined. □

**Theorem 33** (Dugundji – special case). *Let  $K$  be a metrizable compact space and let  $L \subset K$  be a closed subset. Then there exists a linear mapping  $E : C(L) \rightarrow C(K)$  such that  $Ef|_L = f$  and  $\|Ef\|_\infty \leq \|f\|_\infty$  for every  $f \in C(L)$ . Moreover,  $Ef \geq 0$  whenever  $f \geq 0$ .*

*Důkaz.* The proof was presented, it can be examined. □

### the end of 7. lecture (2. 4. 2026)

**Theorem 34** (Bing, Nagata, Smirnov). *For a  $T_{3\frac{1}{2}}$  space  $X$ , the following are equivalent.*

- (a)  $X$  is metrizable.
- (b)  $X$  has a  $\sigma$ -discrete base.
- (c)  $X$  has a  $\sigma$ -locally finite base.

*Důkaz.* The proof was presented, it can be examined. □

## 4. Connectedness

**Definition.** A topological space is called *connected* if it cannot be expressed as a disjoint union of two nonempty open sets.

**Remark.** There are also the notions of path connectedness and arc connectedness; however, we will not discuss these notions further in this lecture.

Note that some authors (e.g. Engelking) consider the empty set to be connected, while others do not.

**Proposition 35.** *For a topological space  $X$ , the following conditions are equivalent.*

- (a) *The space  $X$  is connected.*
- (b) *If  $X = A \cup B$  and  $\overline{A} \cap B = \emptyset = A \cap \overline{B}$ , then  $A = \emptyset$  or  $B = \emptyset$ .*
- (c) *The space  $X$  contains no nontrivial clopen subset.*
- (d) *Every continuous mapping  $f: X \rightarrow \{0, 1\}$  is constant (where  $\{0, 1\}$  is the two-point discrete space).*

*Důkaz.* The proof was presented, it can be examined. □

**Proposition 36.** *The continuous image of a connected space is connected.*

*Důkaz.* The proof was presented, it can be examined. □

**Proposition 37** (Union of connected sets). *Let  $\{C_i: i \in I\}$  be a family of connected subsets of a space  $X$  and suppose that one of the following conditions holds:*

- (a)  $\exists i_0 \in I \forall i \in I: C_i \cap C_{i_0} \neq \emptyset;$
- (b)  $\bigcap_{i \in I} C_i \neq \emptyset.$

*Then  $\bigcup C_i$  is connected.*

*Důkaz.* The proof was presented, it can be examined. □

**Corollary 38.** *If  $X$  is a topological space,  $A \subseteq X$  is connected, and  $A \subseteq M \subseteq \overline{A}$ , then  $M$  is connected.*

*Důkaz.* The proof was presented, it can be examined. □

**Theorem 39.** *Let  $X$  be a  $T_{3\frac{1}{2}}$  topological space. Then  $X$  is connected if and only if  $\beta X$  is connected.*

*Důkaz.* The proof was presented, it can be examined. □

**Theorem 40.** *Let  $X_i, i \in I$ , be nonempty topological spaces. Then  $\prod_I X_i$  is connected if and only if all spaces  $X_i, i \in I$ , are connected.*

*Důkaz.* The proof was presented, it can be examined. □

**Definition.** Let  $X$  be a topological space and let  $x \in X$ . The *connected component* of the point  $x$  is the largest connected set containing  $x$ . It is denoted by  $C_x$ .

**Remark.** By Proposition 37, the connected component of every point exists. If  $C_x$  and  $C_y$  are two components, then either  $C_x = C_y$  or  $C_x \cap C_y = \emptyset$ . Thus, connected components form a partition of the space  $X$ .

**Proposition 41.** If  $X_i, i \in I$ , are topological spaces and  $x = (x_i) \in \prod_I X_i$ , then

$$C_x = \prod_I C_{x_i}.$$

(That is, the component of  $x = (x_i)$  is the product of the components of the corresponding  $x_i, i \in I$ .)

*Důkaz.* The proof was presented, it can be examined.  $\square$

**Definition.** Let  $X$  be a topological space. A set  $Q$  is called the *quasicomponent* of the point  $x$  if

$$Q = \bigcap \{Z: x \in Z, Z \text{ is clopen}\}.$$

It is denoted by  $Q_x$ .

**Remark.** For every  $x \in X$  we have  $C_x \subseteq Q_x$ . Quasicomponents are closed, since they are defined as intersections of closed sets. Moreover, quasicomponents also form a partition of the space.

**Example.** Let  $X$  be a subset of the plane consisting of the points  $a = (0,0)$ ,  $b = (0,1)$ , and a countable family of line segments joining the points  $(2^{-n}, 0)$  and  $(2^{-n}, 1)$ . Then  $C_a = \{a\} \neq \{a, b\} = Q_a$ .

*Důkaz.* The proof was presented, it can be examined.  $\square$

**Lemma 42** (On intersections in compact spaces). *Let  $X$  be a compact space and let  $\mathcal{A}$  be a family of closed sets. If  $\bigcap \mathcal{A} \subseteq U$  for some open set  $U$ , then there exists a finite subfamily  $\mathcal{F} \subseteq \mathcal{A}$  such that  $\bigcap \mathcal{F} \subseteq U$ .*

*Důkaz.* The proof was presented, it can be examined.  $\square$

**Theorem 43.** *In a compact  $T_2$  space, components and quasicomponents coincide.*

*Důkaz.* The proof was presented, it can be examined.  $\square$

## 4.1. Continua

**Definition.** A compact, connected, nonempty  $T_2$  space is called a *continuum*. If it consists of a single point, it is called *degenerate*.

**Remark.** Continuous images and arbitrary products of continua are continua.

**Proposition 44.** *If  $\mathcal{H}$  is a family of continua closed under finite intersections, then  $\bigcap \mathcal{H}$  is a continuum.*

(In particular, the intersection of a decreasing sequence of continua is a continuum.)

*Důkaz.* The proof was presented, it can be examined.  $\square$

**Proposition 45** (Boundary bumping). *If  $A$  is a proper closed subset of a continuum  $X$ , then every component of  $A$  intersects the boundary of  $A$ .*

*Důkaz.* The proof was presented, it can be examined.  $\square$

**Theorem 46** (Sierpiński). *Let  $X$  be a continuum and let  $X_n, n \in \mathbb{N}$ , be pairwise disjoint closed subsets whose union is  $X$ . Then  $X_n = \emptyset$  for all  $n$  except one.*

*Důkaz.* The proof was presented, it can be examined.  $\square$

the end of 8. lecture (9. 4. 2026)

**Definition.** A continuum is called *decomposable* if it can be written as the union of two proper subcontinua. Otherwise, it is called *indecomposable*.

**Example.** There exists an indecomposable continuum in  $\mathbb{R}^2$ .

the end of 9. lecture (16. 4. 2026)

## 4.2. Disconnectedness

**Definition.** A Hausdorff topological space  $X$  is called

- *hereditarily disconnected* if all components are singletons;
- *totally disconnected* if for  $x \neq y$  there exists a clopen set  $Z \subseteq X$  such that  $x \in Z$  and  $y \notin Z$ ;
- *zero-dimensional* (sometimes written *0-dim*) if it has a base consisting of clopen sets;
- *strongly zero-dimensional* (sometimes written *strongly 0-dim*) if for every two disjoint closed sets  $E, F$  there exists a clopen set  $Z$  such that  $E \subseteq Z \subseteq X \setminus F$ .

**Remarks.** • The terminology is not completely uniform; we use the terminology from Engelking (different terminology is used in the lecture notes). The most important notion will be zero-dimensionality (where the terminology is uniform).

- Strong zero-dimensionality as defined here automatically implies normality, but it can also be naturally defined in a reasonable way already in Tychonoff spaces (see, for example, the lecture notes for details).

**Proposition 47.** *Let  $X$  be a  $T_2$  topological space. Then*

*$X$  is strongly 0-dim  $\implies X$  is 0-dim  $\implies X$  is totally disconnected  $\implies X$  is hereditarily disconnected.*

**Examples.** • Consider  $X = \mathbb{R}^2$  with a topology  $\tau$  defined as follows: the points  $\mathbb{Q}^2$  are isolated, and the remaining points  $x$  have basic neighborhoods of the form  $\{x\} \cup (B(x, \varepsilon) \cap \mathbb{Q}^2)$  for  $\varepsilon > 0$ . Then  $(X, \tau)$  is a  $T_2$  space that is hereditarily disconnected but not totally disconnected. There also exists a metrizable example, but it is considerably more complicated (see the lecture notes, Example 8.46).

- Consider the Erdős space, i.e.  $E := \ell_2 \cap \mathbb{Q}^\omega$  with the topology inherited from  $\ell_2$ . Then  $E$  is a metrizable totally disconnected space that is not zero-dimensional.
- Examples of spaces that are zero-dimensional but not normal (and hence not strongly zero-dimensional according to our definition) were already mentioned in General Topology 1 (the product of the Sorgenfrey line, or the Isbell–Mrówka space).
- In exercises we will present an example of a normal space that is zero-dimensional but not strongly zero-dimensional. There even exists a metrizable example, but it is very complicated.

**Theorem 48** (Disconnectedness in compact spaces). *For a  $T_2$  compact space  $X$  we have:*

*$X$  is strongly 0-dim  $\Leftrightarrow X$  is 0-dim  $\Leftrightarrow X$  is totally disconnected  $\Leftrightarrow X$  is hereditarily disconnected.*

**Theorem 49** (Zero-dimensionality of  $\beta X$ ). *Let  $X$  be  $T_4$ . Then  $\beta X$  is 0-dim if and only if  $X$  is strongly 0-dim.*

**Proposition 50.** *Let  $X$  be  $T_2$ . Then  $X$  is zero-dimensional if and only if it can be embedded into  $2^I$  for some set  $I$ . In that case, one can choose  $I = w(X)$ .*

**Theorem 51.** *Every  $T_2$  compact space is a continuous image of a zero-dimensional compact space of the same weight.*

## 5. Topological Dimension

**Definition** (Small inductive dimension: Menger, Urysohn). For a  $T_3$  space  $X$  we define its small inductive dimension inductively for  $n \in \mathbb{N} \cup \{0\}$  as follows:

- We say that  $\text{ind } X = -1$  if and only if  $X = \emptyset$ .
- $\text{ind } X \leq n$  if for every  $x \in X$  and every neighborhood  $U$  of  $x$  there exists an open set  $V$  such that  $x \in V \subseteq U$  and  $\text{ind}(\partial V) \leq n - 1$ .
- $\text{ind } X = n$  if  $\text{ind } X \leq n$  and  $\text{ind } X \leq n - 1$  does not hold.
- $\text{ind } X = \infty$  if  $\text{ind } X \leq n$  does not hold for any  $n \in \mathbb{N}$ .

We call  $\text{ind } X$  the *small inductive dimension* of the space  $X$ .

**Remarks.** Let  $X$  be a  $T_3$  space. Then:

- $\text{ind } X \leq 0$  if and only if  $X$  is zero-dimensional;
- if  $M \subset X$ , then  $\text{ind } M \leq \text{ind } X$ ;
- $\text{ind}[0, 1] = 1$ .

**Definition** (Large inductive dimension: Brouwer, Čech). For a  $T_4$  space  $X$  we define its large inductive dimension inductively for  $n \in \mathbb{N} \cup \{0\}$  as follows:

- We say that  $\text{Ind } X = -1$  if and only if  $X = \emptyset$ .
- $\text{Ind } X \leq n$  if for every closed set  $E$  and every open set  $U \supseteq E$  there exists an open set  $V$  such that  $E \subseteq V \subseteq U$  and  $\text{Ind}(\partial V) \leq n - 1$ .
- $\text{Ind } X = n$  if  $\text{Ind } X \leq n$  and  $\text{Ind } X \leq n - 1$  does not hold.
- $\text{Ind } X = \infty$  if  $\text{Ind } X \leq n$  does not hold for any  $n \in \mathbb{N}$ .

We call  $\text{Ind } X$  the *large inductive dimension* of the space  $X$ .

**Remarks.** Let  $X$  be a  $T_4$  space. Then:

- if  $M \subset X$  is closed, then  $\text{Ind } M \leq \text{Ind } X$ ;
- $\text{Ind } X \leq 0$  if and only if  $X$  is strongly zero-dimensional;
- $\text{ind } X \leq \text{Ind } X$ ;
- $\text{Ind}[0, 1] = 1$ .

**Definition.** We say that a family  $\mathcal{A}$  of subsets of a set  $X$  has order  $n$  if  $n$  is the largest natural number for which there exist distinct elements  $A_1, \dots, A_{n+1} \in \mathcal{A}$  such that  $\bigcap A_i \neq \emptyset$ .

**Definition** (Covering dimension: Čech, Lebesgue). For a  $T_4$  space  $X$  we define its covering dimension inductively for  $n \in \mathbb{N} \cup \{0\}$  as follows:

- $\dim \emptyset = -1$ .
- $\dim X \leq n$  if every finite open cover of  $X$  is refined by a finite open cover of order at most  $n$ .
- $\dim X = n$  if  $\dim X \leq n$  and  $\dim X \leq n - 1$  does not hold.
- $\dim X = \infty$  if  $\dim X \leq n$  does not hold for any  $n$ .

We call  $\dim X$  the *covering dimension* of the space  $X$ .

**Remarks.** Let  $X$  be a  $T_4$  space. Then:

- if  $M \subset X$  is closed, then  $\dim M \leq \dim X$ ;
- $\dim[0, 1] = 1$ .

**Proposition 52.** In a  $T_4$  topological space  $X$ , we have  $\dim X \leq 0$  if and only if  $X$  is strongly 0-dimensional.

**Definition.** Let  $X$  be a set and let  $\mathcal{S} \subseteq \mathcal{P}(X)$  be a cover of  $X$ . An indexed family  $\{T_S : S \in \mathcal{S}\}$  is called a *shrinking* of the family  $\mathcal{S}$  if it is a cover and  $T_S \subseteq S$  for each  $S \in \mathcal{S}$ .

**Lemma 53** (On shrinking). *Let  $X$  be a  $T_4$  space and let  $\{G_1, \dots, G_n\}$  be an open cover of  $X$ . Then there exists an open cover  $\{H_1, \dots, H_n\}$  of  $X$  such that  $\overline{H_i} \subset G_i$  for  $i \in \{1, \dots, n\}$ . (That is, every finite open cover has a closed shrinking whose interiors also form a cover.)*

**Lemma 54** (On swelling). *Let  $X$  be a  $T_4$  space, let  $\{F_1, \dots, F_n\}$  be a finite family of closed subsets of  $X$  of order at most  $n$ , and let  $\{U_1, \dots, U_n\}$  be open sets such that  $F_i \subset U_i$  for  $i = 1, \dots, n$ . Then there exists a family  $\{V_1, \dots, V_n\}$  of open subsets of  $X$  such that  $\{\overline{V_1}, \dots, \overline{V_n}\}$  has order at most  $n$  and*

$$F_i \subset V_i \subset \overline{V_i} \subset U_i$$

for each  $i = 1, \dots, n$ .

**Theorem 55** (Characterization of covering dimension). *For a  $T_4$  space  $X$ , the following conditions are equivalent.*

- (a)  $\dim X \leq n$ .
- (b) Every finite open cover of  $X$  has an open shrinking of order at most  $n$ .
- (c) Every finite open cover of  $X$  has a closed shrinking of order at most  $n$ .
- (d) Every finite open cover of  $X$  is refined by a finite closed cover of order at most  $n$ .

**Theorem 56** (Sum theorem for the dimension  $\dim$ ). *If a  $T_4$  space  $X$  is the union of countably many closed subspaces  $F_i$  and  $\dim F_i \leq n$ , then  $\dim X \leq n$ .*

**Theorem 57.** *If  $X$  is  $T_4$ , then  $\dim X \leq \text{Ind } X$ .*

## 5.1. Topological Dimension in Metrizable Spaces

**Theorem 58.** *If  $X$  is a metrizable space, then  $\dim X = \text{Ind } X$ .*

**Lemma 59.** *Let  $X$  be a metrizable space and let  $Z \subset X$  be strongly 0-dimensional. Then for every closed set  $F \subset X$  and every open set  $U \subset X$  with  $F \subset U$  there exists an open set  $V \subset X$  such that*

$$F \subset V \subset \overline{V} \subset U$$

and  $Z \cap \partial V = \emptyset$ .

**Theorem 60.** *Let  $X$  be a metrizable separable space. Then*

$$\text{ind } X = \dim X = \text{Ind } X.$$

**Theorem 61.** *Let  $X$  be a metrizable space and let  $n \in \mathbb{N} \cup \{0\}$ . Then the following statements are equivalent.*

- (a)  $\text{Ind } X \leq n$ ,
- (b)  $X = Y \cup Z$ , where  $\text{Ind } Y \leq n - 1$  and  $\text{Ind } Z \leq 0$ .

**Corollary 62** (On separation). *Let  $X$  be a metrizable space and let  $n \in \mathbb{N} \cup \{0\}$ . If  $\text{Ind } X \leq n$ , then for every sequence of  $(n+1)$  pairs of closed disjoint sets  $(F_1, H_1), \dots, (F_{n+1}, H_{n+1})$ , there exist open sets  $U_i$ ,  $i = 1, \dots, n+1$ , such that*

$$F_i \subseteq U_i \subseteq \overline{U_i} \subseteq X \setminus H_i \quad \text{and} \quad \bigcap_{i=1}^{n+1} \partial U_i = \emptyset.$$

**Theorem 63.** *Let  $X$  and  $Y$  be nonempty metrizable spaces. Then*

$$\text{Ind}(X \times Y) \leq \text{Ind } X + \text{Ind } Y.$$

## 5.2. Dimension and Euclidean Spaces

**Theorem 64** (Brouwer's fixed point theorem). *Every continuous mapping  $f: [0, 1]^n \rightarrow [0, 1]^n$  has a fixed point, i.e. there exists  $x \in [0, 1]^n$  such that  $f(x) = x$ .*

**Theorem 65.** *For every  $n \in \mathbb{N}$  we have  $\dim[0, 1]^n = \dim \mathbb{R}^n = n$ .*

**Corollary 66.** *If  $n, m \in \mathbb{N}$  and  $n \neq m$ , then  $\mathbb{R}^n$  is not homeomorphic to  $\mathbb{R}^m$ .*