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Complex Finsler spaces of constant holomorphic curvature

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Abstract. This paper is about complex Finsler metrics of constant holomorphic curvature. For a complex Finsler space (M, F), the holomorphic flag curvature is defined with respect to the Chern complex linear connection on the pull-back tangent bundle. A complex Finsler metric, which satisfies some special conditions on the curvature, is called generalized Einstein. By means of such connection, a special approach is devoted to obtain the equivalence conditions that a complex Finsler space be generalized Einstein, (§3). A Schur type theorem for a generalized Einstein complex Finsler space, weakly Kähler, and other characterizations of the holomorphic curvature of this space are given in section 4. The final section applies our results to an example, better illustrating the interest for this work.

1. Introduction

The study of the complex Finsler metrics of constant holomorphic flag curvature, the corespondent of the constant holomorphic sectional curvature from Hermitian geometry, is a challenging problem in complex Finsler geometry.

M. Abate and G. Patrizio [**AP1**] gave a characterization of the constant holomorphic curvature through complex geodesics, with the main result that any complex Finsler metric of holomorphic curvature $K_F = -4$ and satisfying some regularity conditions is the Kobayashi metric. The first proof is due to J. Faran [**F**], who used the method of equivalence problem in his work.

Another result, due M. Abate and G. Patrizio [**AP2**], asserts that if the complex Finsler spaces satisfay the notion of Kähler and a symmetry condition on the curvature and with positive constant holomorphic curvature, then they are purely Hermitian.

In a previous paper, [**Al**], we started to study the curvature of complex Finsler spaces, with respect to the Chern complex linear connection, briefly Chern $(c.l.c)$, on the pull-back tangent bundle. The purpose of the present paper is to determine the conditions in which a complex Finsler metric has constant holomorphic curvature. We shall solve this problem for a special class of complex Finsler spaces, called generalized Einstein, briefly $(g.E.)$. We prove a Schur type theorem for $(g.E.)$ complex Finsler spaces (Theorem 4.1). We obtain that the $(g.E.)$ complex Finsler spaces of nonzero constant holomorphic curvature are weakly Kähler (Proposition 4.1). Moreover, with additional condition of Kähler, we prove that the $(g.E.)$ complex Finsler

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spaces of nonzero constant holomorphic curvature are purely Hermitian (Proposition 4.6). We remark that the class of $(g.E.)$ complex Finsler spaces which are Kähler of constant holomorphic curvature is included in the class of the E-manifolds from [**S**]. The problem of the $(g.E.)$ complex Finsler spaces with identically vanishing holomorphic curvature is exemplified by a particular form of the complex version of Antonelli-Shimada metric. It show that there exists $(g.E.)$ complex Finsler spaces with identically vanishing holomorphic curvature which are not purely Hermitian.

2. Preliminaries

We recall here only the basic notions which are needed; for more information see [**AP1**], [**Mu**], [**Al**]. For the beginning, we shall make an introduction to the geometry of the pull-back tangent bundle with the Chern $(c.l.c)$, [**Al**].

Let M be a complex manifold, $\dim_C M = n$. The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is in its turn a complex manifold, the local coordinates in a chart will be denoted by (z^k, η^k) and these are changed by the rules

$$z'^k = z'^k(z) \quad ; \quad \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j. \tag{1}$$

The complexified tangent bundle of $T'M$ is decomposed as $T_C(T'M) = T'(T'M) \oplus T''(T'M)$. A natural local frame for $T'_u(T'M)$ is $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\}$, which changes by the rules obtained with Jacobi matrix of (1) transformations. Note that the change rule of $\frac{\partial}{\partial z^k}$ contains the second order partial derivatives.

Considering the restriction of the projection to $\widetilde{T'M} = T'M \setminus \{0\}$, for pulling the holomorphic tangent bundle $T'M$ back, we obtain a holomorphic tangent bundle $\pi' : \pi^*(T'M) \longrightarrow \widetilde{T'M}$, called *the pull-back tangent bundle* over the slit $\widetilde{T'M}$. Because $\pi^*(T'M) = \{(u,s) \mid \pi(u) = \pi(s)\}$, where $s = X^i_u \frac{\partial}{\partial z^i}$ in $u = (z,\eta)$, $\eta \neq 0$, then its local fibre $\pi^*(T'M)_u$ is a copy of $T'_z M$. We denote by $\left\{\frac{\partial}{\partial z^k}^*, \frac{\partial}{\partial \bar{z}^k}^*\right\}$, and by $\left\{dz^{*k}, d\bar{z}^{*k}\right\}$, evaluated in the point $u \in \widetilde{T'M}$, the local frames to $\pi^*_C(T'M) = \pi^*(T'M) \oplus \overline{\pi^*(T'M)}$, and respectively to $\pi^*_C(T'^*M) = \pi^*(T'^*M) \oplus \overline{\pi^*(T'^*M)}$. The frame $\frac{\partial}{\partial z^k}^*$ transforms simply like the vector fields of M manifold, i.e. $\frac{\partial}{\partial z^h}^* = \frac{\partial z'^k}{\partial z^h}\frac{\partial}{\partial z'^k}^*$. Furthermore, by the conjugation operator, $\pi^*_C(T'M)$ and $\pi^*(T_C M)$ are isomorphic.

Let $V_C(T'M) = \ker \pi_* \subset T_C(T'M)$ be the complexified vertical bundle. A local frame for $V_u(T'M)$ is $\{\frac{\partial}{\partial \eta^k}\}$. The vertical lift l^v, locally defined by $l^v\left(\frac{\partial}{\partial z^k}\right) = \frac{\partial}{\partial \eta^k}$, determines an isomorphism between $T'M$ and $V(T'M)$, ([**AP1**], [**Mu**]).

A complex nonlinear connection, briefly $(c.n.c.)$, is a splitting in the exact sequence

$$0 \to V(T'M) \xrightarrow{i} T'(T'M) \xrightarrow{p} \pi^*(T'M) \to 0, \tag{2}$$

with $p(X_u) = (u, \pi_*(X_u))$, $X_u = X^k_u \frac{\partial}{\partial z^k} + \widetilde{X}^k_u \frac{\partial}{\partial \eta^k}$, ([**M-A**], [**Mu**]).

A $(c.n.c.)$ determines a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, thus $T'(T'M) = H(T'M) \oplus V(T'M)$. There is an isomorphism between $T'M$ and $H(T'M)$, $l^h : T'M \to H(T'M)$, called the *horizontal lift*, with local representation

$l^h(\frac{\partial}{\partial z^k}) = \frac{\delta}{\delta z^k}$, where

$$\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j} \tag{3}$$

and the functions $N_k^j(z,\eta)$ on $T'M$ are called the coefficients of the (*c.n.c.*).

The frame $\frac{\delta}{\delta z^k}$ transforms itself by the $\left(\frac{\partial z'^h}{\partial z^k}\right)$ matrix, for which results a certain rule of change for the coefficients $N_k^j(z,\eta)$ of the (*c.n.c.*), (**[Mu]**). It results that the isomorphism $p|_{H(T'M)}$ admits the local representation $p|_{H(T'M)}\left(\frac{\delta}{\delta z^k}\right) = \frac{\partial^*}{\partial z^k}$. The frames $\{\frac{\delta}{\delta z^k}, \frac{\partial}{\partial \eta^k}\}$, $\{dz^k, \delta\eta^k\}$ are called *adapted local frames* of $T'_u(T'M)$, respectively of $T'_u(T'^*M)$, where $\delta\eta^k = d\eta^k + N_j^k dz^j$. By conjugation, we obtain the adapted local frames of $T''_u(T'M)$, respectively of $T''_u(T'^*M)$. Further on, as a rule, we shall use the abbreviations $\delta_i = \frac{\delta}{\delta z^i}$, $\dot{\partial}_i = \frac{\partial}{\partial \eta^i}$, $\delta_{\bar{i}} = \frac{\delta}{\delta \bar{z}^i}$, $\dot{\partial}_{\bar{i}} = \frac{\partial}{\partial \bar{\eta}^i}$, (**[AP1]**, **[Ai]**, **[Mu]**).

On $T'M$ let $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ be the fundamental metric tensor of a complex Finsler space (M,F), with $F^2(z,\eta) = L(z,\eta)$. The complex Cartan tensors $C_{i\bar{j}k} := \frac{\partial g_{i\bar{j}}}{\partial \eta^k}$, $C_{i\bar{j}\bar{k}} := \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}$ satisfy the properties:

$$\begin{aligned} C_{i\bar{j}k} &= C_{k\bar{j}i};\ \ C_{i\bar{j}\bar{k}} = C_{i\bar{k}\bar{j}};\ \ C_{i\bar{j}k} = \overline{C_{j\bar{i}\bar{k}}}; \\ C_{i\bar{j}k}\eta^k &= C_{i\bar{j}\bar{k}}\bar{\eta}^j = C_{i\bar{j}k}\eta^i = C_{i\bar{j}\bar{k}}\bar{\eta}^k = 0. \end{aligned} \tag{4}$$

The isomorphism between $\pi^*(T'M)$ and $T'M$ induces an isomorphism between $\pi^*(T_C M)$ and $T_C M$. Thus, $g_{i\bar{j}}$ defines an Hermitian metric structure

$$\mathcal{G}(z,\eta) := g_{j\bar{k}} dz^{*j} \otimes d\bar{z}^{*k} \tag{5}$$

on $\pi^*(T_C M)$, with respect to the J natural complex structure, *i.e.* $\mathcal{G}(X,\bar{Y}) = \overline{\mathcal{G}(X,\bar{Y})}$, $\det(\mathcal{G}) \neq 0$ and $\mathcal{G}(JX,JY) = \mathcal{G}(X,Y)$, for any X, Y the sections on $\pi^*(T_C M)$.

On the other hand, $H(T'M)$ and $\pi^*(T'M)$ are isomorphic by $p|_{H(T'M)}$, which it may be extended to the complexified bundles. Therefore the structures on $\pi^*(T_C M)$ can be pulled-back to $H(T'M) \oplus \overline{H(T'M)}$. By this isomorphism the natural cobasis dz^{*j} is identified with dz^j.

In view of this construction we have, [**Al**]:

THEOREM 1. *Let* (M,F) *be a complex Finsler space. The pull-back tangent bundle* $\pi^*(T'M)$ *admits a unique* (*c.l.c.*) ∇, *called the Chern* (*c.l.c.*), *which is metric with respect to* $\mathcal{G}$ *and of* $(1,0)-$*type.*

The pull-back expression of ∇ connection form is

$$\begin{aligned} \omega_j^i(z,\eta) &= L_{jk}^i(z,\eta)dz^k + C_{jk}^i(z,\eta)\delta\eta^k; \\ L_{jk}^i &= g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta z^k}\ ;\ C_{jk}^i = g^{\overline{m}i}\frac{\partial g_{j\overline{m}}}{\partial \eta^k}. \end{aligned} \tag{6}$$

The Chern (*c.l.c.*) on $\pi^*(T'M)$ determines the Chern-Finsler (*c.n.c.*) on $T'M$, with the coefficients $\overset{CF}{N_k^i} = g^{\overline{m}i}\frac{\partial g_{j\overline{m}}}{\partial z^k}\eta^j$, and its $1-$form ω_j^i satisfies the following structure equations

$$d(dz^i) - dz^k \wedge \omega_k^i = \Omega^i\ ;\ \ d\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i, \tag{7}$$

and their conjugates. The $2-$ forms of torsion and curvature are:

$$\begin{aligned}\Omega^i &= -\frac{1}{2}T^i_{jk}dz^j\wedge dz^k - C^i_{jk}dz^j\wedge\delta\eta^k;\\ \Omega^i_j &= R^i_{j\overline{h}k}dz^k\wedge d\overline{z}^h + \Xi^i_{j\overline{h}k}\delta\eta^k\wedge d\overline{z}^h + P^i_{j\overline{h}k}dz^k\wedge\delta\overline{\eta}^h + S^i_{j\overline{h}k}\delta\eta^k\wedge\delta\overline{\eta}^h,\end{aligned}$$

where

$$\begin{aligned}T^i_{jk} &:= L^i_{jk} - L^i_{kj}\ ;\\ R^i_{j\overline{h}k} &:= -\delta_{\overline{h}}L^i_{jk} - \delta_{\overline{h}}(\overset{CF}{N^l_k})C^i_{jl}\ ;\ \Xi^i_{j\overline{h}k} := -\delta_{\overline{h}}C^i_{jk} = \Xi^i_{k\overline{h}j};\\ P^i_{j\overline{h}k} &:= -\dot{\partial}_{\overline{h}}L^i_{jk} - \dot{\partial}_{\overline{h}}(\overset{CF}{N^l_k})C^i_{jl}\ ;\ S^i_{j\overline{h}k} := -\dot{\partial}_{\overline{h}}C^i_{jk} = S^i_{k\overline{h}j}.\end{aligned}\tag{8}$$

The coefficients $R^i_{j\overline{h}k}$, $\Xi^i_{j\overline{h}k}$, $P^i_{j\overline{h}k}$, $S^i_{j\overline{h}k}$ are $h\overline{h}-$, $v\overline{h}-$, $h\overline{v}-$, $v\overline{v}-$curvatures of the Chern-Finsler $\overset{CF}{N}$ $-(c.l.c.)$ on $T'M$ from [**AP1**] and [**Mu**], induced by the Chern $(c.l.c.)$ on $\pi^*(T'M)$, see [**Al**].

We recall here the following properties

i) $R^i_{\overline{h}k} := R^i_{j\overline{h}k}\eta^j = -\delta_{\overline{h}}\overset{CF}{N^i_k}$; $P^i_{\overline{h}k} := P^i_{j\overline{h}k}\eta^j = -\dot{\partial}_{\overline{h}}\overset{CF}{N^i_k}$;

ii) $\Xi^i_{j\overline{h}k}\eta^j = S^i_{j\overline{h}k}\eta^j = 0$;

iii) $P^i_{\overline{h}k} = -g^{\overline{j}i}(C_{l\overline{j}\overline{h}}\eta^l)_{|k}$; $P_{\overline{r}\overline{h}k} := P^i_{\overline{h}k}g_{i\overline{r}} = -(C_{l\overline{r}\overline{h}}\eta^l)_{|k}$;

$P^i_{\overline{h}k}\overline{\eta}^h = P_{\overline{r}\overline{h}k}\overline{\eta}^r = P_{\overline{r}\overline{h}k}\overline{\eta}^h = 0,$

where by " $_{|}$ " , " $|$ " , "$\overline{_{|}}$" and "$\overline{|}$", we denoted the $h-$, $v-$, $\overline{h}-$, $\overline{v}-$ covariant derivatives with respect to the Chern $(c.l.c)$, respectively. By differentiating the structure equations and the condition of $\mathcal{G}-$ compatibility we obtained Bianchi identities. See [**Al**] for more details.

The Riemann type tensor

$$\begin{aligned}R(W,\overline{Z},X,\overline{Y}) &:= \mathcal{G}(R(X,\overline{Y})W,\overline{Z});\\ R(X,\overline{Y})W &:= W^iX^k\overline{Y}^hR^l_{i\overline{h}k}\frac{\partial}{\partial z^l}^*,\end{aligned}\tag{9}$$

where $W := W^i\frac{\partial}{\partial z^i}^*$, $X := X^k\frac{\partial}{\partial z^k}^*$ and $\overline{Z} := \overline{Z}^j\frac{\partial}{\partial\overline{z}^j}^*$; $\overline{Y} := \overline{Y}^h\frac{\partial}{\partial\overline{z}^h}^*$ are local sections on $\pi^*(T'M)$, $\overline{\pi^*(T'M)}$, respectively, has the properties:

$$\begin{aligned}R(W,\overline{Z},X,\overline{Y}) &= W^i\overline{Z}^jX^k\overline{Y}^hR_{i\overline{j}k\overline{h}};R_{\overline{j}i\overline{h}k} := R^l_{i\overline{h}k}g_{l\overline{j}};\\ R_{i\overline{j}k\overline{h}} &= -R_{i\overline{j}\overline{h}k} = \overline{R_{\overline{j}i\overline{h}\overline{k}}} = R_{\overline{j}i\overline{h}k};\\ \text{If }\ R^i_{j\overline{h}k} &= R^i_{k\overline{h}j}\ \text{ then }\ R_{i\overline{j}k\overline{h}} = R_{k\overline{j}i\overline{h}} = R_{k\overline{h}i\overline{j}}.\end{aligned}\tag{10}$$

By setting $R_{\overline{j}k} := R_{i\overline{j}k\overline{h}}\eta^i\overline{\eta}^h = -g_{l\overline{j}}\delta_{\overline{h}}(\overset{CF}{N^l_k})\overline{\eta}^h$, the *Ricci scalar* and the *Ricci tensor* associated to the Chern $(c.l.c.)$ on $\pi^*(T'M)$ are defined by

$$\begin{aligned}Ric &:= g^{\overline{j}k}R_{\overline{j}k} = R^k_{i\overline{h}k}\eta^i\overline{\eta}^h;\\ Ric_{i\overline{j}} &:= \frac{\partial^2 Ric}{\partial\eta^i\partial\overline{\eta}^j}.\end{aligned}\tag{11}$$

An easy computation shows that $R_{\overline{j}k}$ and Ric are $1-$ homogeneous with respect to η, hence $Ric_{i\overline{j}}\eta^i\overline{\eta}^j = Ric$ and $\left(Ric_{i\overline{j}} - R^k_{i\overline{j}k}\right)\eta^i\overline{\eta}^j = 0$.

Now, let us recall that in [**AP1**]'s terminology, the complex Finsler space (M, F) is *strongly Kähler* iff $T^i_{jk} = 0$, *Kähler* iff $T^i_{jk}\eta^j = 0$ and *weakly Kähler* iff $g_{i\overline{l}}T^i_{jk}\eta^j\overline{\eta}^l = 0$. We notice that in the particular case of complex Finsler metrics which come from Hermitian metrics on M, so-called *purely Hermitian metrics* in [**Mu**], that is $g_{i\overline{j}} = g_{i\overline{j}}(z)$, those three nuances of Kähler coincide, [**S**]. For purely Hermitian metrics with the additional condition of Kähler, the Ricci tensor defined in (11) coincides with the Ricci tensor in Kähler geometry.

3. Holomorphic flag curvature

The notion of holomorphic flag curvature is a generalization of the holomorphic sectional curvature to complex Finsler geometry. Following the ideas from the real case, ([**BCS**]), we study the holomorphic flag curvature with respect to the Chern (*c.l.c.*).

We consider $z \in M$ and $\eta \in T'_zM$, $\eta \neq 0$. A flag is described by the tangent vector field η, called flagpole, and another transversal vector field χ. Let (η, χ) be the flag in which, η and χ are viewed as local sections of $\pi^*(T'M)$, *i.e.* $\eta := \eta^i \frac{\partial}{\partial z^i}^*$ and $\chi := \chi^j \frac{\partial}{\partial z^j}^*$.

DEFINITION 1. *The holomorphic flag curvature of the complex Finsler metric F, along of the flag (η, χ), is given by*

$$\mathcal{K}_F(z, \eta, \chi) := \frac{R(\eta, \overline{\chi}, \eta, \overline{\chi}) + R(\chi, \overline{\eta}, \chi, \overline{\eta})}{\mathcal{G}(\eta, \overline{\eta})\mathcal{G}(\chi, \overline{\chi})}, \tag{12}$$

where, $\mathcal{G}(\chi, \overline{\chi}) \neq 0$.

The holomorphic flag curvature $\mathcal{K}_F(z, \eta, \chi)$ depends both on the position $z \in M$ and the flag (η, χ), and by Proposition 2.5.2 from [**AP1**], p.107, if $R^i_{j\overline{h}k} = R^i_{k\overline{h}j}$, then it completely determines the curvature tensor $R^i_{j\overline{h}k}$. $\mathcal{K}_F(z, \eta, \chi)$ is real valued.

In particular, choosing $\eta \equiv \chi$, we obtain

$$\mathcal{K}_F(z, \eta) := \frac{2R(\eta, \overline{\eta}, \eta, \overline{\eta})}{\mathcal{G}^2(\eta, \overline{\eta})}, \tag{13}$$

called in [**AP1**] the holomorphic curvature of F in direction η. In local coordinates it is

$$\mathcal{K}_F(z, \eta) = \frac{2\overline{\eta}^j\eta^k R_{\overline{j}k}}{L^2(z, \eta)}. \tag{14}$$

Our goal in the sequel is to find the circumstances in which the holomorphic curvature of F in direction η is constant. This means that $\mathcal{K}_F(z, \eta)$ has the same constant value for any choice of z and η. We solve this problem for a particular class of complex Finsler spaces, the general case remaining unsettled for the moment. However this class of spaces is fairly generous and offers many significant solutions. Further on, we shall use the holomorphic curvature terminology for holomorphic curvature in direction η.

Let us introduce the following two $d-$tensors

$$h_{k\bar{j}} := g_{k\bar{j}} - \frac{1}{L(z,\eta)}\eta_k\overline{\eta}_j \text{ and } t_{k\bar{j}} := L(z,\eta)g_{k\bar{j}} + \eta_k\overline{\eta}_j. \tag{15}$$

Note that $h_{k\bar{j}}$ is not invertible, because $h_{k\bar{j}}\eta^k = h_{k\bar{j}}\overline{\eta}^j = 0$, while the inverse of $t_{k\bar{j}}$ is

$$t^{\bar{j}k} = \frac{1}{L}g^{\bar{j}k} - \frac{1}{2L^2}\overline{\eta}^j\eta^k. \tag{16}$$

A direct computation yields

Proposition 1. *i)* $\overline{h_{k\bar{j}}} = h_{j\bar{k}}$; $\overline{t_{k\bar{j}}} = t_{j\bar{k}}$;

ii) $h_{k\bar{j}}g^{\bar{j}k} = n-1$; $t_{k\bar{j}}g^{\bar{j}k} = L(n+1)$; $g_{k\bar{j}} = \frac{t_{k\bar{j}}+Lh_{k\bar{j}}}{2L}$;

iii) $t_{k\bar{j}}\eta^k = 2L\overline{\eta}_j$; $t_{k\bar{j}}\overline{\eta}^j = 2L\eta_k$; $t_{k\bar{j}}\eta^k\overline{\eta}^j = 2L^2$;

iv) $t_{k\bar{j}|r} = t_{k\bar{j}|\bar{r}} = 0$; $t_{k\bar{j}}|_r = \eta_r g_{k\bar{j}} + \eta_k g_{r\bar{j}}$; $t_{k\bar{j}}|_{\bar{r}} = \overline{\eta}_r g_{k\bar{j}} + \overline{\eta}_j g_{k\bar{r}}$;

v) $t_{k\bar{j}}|_r\eta^r = t_{k\bar{j}}$; $t_{k\bar{j}}|_r\overline{\eta}^j = 2\eta_k\eta_r$; $t_{k\bar{j}}|_r\eta^k = t_{r\bar{j}}$; $t_{k\bar{j}}|_{\bar{r}}\overline{\eta}^r = t_{k\bar{j}}$;

$t_{k\bar{j}}|_{\bar{r}}\eta^k = 2\overline{\eta}_j\overline{\eta}_r$; $t_{k\bar{j}}|_{\bar{r}}\overline{\eta}^j = t_{k\bar{r}}$.

Definition 2. *The complex Finsler space (M,F) is called generalized Einstein if $R_{\bar{j}k}$ is proportional to $t_{k\bar{j}}$, i.e. if there exists a real valuated function $K(z,\eta)$, such that*

$$R_{\bar{j}k} = K(z,\eta)t_{k\bar{j}}. \tag{17}$$

Under the (1) changes of complex coordinates on $T'M$, the function $K(z,\eta)$ is well defined on $T'M$. As a consequence of the 1−homogeneity property of $t_{k\bar{j}}$ and $R_{\bar{j}k}$, we obtain $K(z,\lambda\eta) = K(z,\eta)$, for any $\lambda \in \mathbf{C}$. Moreover, from (17) formula we deduce:

Proposition 2. *If (M,F) is a (g.E.) complex Finsler space then*

i) $K(z,\eta) = \frac{1}{4}\mathcal{K}_F(z,\eta)$;
ii) $K(z,\eta)|_k\eta^k = K(z,\eta)|_{\bar{j}}\overline{\eta}^j = K(z,\eta)|_{\bar{j}}|_k\eta^k = K(z,\eta)|_k|_{\bar{j}}\overline{\eta}^j = 0$;

iii) $\overline{R_{\bar{j}k}} = R_{\bar{k}j}$.

Proposition 3. *Let (M,F) be a complex Finsler space. The following statements are equivalent:*
i) (M,F) *is (g.E.);*
ii)

$$\begin{aligned} R_{\bar{j}\bar{h}k} \ : &= R^l_{\bar{h}k}g_{l\bar{j}} = K(z,\eta)\left(g_{k\bar{j}}\overline{\eta}_h + g_{k\bar{h}}\overline{\eta}_j\right) + \\ &K(z,\eta)|_{\bar{h}}t_{k\bar{j}} + C_{\bar{j}\bar{h}|k|\bar{m}}\overline{\eta}^m; \end{aligned} \tag{18}$$

iii)

$$R_{\bar{j}l\bar{h}k} = K(z,\eta)\left(C_{k\bar{j}l}\overline{\eta}_h + C_{k\bar{h}l}\overline{\eta}_j + g_{l\bar{h}}g_{k\bar{j}} + g_{l\bar{j}}g_{k\bar{h}}\right) + K(z,\eta)|_l\left(g_{k\bar{j}}\overline{\eta}_h + g_{k\bar{h}}\overline{\eta}_j\right) + K(z,\eta)|_{\overline{h}}\left(L(z,\eta)C_{k\bar{j}l} + C_{kl}\overline{\eta}_j + g_{k\bar{j}}\eta_l + g_{l\bar{j}}\eta_k\right) + K(z,\eta)|_{\overline{h}}|_l t_{k\bar{j}} + C_{\overline{j}\overline{h}|r|\bar{m}}C^r_{kl}\bar{\eta}^m + C_{\overline{j}\overline{h}|k|\bar{m}}|_l\bar{\eta}^m - C_{\overline{j}\overline{r}|k}C^{\overline{r}}_{l|\overline{h}}. \tag{19}$$

Given any of these equivalent conditions, we have

$$2L(z,\eta)K(z,\eta)|_k\overline{\eta}_j - C_{\overline{j}\overline{r}|l}C^{\overline{r}}_{k|\bar{h}}\eta^l\bar{\eta}^h - \dot{T}_{\bar{j}k} = 0, \tag{20}$$

where

$$\eta_i := g_{i\bar{j}}\overline{\eta}^j;\ C_{\overline{i}\overline{j}} := C_{h\overline{i}\overline{j}}\eta^h\ ;\ C^{\overline{r}}_l := g^{\overline{r}j}C_{jl}; \qquad T_{\bar{j}k} := g_{i\bar{j}}T^i_{lk}\eta^l\ ;\ \dot{T}_{\bar{j}k} := T_{\bar{j}k|\overline{m}}\overline{\eta}^m. \tag{21}$$

Proof. By a direct computation, we obtain:

$$R_{\bar{j}k}|_{\overline{h}} = K(z,\eta)\left(g_{k\bar{j}}\overline{\eta}_h + g_{k\bar{h}}\overline{\eta}_j\right) + K(z,\eta)|_{\overline{h}}t_{k\bar{j}}; \tag{22}$$

and

$$R_{\bar{j}k}|_{\overline{h}}|_l = K(z,\eta)\left(g_{l\bar{h}}g_{k\bar{j}} + g_{l\bar{j}}g_{k\bar{h}}\right) + K(z,\eta)|_l\left(g_{k\bar{j}}\overline{\eta}_h + g_{k\bar{h}}\overline{\eta}_j\right) + K(z,\eta)|_{\overline{h}}\left(g_{k\bar{j}}\eta_l + g_{l\bar{j}}\eta_k\right) + K(z,\eta)|_{\overline{h}}|_l t_{k\bar{j}}. \tag{23}$$

Now let us prove that **i)** $\Longleftrightarrow$ **ii)**.

Given $R_{\bar{j}k}$ as in (17), we can reconstruct $R^i_{\bar{j}k}$. For this, contracting the Bianchi identity, (see [**Al**]),

$R^i_{\bar{j}\bar{h}k}|_{\bar{l}} - P^i_{\bar{j}\bar{l}k|\bar{h}} - \Xi^i_{\bar{j}\bar{h}l}P^r_{\bar{l}k} + S^i_{\bar{j}\bar{l}r}R^r_{\bar{h}k} + R^i_{\bar{j}\bar{r}k}C^{\overline{r}}_{\bar{h}\bar{l}} = 0$

with $\eta^j\overline{\eta}^h$, we obtain $R^i_{\bar{h}k}|_{\bar{l}}\overline{\eta}^h = -C^i_{\bar{l}|k|\bar{h}}\overline{\eta}^h$.

On the other hand, $R^i_{\bar{h}k}|_{\bar{l}}\overline{\eta}^h = R^i_k|_{\bar{l}} - R^i_{\bar{l}k}$ and so, $R^i_{\bar{l}k} = C^i_{\bar{l}|k|\bar{h}}\overline{\eta}^h + R^i_k|_{\bar{l}}$. Indeed, $R_{\bar{i}\bar{l}k} = C^i_{\bar{l}|k|\bar{h}}\overline{\eta}^h + R_{\bar{i}k}|_{\bar{l}}$ which, together with (22) implies *ii*).

Conversely, *ii*) and the fact that $R_{\bar{j}\bar{h}k}\overline{\eta}^h = R_{\bar{j}k}$, $K(z,\eta)|_{\bar{h}}\overline{\eta}^h = 0$ and $C_{\bar{j}\bar{h}|k|\bar{m}}\overline{\eta}^m\overline{\eta}^h = 0$ give *i*).

i) $\Longleftrightarrow$ **iii)**. Given $R_{\bar{j}k}$ as in (17), we use the following Bianchi identity $R^i_{\bar{j}\bar{h}k}|_l - \Xi^i_{\bar{j}\bar{h}l|k} - P^i_{\bar{j}\bar{r}k}P^{\overline{r}}_{l\bar{h}} + S^i_{\bar{j}\bar{r}l}R^{\overline{r}}_{k\bar{h}} + R^i_{\bar{j}\bar{h}r}C^r_{kl} = 0$ to reconstruct $R_{\bar{j}l\bar{h}k}$.

If we contract this with η^j, we obtain

$R^i_{\bar{j}\bar{h}k}|_l\eta^j = -C^i_{\bar{r}|k}C^{\overline{r}}_{l|\bar{h}} - R^i_{\bar{h}r}C^r_{kl}$.

But, $R^i_{\bar{j}\bar{h}k}|_l\eta^j = R^i_{\bar{h}k}|_l - R^i_{l\bar{h}k}$, so that $R^i_{l\bar{h}k} = R^i_{\bar{h}k}|_l - C^i_{\bar{r}|k}C^{\overline{r}}_{l|\bar{h}} + R^i_{\bar{h}r}C^r_{kl}$.

It results that

$R_{\bar{j}l\bar{h}k} = R_{\bar{j}r}|_{\bar{h}}C^r_{kl} + R_{\bar{j}k}|_{\bar{h}}|_l + C_{\bar{j}\bar{h}|r|\bar{m}}C^r_{kl}\overline{\eta}^m - C_{\bar{j}\bar{r}|k}C^{\overline{r}}_{l|\bar{h}} + C_{\bar{j}\bar{h}|k|\bar{m}}|_l\overline{\eta}^m$.

Plugging (22) and (23) into the last relation, we obtain *iii*).

The converse follows from $K(z,\eta)|_{\bar{j}}\overline{\eta}^j = 0$ and $K(z,\eta)|_{\bar{j}}|_l\eta^l = 0$.

To prove (20) we compute $(R_{\bar{j}l\bar{h}k}-R_{\bar{j}k\bar{h}l})\eta^l\overline{\eta}^h$ in two ways. By $iii)$,
$(R_{\bar{j}l\bar{h}k}-R_{\bar{j}k\bar{h}l})\eta^l\overline{\eta}^h=-2L(z,\eta)K(z,\eta)|_k\overline{\eta}_j+C_{\overline{jr}|l}C^{\overline{r}}_{k|\bar{h}}\eta^l\overline{\eta}^h,$
and by Bianchi identity $T^i_{jk|\bar{h}}+\mathcal{A}_{jk}\left\{R^i_{j\bar{h}k}-C^i_{jl}R^l_{\bar{h}k}\right\}=0$, we obtain
$(R_{\bar{j}l\bar{h}k}-R_{\bar{j}k\bar{h}l})\eta^l\overline{\eta}^h=-\dot{T}_{\bar{j}k}$. So, we have (20). □

Proposition 4. *Let (M,F) be a $(g.E.)$ complex Finsler space. Then*
$i)$ $K:=K(z)$ depends on z alone.
$ii)$ $C_{\overline{jh}|k|\bar{m}}\eta^k\bar{\eta}^m-2K(z)L(z,\eta)C_{\overline{jh}}=0.$

Proof. Since $\overline{R_{j\bar{l}h\bar{k}}}=R_{\bar{j}l\bar{h}k}$, then $\overline{R_{j\bar{l}h\bar{k}}\overline{\eta}^l\overline{\eta}^k}=R_{\bar{j}l\bar{h}k}\eta^l\eta^k$. If we contract (19) by $\eta^l\eta^k$, taking into account Proposition 3.2, $ii)$ we deduce

$$R_{\bar{j}l\bar{h}k}\eta^l\eta^k=C_{\overline{jh}|k|\bar{m}}\eta^k\bar{\eta}^m+2L(z,\eta)K(z,\eta)|_{\bar{h}}\bar{\eta}_j+2K(z,\eta)\bar{\eta}_j\bar{\eta}_h. \tag{24}$$

On the other hand, $R_{j\bar{l}h\bar{k}}\overline{\eta}^l\overline{\eta}^k=R_{\bar{l}j\bar{k}h}\overline{\eta}^l\overline{\eta}^k$ and by (19), we have
$R_{\bar{l}j\bar{k}h}\overline{\eta}^l\overline{\eta}^k=2K(z,\eta)\left(L(z,\eta)C_{jh}+\eta_j\eta_k\right)+2L(z,\eta)K(z,\eta)|_j\eta_h.$
By conjugation,

$$\overline{R_{\bar{l}j\bar{k}h}\overline{\eta}^l\overline{\eta}^k}=2K(z,\eta)\left(L(z,\eta)C_{\overline{jh}}+\overline{\eta}_j\overline{\eta}_k\right)+2L(z,\eta)K(z,\eta)|_{\bar{j}}\overline{\eta}_h. \tag{25}$$

So, (24) and (25) lead to

$$C_{\overline{jh}|k|\bar{m}}\eta^k\bar{\eta}^m-2L(z,\eta)\left(K(z,\eta)C_{\overline{jh}}-K(z,\eta)|_{\bar{h}}\bar{\eta}_j+K(z,\eta)|_{\bar{j}}\bar{\eta}_h\right)=0. \tag{26}$$

To prove $i)$ we contract (26) with $\bar{\eta}^j$ and we have $2L^2(z,\eta)K(z,\eta)|_{\bar{h}}=0$. Hence $K(z,\eta)|_{\bar{h}}=0$, i.e. $\frac{\partial K(z,\eta)}{\partial\bar{\eta}^h}=0$. By conjugation, $\frac{\partial K(z,\eta)}{\partial\eta^h}=0$, and so K does not depends on η. As a consequence of $i)$, (26) relation brings to $ii)$. □

Proposition 5. *Let (M,F) be a $(g.E.)$ complex Finsler space. Then*
$i)$ $Ric=(n+1)L(z,\eta)K(z)$;
$ii)$ $Ric_{i\bar{j}}=(n+1)K(z)g_{i\bar{j}}$.

Note that in any $(g.E.)$ complex Finsler space the holomorphic curvature depends on z only, $\mathcal{K}_F(z):=\mathcal{K}_F(z,\eta)=4K(z)$ and the Ricci scalar, associated to the Chern $(c.l.c.)$ on $\pi^*(T'M)$, is real valued. Moreover, the assertion $ii)$ from above proposition shows that the Ricci tensor is proportional with $g_{i\bar{j}}$. This motivates the name 'generalized Einstein'.

In the remainder of this section, we prove a necessary and sufficient condition that a complex Finsler space be $(g.E.)$.

Proposition 6. *Let (M,F) be a complex Finsler space. The following assertions are equivalent:*
$i)$ The holomorphic curvature of F depends only on the position $z\in M$ and satisfies

$$\frac{\partial R_{\bar{j}i}}{\partial\eta^k}\eta^i=\frac{\partial R_{\bar{i}k}}{\partial\overline{\eta}^j}\overline{\eta}^i=R_{\bar{j}k}, \tag{27}$$

$ii)$ The complex Finsler space is $(g.E.)$.

PROOF. If $\frac{\partial \mathcal{K}_F(z,\eta)}{\partial \eta^i} = \frac{\partial \mathcal{K}_F(z,\eta)}{\partial \overline{\eta}^i} = 0$ and taking into account (14) then,
$\frac{\partial^2 R_{\overline{j}k}}{\partial \eta^i \partial \overline{\eta}^h}\eta^k \overline{\eta}^j + \frac{\partial R_{\overline{h}k}}{\partial \eta^i}\eta^k + \frac{\partial R_{\overline{j}i}}{\partial \overline{\eta}^h}\overline{\eta}^j + R_{\overline{h}i} = \mathcal{K}_F(z,\eta)t_{i\overline{h}}$,
and plugging (27) into this, it yields that F is $(g.E.)$.

For the converse, it is obvious that $\mathcal{K}_F$ depends on z only and from Proposition 3.1, v), the conditions (27) are easily deduced. □

4. Generalized Einstein spaces with constant holomorphic curvature

As noticed in the previous section, for a $(g.E.)$ complex Finsler space the proportionality factor is $\frac{1}{4}\mathcal{K}_F(z)$ and it depends on z only. Therefore, the problem of $(g.E.)$ spaces with constant holomorphic curvature reduces to determine the conditions in which $K(z)$ is constant on M.

For the beginning, we prove a Schur type theorem for $(g.E.)$ complex Finsler spaces, namely:

THEOREM 2. *Let (M,F) be a $(g.E.)$ connected complex Finsler space, weakly Kähler, of complex dimension $\geqslant 2$. Then it is a space with constant holomorphic curvature.*

PROOF. By a direct computation, using Proposition 3.1, iv), we obtain

$$\begin{aligned} R_{\overline{j}k|l} &= t_{k\overline{j}}K(z)_{|l}; \\ R^{\overline{s}}_{k\overline{h}} &= C^{\overline{s}}_{k|\overline{h}|m}\eta^m + R^{\overline{s}}_{\overline{h}}|_k; \ \text{ where} \\ R^{\overline{s}}_{\overline{h}} :=& \ R_{\overline{h}k}g^{\overline{s}k} = \left(L(z,\eta)\delta^{\overline{s}}_{\overline{h}} + \overline{\eta}_h\overline{\eta}^s\right)K(z); \\ R^{\overline{s}}_{\overline{h}}|_k &= \left(\eta_k\delta^{\overline{s}}_{\overline{h}} + g_{k\overline{h}}\overline{\eta}^s\right)K(z). \end{aligned} \tag{28}$$

The contraction of Bianchi identity
$\mathcal{A}_{kl}\left\{R^i_{j\overline{h}k|l} - P^i_{j\overline{r}k}R^{\overline{r}}_{l\overline{h}}\right\} + R^i_{j\overline{h}m}T^m_{kl} = 0$ with $g_{i\overline{r}}\eta^j\eta^l\overline{\eta}^h$, leads to
$R_{\overline{r}k|l}\eta^l - R_{\overline{r}l|k}\eta^l + C_{\overline{r}\overline{s}|k}R^{\overline{s}}_{\overline{h}}\overline{\eta}^h - C_{\overline{r}\overline{s}|l}R^{\overline{s}}_{k\overline{h}}\eta^l\overline{\eta}^h + R_{\overline{r}m}T^m_{kl}\eta^l = 0.$
The last result and (28) give

$$K(z)g_{m\overline{r}}T^m_{kl}\eta^l\overline{\eta}^r + \eta_k K(z)_{|l}\eta^l - L(z,\eta)K(z)_{|k} = 0. \tag{29}$$

Since F is weakly Kähler, then from (29) we get
$\eta_k K(z)_{|l}\eta^l - L(z,\eta)K(z)_{|k} = 0.$
So, by conjugation we have

$$K(z)_{|\overline{h}} = \frac{1}{L(z,\eta)}\overline{\eta}_h K(z)_{|\overline{l}}\overline{\eta}^l. \tag{30}$$

Because of $K(z)_{|\overline{h}}|_j = K(z)|_{j|\overline{h}} = 0$, deriving (30) we easily deduce

$0 = K(z)_{|\overline{h}}|_j = \frac{K(z)_{|\overline{l}}\overline{\eta}^l}{L(z,\eta)}h_{j\overline{h}}$, which multiplied by $g^{\bar{h}j}$, in view of ii) in Proposition 3.1, for $n \geqslant 2$ we obtain $K(z)_{|\overline{l}}\overline{\eta}^l = 0$. Plugging it into (30), it follows that $K(z)_{|\overline{h}} = 0$, i.e. $\frac{\partial K(z)}{\partial \overline{z}^h} = 0$. By conjugation, $\frac{\partial K(z)}{\partial z^h} = 0$ and so, $K(z)$ is a constant on M. □

By (29), we deduce the following

PROPOSITION 7. *If (M,F) is a $(g.E.)$ complex Finsler space with $K(z)$ a nonzero constant, then F is weakly Kähler.*

Proof. Since F is $(g.E.)$, with $K(z)$ a nonzero constant, then $K(z)_{|l}=0$ and (29) becomes $g_{m\bar{r}}T^m_{kl}\eta^l\overline{\eta}^r=0$, i.e. F is weakly Kähler. □

Particularly, if (M,F) is a $(g.E.)$ complex Finsler space, with $K(z)=0$, then it is a flat complex Finsler space, i.e. $\mathcal{K}_F=0$, and $C_{\overline{j}\overline{h}|k|\bar{m}}\eta^k\bar{\eta}^m=0$. We note that if the complex Finsler metric F is locally Minkowski, i.e. it depends on η only, then it has $R_{\bar{j}l\bar{h}k}=0$. So, it is $(g.E.)$ metric with zero holomorphic curvature. We shall see in the final section that there exists $(g.E.)$ complex Finsler spaces of zero holomorphic curvature which are not weakly Kähler or locally Minkowski.

Moreover, taking $K(z)=c$, $c\in\mathbf{R}$, in Proposition 3.3, we have proved

Proposition 8. *Let (M,F) be a complex Finsler space. The following statements are equivalent:*

i) (M,F) is $(g.E.)$, with $K(z)=c$;

ii)

$$R_{\overline{j}\overline{h}k}=c\left(g_{k\bar{j}}\overline{\eta}_h+g_{k\bar{h}}\overline{\eta}_j\right)+C_{\overline{j}\overline{h}|k|\bar{m}}\bar{\eta}^m; \tag{31}$$

iii)

$$\begin{aligned} R_{\bar{j}l\bar{h}k} &= c\left(C_{k\bar{j}l}\overline{\eta}_h+C_{k\bar{h}l}\overline{\eta}_j+g_{l\bar{h}}g_{k\bar{j}}+g_{l\bar{j}}g_{k\bar{h}}\right)+\\ &\quad C_{\overline{j}\overline{h}|r|\bar{m}}C^r_{kl}\bar{\eta}^m+C_{\overline{j}\overline{h}|k|\bar{m}}|_l\bar{\eta}^m-C_{\overline{j}\overline{r}|k}C^{\overline{r}}_{l|\overline{h}}. \end{aligned} \tag{32}$$

From Proposition 3.2, $i)$, we obtain

Proposition 9. *Let (M,F) be a complex Finsler space. If a statement from Proposition 4.2 is satisfied, then (M,F) has constant holomorphic curvature $4c$.*

Proposition 10. *Let (M,F) be a $(g.E.)$ complex Finsler space of constant holomorphic curvature $4c$. Then,*

i) $Ric=c(n+1)L(z,\eta)$;

ii) $Ric_{i\bar{j}}=c(n+1)g_{i\bar{j}}$;

iii) $R^l_{\bar{j}k}\overline{\eta}^j\eta^k=2cL(z,\eta)\eta^l$; $R_{\bar{j}l\bar{h}k}\overline{\eta}^j\eta^l\overline{\eta}^h=2cL(z,\eta)\eta_k$;

iv) $(R_{\bar{j}l\bar{h}k}-R_{\bar{j}k\bar{h}l})\overline{\eta}^j\eta^l\overline{\eta}^h=0$;

v) $C_{\overline{j}\overline{h}|k|\bar{m}}\eta^k\bar{\eta}^m-2cLC_{\overline{j}\overline{h}}=0$.

Proof. It follows from Propositions 3.3, 3.5 and 4.2. □

Observe that the above conditions $iii)$ and $iv)$, with $c=-1$, are equivalent to the conditions of Theorem 3.1.15, from [**AP1**], p. 146. Therefore, the following Proposition gives a particular form of this Theorem.

Proposition 11. *Let (M,F) be a complex Finsler space. If one of equivalent conditions from Proposition 4.2 holds for $c=-1$, then F is the Kobayashi metric on M.*

Proposition 12. *If (M,F) is a $(g.E.)$ complex Finsler space, Kähler with $K(z)=c$ a nonzero constant, then F is purely Hermitian.*

Proof. By Iacobi identity
$\left[\dot{\partial}_i,[\delta_j,\delta_{\bar{k}}]\right]+\left[\delta_j,[\delta_{\bar{k}},\dot{\partial}_i]\right]+\left[\delta_{\bar{k}},[\dot{\partial}_i,\delta_j]\right]=0$, we have
$\dot{\partial}_i R^{\bar{l}}_{j\bar{k}}-\dot{\partial}_j R^{\bar{l}}_{i\bar{k}}-T^r_{ij}P^{\bar{l}}_{r\bar{k}}=0.$
Multiplying the last relation by $g_{m\bar{l}}\eta^i\bar{\eta}^k$, we obtain
$4cL(z,\eta)C_{mj}=T^r_{ij}\eta^i C_{mr|\bar{k}}\bar{\eta}^k.$
But the space is Kähler, so that $C_{mj}=0$. This leads to $\frac{\partial g_{m\bar{k}}}{\partial\eta^j}=0$, *i.e.* F is purely Hermitian. □

Taking into account Propositions 4.2, *iii)*,4.6 and 3.6 we may conclude that for the particular case of the complex Kähler-Finsler spaces of nonzero constant holomorphic curvature, the notions of $(g.E.)$ and purely Hermitian coincide.

5. An example

In this section we depict an example which illustrate our theory. We consider a complex Finsler space of complex dimension two. In order to reduce clutter, let us relabel the local coordinates z^1, z^2, η^1, η^2 as z, w, η, θ, respectively.

Let

$$L_{AS}(z,w;\eta,\theta):=e^{2\sigma(z,w)}\left(|\eta|^4+|\theta|^4\right)^{\frac{1}{2}},\ \text{with}\ \ \eta,\theta\neq 0, \tag{33}$$

be the complex version of *Antonelli-Shimada metric* on a domain from $\widetilde{T'M}$, $\dim_C M=2$, such that its metric tensor be nondegenerated, where $\sigma(z,w)$ is a real valued function and $|\eta^i|^2:=\eta^i\bar{\eta}^i$, $\eta^i\in\{\eta,\theta\}$, ([**Mu**]).
A direct computation leads to

$$\begin{aligned} g_{z\bar{z}} &= \frac{e^{2\sigma(z,w)}|\eta|^2(|\eta|^4+2|\theta|^4)}{(|\eta|^4+|\theta|^4)^{\frac{3}{2}}};\ g^{\bar{z}z}=\frac{e^{-2\sigma(z,w)}(2|\eta|^4+|\theta|^4)}{2|\eta|^2(|\eta|^4+|\theta|^4)^{\frac{1}{2}}};\\ g_{z\overline{w}} &= -\frac{e^{2\sigma(z,w)}|\eta|^2|\theta|^2\bar{\eta}\theta}{(|\eta|^4+|\theta|^4)^{\frac{3}{2}}};\ g^{\overline{w}z}=\frac{e^{-2\sigma(z,w)}\eta\bar{\theta}}{2(|\eta|^4+|\theta|^4)^{\frac{1}{2}}};\\ g_{w\bar{w}} &= \frac{e^{2\sigma(z,w)}|\theta|^2(2|\eta|^4+|\theta|^4)}{(|\eta|^4+|\theta|^4)^{\frac{3}{2}}};\ g^{\bar{w}w}=\frac{e^{-2\sigma(z,w)}(|\eta|^4+2|\theta|^4)}{2|\theta|^2(|\eta|^4+|\theta|^4)^{\frac{1}{2}}}. \end{aligned} \tag{34}$$

The nonzero coefficients of the Chern $(c.l.c.)$ are

$$\begin{aligned} &L^z_{zz}=L^w_{wz}=2\frac{\partial\sigma}{\partial z};\ L^z_{zw}=L^w_{ww}=2\frac{\partial\sigma}{\partial w};\\ &C^z_{zz}=\frac{|\theta|^8\bar{\eta}}{|\eta|^2(|\eta|^4+|\theta|^4)^2};\ C^z_{zw}=C^z_{wz}=-\frac{|\theta|^6\bar{\theta}}{(|\eta|^4+|\theta|^4)^2};\ C^z_{ww}=\frac{|\theta|^4\bar{\theta}^2\eta}{(|\eta|^4+|\theta|^4)^2};\\ &C^w_{ww}=\frac{|\eta|^8\bar{\theta}}{|\theta|^2(|\eta|^4+|\theta|^4)^2};\ C^w_{zw}=C^w_{wz}=-\frac{|\eta|^6\bar{\eta}}{(|\eta|^4+|\theta|^4)^2};\ C^w_{zz}=\frac{|\eta|^4\bar{\eta}^2\theta}{(|\eta|^4+|\theta|^4)^2}. \end{aligned} \tag{35}$$

From here, we obtain

$$
\begin{aligned}
(36) \qquad R_{\bar{z}z} &= -\frac{2e^{2\sigma(z,w)}|\eta|^2}{(|\eta|^4+|\theta|^4)^{\frac{1}{2}}}\left(\frac{\partial^2\sigma}{\partial z\partial\bar{z}}|\eta|^2+\frac{\partial^2\sigma}{\partial z\partial\bar{w}}\eta\bar{\theta}\right);\\
R_{\bar{z}w} &= -\frac{2e^{2\sigma(z,w)}|\eta|^2}{(|\eta|^4+|\theta|^4)^{\frac{1}{2}}}\left(\frac{\partial^2\sigma}{\partial w\partial\bar{z}}|\eta|^2+\frac{\partial^2\sigma}{\partial w\partial\bar{w}}\eta\bar{\theta}\right);\\
R_{\bar{w}z} &= -\frac{2e^{2\sigma(z,w)}|\theta|^2}{(|\eta|^4+|\theta|^4)^{\frac{1}{2}}}\left(\frac{\partial^2\sigma}{\partial z\partial\bar{w}}|\theta|^2+\frac{\partial^2\sigma}{\partial z\partial\bar{z}}\bar{\eta}\theta\right);\\
R_{\bar{w}w} &= -\frac{2e^{2\sigma(z,w)}|\theta|^2}{(|\eta|^4+|\theta|^4)^{\frac{1}{2}}}\left(\frac{\partial^2\sigma}{\partial w\partial\bar{w}}|\theta|^2+\frac{\partial^2\sigma}{\partial w\partial\bar{z}}\bar{\eta}\theta\right).
\end{aligned}
$$

We conclude that the (33) metric is not $(g.E.)$ and its holomorphic curvature is $\mathcal{K}_F = -\frac{4}{L_{AS}}\frac{\partial^2\sigma}{\partial z^k\partial\bar{z}^h}\eta^k\overline{\eta}^h$, where $z^i \in \{z,w\}, \eta^i \in \{\eta,\theta\}$.

Particularly, if $\frac{\partial^2\sigma}{\partial z^k\partial\bar{z}^h} = 0$ then the (33) metric is $(g.E.)$ with $K = 0$. In this particular case, it is not purely Hermitian or weakly Kähler. Moreover, the (33) metric is locally Minkowski if and only if σ is a constant.

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