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Regularized volume form of the sphere of a Hilbert space with the determinant bundle

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Abstract. Regularizations of the volume form and the volume of the sphere S^∞ in a Hilbert space H are presented by using ζ-function of a preassigned positive Schatten class operator G on H. It is shown the regularized volume form does not defined on S^∞ but defined on the sphere $\hat{S}^\infty$ of an extended space $\hat{H} = H \oplus \mathbb{R}$ of H, where the extra space $\mathbb{R}$ may be regarded as the fibre of the determinant bundle of H.

1. Introduction

In [3], a systematic way of the applications of ζ-regularization to the calculus on a Hilbert space H was proposed under the name *regularized calculus* provided H is equipped with a positive Schatten class operator G. Precisely saying, G induces k-Sobolev metric $\|\cdot\|_k$ to elements of H, and regularized calculus is performed on $W^{k-0}(finite) = W^k(0) \oplus \mathbb{R}$, where W^k is the Sobolev space constructed by $\|\cdot\|_k$ and H, $W^{k-0}(0)$ is an extended space of W^k. $\mathbb{R}$ is the fibre of the determinant bundle of H (or $W^{k-0}(0)$).

In [4], regularized Jacobian of a scaling transformation T_a; $T_a e_n = a_n e_n$, $a = (a_1, a_2, \ldots)$, $e_1, e_2, \ldots$, the complete orthonormal basis of H given by the eigenfunctions of G, $Ge_n = \mu_n e_n$, is shown to be the regularized determinant $det_G T_a$ of T_a with respect to G. Here $det_G T$ is defined by

$$det_G T = e^{tr(G^s S)}|_{s=0}, \quad T = e^S,$$

where $|_{s=0}$ means analyitc continuation to $s = 0$. Then $det_G T_a$ is the regularized infinite product $\prod_n a_n$ defined in [2];

$$det_G T_a = \prod_{s=0}^{\infty} a_n^{\mu_n^s}|_{s=0} = \prod_{n=1}^{\infty} a_n.$$

Let $x = \sum_n x_n e_n$ be an element of H or some extended space of H. We regard $\prod_n x_n$ to be a function on H, *etc.*. Then $\prod_n x_n$ is a non-trivial function on $H^-(finite)$, but can not be defined on $H^-(0)$. Here H^- means W^{0-0}. It is linear in each varible

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x_n. Moreover, it is shown

$$\lim_{N\to\infty}\frac{\partial^N}{\partial x_1\cdots\partial x_N}\prod_{n=1}^{\infty}x_n := 1,$$

in suitable weak sense ([4]). Regularized Jacobian of T_a; $T_a^*f(x) = f(T_ax)$, is computed by this equality and

$$\begin{aligned}&\lim_{N\to\infty}\int_{\mathbb{R}^N}\Big(\frac{\partial^N}{\partial x_1\cdots\partial x_N}|x_1|^{\mu_1^s}\cdots|x_N|^{\mu_N^s}\Big)T_a^*f(x)d^Nx|_{s=0}\\ =\ &\lim_{N\to\infty}\int_{\mathbb{R}^N}\Big(\prod_{n=1}^{\infty}|a_n|\Big)^{-1}f(x)d^Nx.\end{aligned}$$

Here, we note that the limit of this integral should be considered integral on $H^-(finite)$. Because regularized infinite product is defined on $H^-(finite)$, but does not defined on H.

As a corollary, since $(x,Dx) = I^*_{\sqrt{D}}r^2(x)$, $r^2(x) = r(x)^2 = (x,x)$, we get

$$\int e^{-\pi(x,Dx)}\mathcal{D}x = \frac{1}{\sqrt{det.D}},$$

where $det.D$ is the Ray-Singer determinat of D (cf.[7],[13]). Here, we skip precise discussions on the domain of the integral.

Rewriting above calculation of regularized Jacobian by using polar coordinate of H, we compute regualrized volume form and regularized volume of the sphere of H. Owing to the lack of the last coordinate of H, the polar coordiante

$$x_1 = r\cos\theta_1,\dots,x_n = r\sin\theta_1\cdots\sin\theta_{n-1}\cos\theta_n,\dots,\quad r = \|x\|,$$

of H lacks the longitude, and latitude variables $\theta_1,\theta_2,\dots$, are not independent. They need to satisfy

$$\lim_{n\to\infty}\sin\theta_1\cdots\sin\theta_n = 0.$$

In the regularization procedure, we regard $\theta_1,\theta_2,\dots$, to be independent variables. This means we add new coordinate $t_\infty = \prod_n\sin\theta_n$ to H. Hence we get an extended space $\hat{H} = H\oplus\mathbb{R}$, where $\mathbb{R}$ may be regarded as the fibre of the determinant bundle of H. Let $\hat{S}^\infty$ be the (unit) sphere of $\hat{H}$. Then the regularized volume form and the regularized volume of $\hat{S}^\infty$ are computed to be

$$d\omega := \prod_{n=1}^{\infty}\sin^{\nu-n-2}\theta_n d\theta_n,\quad \mathrm{vol}(\hat{S}^\infty) = \frac{2\pi^{\nu/2}}{\Gamma(\frac{\nu}{2})},$$

where $\nu = \zeta(G,0)$.

This paper is organized as follows: In Sec.2, we review Ray-Singer determinant and regularized determinant. Hilbert space equipped with a Schatten class operator and regularized infinite product are reviewed in Sec.3. Weak derivation of regularized infinite product is discussed in Sec.4. Then in Sec.5, we compute regularized Jacobian of a scaling tranasfoemation. After exposing on polar coordiante of a Hilbert space in Sec.6, we compute regularized volume form and regularized volume of $\hat{S}^\infty$ in Sec.7.

2. Ray-Singer determinants and regularized determinants

Let D be a (pseudo) differential opertor with the spectres $\lambda_1, \lambda_2, \ldots$. Then by using $\zeta(D,s) = \sum_n \lambda_n^{-s}$, the Ray-Singer dtermiant $det.D$ of D is defined by

$$det.D = e^{-\zeta'(D,0)}, \quad \zeta'(D,s) = \frac{d}{ds}\zeta(D,s), \tag{1}$$

if $\zeta(D,s)$ allows analytic continuation and holomorphic at $s=0$ ([13], cf [11]).

Of course, this definition involves several ambuiguities (cf.[1]). For example, the Dirac operator $\not{D}$ on a compact spin manifold has two differnt ζ-functions. Consequently, there are two kinds of Ray-Singer determinants of $\not{D}$. They are complex conjugates each other. Hence $\not{D}$ has unique Ray-Singer determinat if and only if it is a real number ([1]). But if D is a positive elliptic operator on a compact Riemannian manifold X, then $det.D$ is uniquely defined positive number ([8],[13]). It is also known $\zeta(D,s)$ has poles at most order 1 at $s = d/m, (d-1)/m, \ldots$, where $d = \dim X$ and $m = \operatorname{ord} D$ ([8]).

Note. $\nu = \zeta(D,0)$ is important to the application of ζ-regularization to infinite dimensional calculus ([3]). For example, to construct infinite dimensional real Grassmann algebra with ∞-degree elements, ν needs to be a real number (Complex Grassmann algebra with ∞-degree elements can be constructed for arbitrary ν (cf.[3])). Integrity of ν can not be expected for an arbitrary D. But it is known $\zeta(D+mI,0)$ is a polynomial of m, where m is a scalar (mass term) and I is the identity. Hence selecting m and repalce D by $D+mI$, we can assume integrity of ν ([1],[3]).

We rewrite $det.D$ as follows:

$$det.D = e^{tr(D^{-s}\log D)}|_{s=0} = \prod_{n=1}^{\infty} \lambda_n^{\lambda_n^{-s}}|_{s=0}.$$

Motivated the formula $det.D = \exp tr(D^{-s}\log D)|_{s=0}$, we define regualrized determinant of $T = e^S$ with respect to D as follows:

Definition 1. *Let T be a linear operator defined on a dense subspace of H. Assume T has $S = \log T$; $T = e^S$. Then we define regularized determinant $det_D T$ of T with respect to D by*

$$det_D T = e^{tr(D^{-s}S)}|_{s=0}. \tag{2}$$

Note. $tr(D^{-s}S)$ is the ζ-regularized trace of S in the sense of Paycha ([5],[12]). Hence regularized determinant is in the scope of Paycha's regularized trace.

Regularized determinant is neither defined nor unique for general T. But if it is defined, fixing $det_D T$, we have

$$det_D I = 1, \quad det_D(tT) = t^{\nu} det_D T, \quad det_D(T^m) = (det_D T)^m. \tag{3}$$

Here $\nu = \zeta(D,0)$ and t is a positive real number.

Let $a = (a_1, a_2, \ldots)$ be an ∞-vector, and T_a be the operator

$$T_a e_n = a_n e_n, \ n = 1, 2, \ldots, \quad De_n = \lambda_n e_n.$$

Note that T_a may not be a bounded operator. For example, we have

$$D = I_D; \ I_D e_n = \lambda_n e_n, \ n = 1, 2, \ldots.$$

Since $\log T_a e_n = \log a_n e_n$, $n = 1, 2, \dots$, we have

$$det_D T_a = e^{tr(D^{-s}\log T_a)}|_{s=0} = \prod_{n=1}^{\infty} a_n^{\lambda_n^{-s}}|_{s=0}.$$

Hence we get

LEMMA 1. *Regularized determiant of a scaling operator T_a is the regularized infinite product of components of a:*

$$det_D T_a = \prod_{n=1}^{\infty} a_n. \tag{4}$$

As a special case of Lemma 1, we have

$$det_D D = det_D I_D = det.D.$$

Here $det.D$ means the Ray-Singer determinant of D.

In general, if $T = T_a + N$, N a generalized nilpotent element, that is

$$\lim_{n\to\infty} \|N^n\|^{1/n} = 0,$$

and $T_a N = N T_a$, then we have

$$det_D T = det_D T_a = \prod_{n=1}^{\infty} a_n.$$

Note. In the rest, we use the Green operator G of D instead of D for the convenience. We set $\lambda_n^{-1} = \mu_n$. Then we have

$$\zeta(G, s) = \zeta(D, s);\ \zeta(G, s) = tr(G^s) = \sum_{n=1}^{\infty} \mu_n^s.$$

We also denote $det_G T$ instead of $det_D T$, that is

$$det_G T = e^{tr(G^s S)}|_{s=0} = det_D T.$$

3. Hilbert space equipped with a Schatten class operator and regualrized infinite product

Let $\{H, G\}$ be a pair of a (real) Hilbert space and a positive Schatten class operator on H, such that $\zeta(G, s) = tr(G^s)$ allows analytic continuation and holomorphic at $s = 0$ (As for Schatten class operator, we refer [14]).

$\{H, G\}$ has numerical invariants

(1) $\nu = \zeta(G, 0)$: The regularized dimension of H.

(2) d: The location of the first pole of $\zeta(G, s)$.

The pair $\{H, G\}$ is closely related to Connes' spectral triple ([6]). Spectral triple is wider than $\{H, G\}$, but $\{H, G\}$ is more convenient for practical use. Typical example of $\{H, G\}$ is $H = L^2(X, E)$, X a compact Riemannian manifold E a vector bundle over X, and G the Green operator of a positive elliptic (pseudo) differntial operator acting on the sections of E. In this case, d is $\dim X/\mathrm{ord} D$.

We fix the complete ortho-normal basis $e_1, e_2, \dots$, of H to be the eigenfunctions of G; $Ge_n = \mu_n e_n$, $\mu_1 \geqslant \mu_2 \geqslant \cdots > 0$. Let

$$\|x\|_k = \|G^{-k}x\|, \tag{5}$$

be the k-th Sobolev type norm of H. The k-Sobolev space constructed from H and $\| \|_k$ is denoted by W^k. The complete ortho-normal basis of W^k is fixed to be $e_{1,k}, e_{2,k}, \ldots, e_{n,k} = \mu_n^k e_k$. As the sets, we have $W^k \subset W^l$, $k > l$. We set

$$W^{k-0} = \bigcap_{l<k} W^l, \quad H^- = W^{0-0} = \bigcap_{l<0} W^l. \tag{6}$$

Note. We set $W^{k+0} = \bigcup_{j>k} W^j$. Then by virtue of Sobolev dulaity, we have $W^{k-0,\dagger} = W^{-k+0}$. But in the rest, we do not use W^{k+0}.

Since $e_{\infty,k}$ does not belong to W^k but belongs to W^{k-0}, we have $W^{k-0} \supsetneqq W^k$. Here

$$e_{\infty,k} = \sum_{n=1}^{\infty} \mu_n^{d/2} e_{n,k}, \quad e_\infty = e_{\infty,0} = \sum_{n=1}^{\infty} \mu_n^{d/2} e_n. \tag{7}$$

Definition 2. *We define the spaces $W^{k-0}(finite)$ and $W^{k-0}(0)$ by*

$$W^{k-0}(finite) = \{\sum_{n=1}^{\infty} x_n e_{n,k} \in W^{k-0} | \lim_{n\to\infty} \mu_n^{-d/2} x_n \text{ exists}\}, \tag{8}$$

$$W^{k-0}(0) = \{\sum_{n=1}^{\infty} x_n e_{n,k} \in W^{k-0} | \lim_{n\to\infty} \mu_n^{-d/2} x_n = 0\}. \tag{9}$$

$W^{0-0}(finite)$ and $W^{0-0}(0)$ are denoted by $H^-(finite)$ and $H^-(0)$.

By definition, $W^{k-0}(0)$ is close (or almost equal) to W^k. By defintions of $W^{k-0}(finite)$ and $W^{k-0}(0)$, we have

$$W^{k-0}(finite) = W^{k-0}(0) \oplus \mathbb{R} e_{\infty,k}. \tag{10}$$

As a topological space, we regard $W^{k-0}(0)$ to be a subspace of W^{k-0}. But $W^{k-0}(finite)$ is regarded as the product space of $W^{k-0}(0)$ and $\mathbb{R}$.

We may interprete $\mathbb{R}e_{\infty,k}$ as the fibre of the determinant bundle of $W^{k-0}(0)$, which is close to W^k. But since $W^{k-0}(0)$ is a flat space, its determinat bundle is trivial, and we do not have further geometric informations. Therefore we do not discuss on geometric properties of $\mathbb{R}e_{\infty,k}$.

By (10), $x \in W^{k-0}(finite)$ is uniquely written as

$$x = y + te_{\infty,k}, \quad y \in W^{k-0}(0), \quad x = \sum_{n=1}^{\infty} x_n e_{n,k},\ y = \sum_{n-1}^{\infty} y_n e_{n,k}.$$

Then, if $t \neq 0$, we have

$$\prod_{n=1}^{\infty} x_n^{\mu_n^s} = \prod_{n=1}^{\infty} (y_n + t\mu_n^{d/2})^{\mu_n^s} = t^{\zeta(G,s)} \prod_{n=1}^{\infty} (\mu_n^{\mu_n^s})^{d/2} \prod_{n=1}^{\infty} (1 + \frac{\mu_n^{-d/2} y_n}{t})^{\mu_n^s}.$$

Here, if some of x_n's are negaive, we fix Augx_n to be π or $-\pi$. Hence there may exist two kinds of regularized infinite product if ν is not an integer.

By the above calculation, if $\prod_{n=1}^{\infty}(1 + \frac{\mu_n^{-d/2} y_n}{t})$ converges, we have

$$\prod_{n=1}^{\infty} x_n = t^{\nu} (det.G)^{d/2} \prod_{n=1}^{\infty} (1 + \frac{\mu_n^{-d/2} y_n}{t}). \tag{11}$$

Therefore $\prod_n x_n$ is a function of $x_1, x_2, \ldots$, defined on dense subset of $W^{k-0}(finite)$, but does not defined on $W^{k-0}(0)$.

Note. We set $x = \sum_n x_n e_{n,k} = \sum_n x_{n,k} e_n$. Then $x_{n,k} = \mu_n^{-k} e_n$. Hence we have $\prod_n x_{n,k} = (det.G)^{-k} \prod_n x_n$.

Note. It is shown in [2], that $\prod_n x_n$ is linear in each variable x_n, positive if each x_n is positive and

$$|\prod_{n=1}^{\infty} x_n| = \prod_{n=1}^{\infty} |x_n|. \tag{12}$$

Note. Regularized infinite product can be defined for complex variables $z_1, z_2, \ldots$. But in this case, we need to fix the Agmon angle θ such that $\mathrm{Arg} z_n \neq \theta$, $n = 1, 2, \ldots$ (cf.[11]). Regularized infinite product of $z_1, z_2, \ldots$ depends on the choice of the Agmon angle θ.

4. Weak derivative of regularized infinite product

Since $\prod_n x_n$ is linear in each variable, we may consider

$$\frac{\partial^N}{\partial x_1 \cdots \partial x_N} \prod_{n=1}^{\infty} x_n = \prod_{n=N+1}^{\infty} x_n. \tag{13}$$

In ordinary sense, we can not compute $\lim\limits_{N\to\infty} \dfrac{\partial^N}{\partial x_1 \cdots \partial x_N} \prod\limits_{n=1}^{\infty} x_n$. But assuming G to be the Green operator of an elliptic operator, we have in suitable weak sense

$$\lim_{N\to\infty} \frac{\partial^N}{\partial x_1 \cdots \partial x_N} \prod_{n=1}^{\infty} x_n = 1. \tag{14}$$

In the rest of this paper, we assume *H is the Hilbert space of sections of a bundle E over a compact Riemannian manifold, and G is the Green operator of an elliptic (pseudo) differential operator D acting on the sections of E.*

To show (14), first we take a C^1-class poasitive function $f(t)$ on $\mathbb{R}_+ = \{t \in \mathbb{R} | t \geqslant 0\}$, such that $f(0) = 1$, $\int_0^\infty f(t)dt = 1$. Let $\mathcal{M}[f'](x)$ be the Mellin transformation $\displaystyle\int_0^\infty t^{x-1}\frac{df}{dt}dt$ of f (cf.[15]). Then $-\mathcal{M}[f'](2) = 1$, and $\log(-\mathcal{M}[f'](1+x))$ has the following type Taylor expansion;

$$\log(-\mathcal{M}[f'](1+x)) = \sum_{n\geqslant 1} a_n x^n, \ |x| < 1, \tag{15}$$

An example of $f(t)$ is e^{-t}. In this case, $-\mathcal{M}[f'] = \mathcal{M}[f] = \Gamma(t)$. Hence

$$\log(-\mathcal{M}[f'](1+x)) = -\gamma x + \sum_{n\geqslant 2} (-1)^n \frac{\zeta(n)}{n} x^m,$$

where γ is the Euler constant and $\zeta(s)$ is the Riemann ζ-function.

Since we get

$$\int_0^\infty (\frac{d}{dt} t^x) f(t)dt = -\int_0^\infty t^x \frac{df}{dt} dt = -\mathcal{M}(f')(1+x),$$

we have

$$\lim_{N\to\infty}\int_{\mathbb{R}_+^N}\Big(\frac{\partial^N}{\partial x_1\cdots\partial x_N}x_1^{\mu_1^s}\cdots x_N^{\mu_N^s}\Big)\prod_{n=1}^N f(x_n)d^Nx$$
$$= \prod_{n=1}^{\infty}(-\mathcal{M}(f')(1+\mu_n^s).$$

Here $\mathbb{R}_+^N$ is $\{(x_1,\cdots,x_N)|x_1\geqslant 0,\cdots,x_N\geqslant 0\}$. Then by (15), we have

$$\log\Big(\prod_{n=1}^{\infty}(-\mathcal{M}(f')(1+\mu_n^s)\Big)=\sum_{n=1}^{\infty}a_n\zeta(G,ns). \tag{16}$$

Hence $\prod_n(-\mathcal{M}(f')(1+^m u_n^s))$ allows analytic continuation to the right half palne $\{z|\operatorname{Re}z>0\}$ with singularities on real axis which concentrates to 0. Because, since G is the Green function of an elliptic operator D, $\zeta(G,s)$ has poles only at $s=d,d-1/m,d-2/m,\ldots$. Hence if $s=\gamma(t)$ tends to $0=\gamma(0)$, where γ is not tangent to the real and imaginary axes, we have

$$\lim_{s\to 0}\prod_{n=1}^{\infty}\big(\mathcal{M}(f')(1+\mu_n^s)\big)=\prod_{n=1}^{\infty}\big(\mathcal{M}(f')(2)\big)=1. \tag{17}$$

That is, we obtain

$$\lim_{N\to\infty}\int_{\mathbb{R}_+^N}\Big(\frac{\partial^N}{\partial x_1\cdots\partial x_N}x_1^{\mu_1^s}\cdots x_N^{\mu_N^s}\Big)\prod_{n=1}^N f(x_n)d^Nx|_{s=0}=1. \tag{18}$$

Let $L^{1,1}(\mathbb{R}_+^\infty)$ be the L^1 1-Sobolev space generated by

$$\{F(x_1,\ldots,x_m)\prod_{n>m}f(x_n)|F\in L^{1,1}(\mathbb{R}_+^m)\},\ m=1,2,\ldots,.$$

Then by (18), (14) is hold as the weak derivation in the dual space of $L^{1,1}(\mathbb{R}_+^\infty)$.

Let $g(t)$ be $f(t^2)$. Then we have $\mathcal{M}[g](x)=\dfrac{1}{2}\mathcal{M}[f](\dfrac{x}{2})$. Hence if a C^1-class function $g(x)=f(x^2)$ satisfies $g(0)=1$ and $\displaystyle\int_{-\infty}^{\infty}g(x)dx=1$, then we get

$$\lim_{N\to\infty}\int_{\mathbb{R}^N}\Big(\frac{\partial^N}{\partial x_1\cdots\partial x_N}|x_1|^{\mu_1^s}\cdots|x_N|^{\mu_N^s}\Big)\prod_{n=1}^N g(x_n)d^Nx|_{s=0}=1. \tag{19}$$

Here the analytic continuation is done along the path γ which does not tangent to real and imaginary axes. Therefore we have

$$\lim_{N\to\infty}\frac{\partial^N}{\partial x_1\cdots\partial x_N}\prod_{n=1}^{\infty}|x_n|=1, \tag{20}$$

as the weak derivation in $W^1(\mathbb{R}^\infty)$, the Sobolev 1-space generated by

$$\{F(x_1,\cdots,x_m)\prod_{n>m}g(x_n)|F\in W^1(\mathbb{R}^m\},\ m=1,2,\ldots,.$$

An example of $g(x)$ is $e^{-\pi x^2}$. This function will be used in the computation of regularized volume of the infinite dimensional sphere $\hat{S}^\infty$ (Sec.7).

Note. The use of weak derivation to the study of regularized infinite product was first suggested by Léandre (cf.[10]).

5. Regularized Jacobian of the scaling transformation

Let T_a be a scaling trnasformation $T_a e_n = a_n e_n$; $a = (a_1, a_2, \ldots)$. T_a need not be a bounded operator on H. For example, if $a_n = \mu_n^{-1} = \lambda_n$, then T_a is unbounded and defined only on W^1, where W^1 is regarded as a subset of H. For a function f on T_aU, U a subset of W^k, or $W^k(finite)$, we define $T_a^* f$, a function on U, by

$$T_a^* f(x) = f(T_a x).$$

We consider $\mathbb{R}^n$ to be the subspace of H spanned by $e_1, \ldots, e_n$. Let $T_a^n \mathbb{R}^n \to \mathbb{R}^n$ be the linear operator defined by $T_a e_k = a_k e_k$, $k = 1, \ldots, n$. If $a \gg 0$ that is $a_1 > 0, a_2 > 0, \ldots$, then T_a^n maps $\mathbb{R}_+^n$ on $\mathbb{R}_+^n$. Then if $f \in L^{1,1}(\mathbb{R}_+^\infty)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^N} \Big(\frac{\partial^N}{\partial x_1 \cdots \partial x_N} x_1^{\mu_1^s} \cdots x_N^{\mu_N^s}\Big) T_a^{n*} f(x) d^N x \\
= &\int_{\mathbb{R}_+^n} a_1^{-\mu_1^s} \cdots a_N^{-\mu_N^s} \Big(\frac{\partial^N}{\partial y_1 \cdots \partial y_N} y_1^{\mu_1^s} \cdots y_N^{\mu_N^s}\Big) f(y) d^N y,
\end{aligned}$$

where $y_n = a_n x_n$, $n = 1, \ldots, N$. Hence we obtain

$$\begin{aligned}
&\lim_{N\to\infty} \int_{\mathbb{R}_+^N} \Big(\frac{\partial^N}{\partial x_1 \cdots \partial x_N} x_1^{\mu_1^s} \cdots x_N^{\mu_N^s}\Big) T_a^* f(x) d^N x|_{s=0} \\
= &\lim_{N\to\infty} \int_{\mathbb{R}_+^N} \Big(\prod_{n=1}^{\infty} a_n\Big)^{-1} f(x) d^N x. \qquad (21)
\end{aligned}$$

Here analytic continuation is done along the path in the right half plane, which does not tangent to real and imaginary axes. Similarly, if $f \in W^1(\mathbb{R}^\infty)$, we get

$$\begin{aligned}
&\lim_{N\to\infty} \int_{\mathbb{R}^N} \Big(\frac{\partial^N}{\partial x_1 \cdots \partial x_N} |x_1|^{\mu_1^s} \cdots |x_N|^{\mu_N^s}\Big) T_a^* f(x) d^N x|_{s=0} \\
= &\lim_{N\to\infty} \int_{\mathbb{R}^N} \Big|\prod_{n=1}^{\infty} a_n\Big|^{-1} f(x) d^N x. \qquad (22)
\end{aligned}$$

By (21) and (22), we obtain

THEOREM 1. *The Jacobian of the scaling transformation T_a is regularized by the regularized determinant of T_a with respect to G, as follows:*

$$\Big|\prod_{n=1}^{\infty} a_n\Big|^{-1} = |det_G T_a|^{-1}. \qquad (23)$$

Note. (23) should be a formula of integral on $H^-(finite)$, because regularized infinite product have maenings on $H^-(finite)$, but does not have meanings on H.

For the simplicity, we denote

$$\int_{H^-(finite)} f(x) d^\infty x = \lim_{n\to\infty} \int_{\mathbb{R}^n} \Big(\frac{\partial^n}{\partial x_1 \cdots \partial x_n} |x_1|^{\mu_1^s} \cdots |x_n|^{\mu_n^s}\Big) f(x) d^n x|_{s=0}.$$

The domain of the integral of $d^\infty x$ needs discussions. But we omit precise discussions. By using this notation, we can write

$$\int_{H^-(finite)} T_a^* f(x) d^\infty x = \int_{T_a^{-1}(H^-(finite))} |\prod_{n=1}^{\infty} a_n|^{-1} f(x) d^\infty x. \tag{24}$$

EXAMPLE. *Since* $(x, Dx) = I_{sqrtD}^* r(x)^2$, $r(x)^2 = \|x\|^2$, *we can claim (23) justifies the formula (cf.[9])*

$$\int_{W^{1/2-0}(finite)} e^{-\pi(x,Dx)} \mathcal{D}x = \frac{1}{\sqrt{D}}. \tag{25}$$

Let $a \in W^{k-0}(finite)$ *and* $a \gg 0$, *then we set*

$$\begin{aligned} Q(c) &= \{x = \sum_{n=1}^{\infty} x_n e_{n,k} \in W^{k-0}(finite) | |x_n| \leqslant c_n\}, \\ Q(c;+) &= \{x = \sum_{n=1}^{\infty} x_n e_{n,k} \in W^{k-0}(finite) | 0 \geqslant x_n \leqslant c_n\}. \end{aligned}$$

Here $c = \sum_n c_n e_{n,k} \in W^{k-0}(finite)$, $c_1 > 0, c_2 > 0, \ldots$. *We define* $d^\infty x$ *on* $Q(c)$, $Q(c;+)$,*etc., similarly. Then we have*

$$\int_{Q(c)} T_a^* f(x) d^\infty x = \int_{(Q(a^{(-1)} \cdot c))} |\prod_{n=1}^{\infty} a_n|^{-1} f(x) d^\infty x, \tag{26}$$

$$\int_{Q(c;+)} T_a f(x) d^\infty x = \int_{(Q(a^{(-1)} \cdot c;+))} (\prod_{n=1}^{\infty} a_n)^{-1} f(x) d^\infty x, \tag{27}$$

where a *is assumed to be* $a \gg 0$. *Here* $a^{(-1)} \cdot c$ *means*

$$a^{(-1)} \cdot c = \sum_{n=1}^{\infty} a_n^{-1} c_n e_{n,k}.$$

Hence $Q(a^{(-1)} \cdot c) = T_a^{-1}(Q(c))$ *and* $Q(a^{(-1)} \cdot c;+) = T_a^{-1}(Q(c;+))$.

Note. The integral by $d^\infty x$ seems can not be expressed by using measure. Because it uses analytic continuation along a path which is not tangent to the real axis.

6. Polar coordiante of a Hilbert space

Similar to the finite dimensional case, the polar coordinate of H is given by

$$\begin{aligned} x_1 &= r\cos\theta_1, \quad x_2 = r\sin\theta_1\cos\theta_2, \ldots, \\ x_n &= r\sin\theta_1 \cdots \sin\theta_{n-1}\cos\theta_n, \ldots, \end{aligned}$$

where $r = \|x\|$ and $0 \leqslant \theta \leqslant \pi$. This coordinate has only latitudes and lacks longitude. Latitudes are not independent. They need to satisfy

$$\lim_{n\to\infty} \sin\theta_1 \sin\theta_2 \cdots \sin\theta_n = 0. \tag{28}$$

Because $r^2 = x_1^2 + \cdots + x_n^2 + (r\sin\theta_1 \cdots \sin\theta_n)^2$. But since

$$t_\infty = \lim_{n\to\infty} \sin\theta_1 \cdots \sin\theta_n = \prod_n \sin\theta_n,$$

exists and $0 \leqslant t_\infty \leqslant 1$, although $\theta_1, \theta_2, \ldots$, are independent with $0 \leqslant \theta_i \leqslant \pi$, $i = 1, 2, \ldots$, we add new coordinate $y = \pm rt_\infty$ to H and set

$$\hat{H} = \{(x, y) | x \in H, y = \pm rt_\infty\} \cong H \oplus \mathbb{R}, \qquad \|(x, y)\|^2 = \|x\|^2 + |y|^2. \tag{29}$$

Note. Adding longitude to H provides a two-dimensional higher space $\tilde{H} = H \oplus \mathbb{R}^2$ with the new variables y and z such that

$$y = rt_\infty \cos\varphi, \quad z = rt_\infty \sin\varphi, \quad 0 \leqslant \varphi \leqslant 2\pi, \tag{30}$$

Then we can regard $\hat{H}$ to be the subspace of $\tilde{H}$ defined by $\varphi = 0, \pi$.

We denote the unit spheres in H and $\hat{H}$ by S^∞ and $\hat{S}^\infty$, respectively. Spheres of radius r are denoted by S^∞_r and $\hat{S}^\infty_r$. $\hat{S}^\infty$ may be regarded as the Greenwitch line of $\tilde{S}^\infty$, the unit sphere of $\tilde{H}$.

To search the relation between the extra 1-dimensional subspaces $\mathbb{R} = \pm rt_\infty$ of $\hat{H}$ and $\mathbb{R}e_\infty$ of $H^-(finite)$, we assume $t_\infty \neq 0$. Then $\lim_{n\to\infty} \theta_n = \pi/2$, because $\lim_{n\to\infty} \sin\theta_n = 1$. Let $x = \sum_n x_n e_n \in H^-(finite)$. Since

$$\sin\theta_1 \cdots \sin\theta_n = \frac{x_n}{\|x\| \cos\theta_n}, \quad \|x\|^2 = \|y\|^2 + t^2,$$

where $x = y + te_\infty$, $y \in H^-(0) \cap H$, we get $x \in W^k$ for some $k < 0$, if $\theta_n = \pi/2 + c\mu_n^{d/2} + o(\mu_n^{d/2})$. Hence $x \in \mathbb{R}e_\infty$, if $c \neq 0$.

Conversely, if $x = \sum_n x_n e_n \in H^-(finite)$, we set $x = y + te_\infty$, $y = \sum_n y_n e_n \in H^-(0)$. Let c be $\mathrm{Res}_{s=d}\zeta(G, s) \neq 0$. Then

$$\lim_{k\to+0} k(\sum_{n>N} \mu_n^{d+k}) = c, \quad k(\sum_{n=1}^{N} \mu_n^{d+k} y_n^2) = 0,$$

for a fixed N such that $\mu_n^{-d}|y_n|^2 < \epsilon/2|c|$, $n > N$. Therefore there exists $\delta > 0$ such that $|k(\sum_{n=1}^N \mu_n^{d+k}|y_n|^2| \leqslant \epsilon$ and $|k(\sum_{n\geqslant N} \mu_n^{d+k}| < 2|c|$, if $k < \delta$. Then if $k < \delta$, we have

$$\begin{aligned} k|y\|^2_{-k} &= \frac{1}{2} 2k(\sum_{n=1}^{\infty} \mu_n^{d+2k} |\mu_n^{-d/2} y_n|^2) \\ &< k\big(\sum_{n=1}^{N} \mu_n^{d+2k} |y_n|^2 + \frac{\epsilon}{|c|}\zeta(G, d+2k)\big) < \frac{1}{2}\big(\epsilon + \frac{\epsilon}{|c|}(|c| + O(k))\big) < \epsilon. \end{aligned}$$

Hence we get

$$\lim_{k\to+0} k\|x\|^2_{-k} = \lim_{k\to+0} k|t|^2 |\zeta(G, d+2k)| = \frac{1}{2}|t|^2|c|,$$

that is we have

$$\lim_{k\to+0} \sqrt{k}\|x\|_{-k} = \sqrt{\frac{|c|}{2}}|t|. \tag{31}$$

Therefore extra coordinates of $\hat{H}$ and $H^-(finite)$ related each other. In other word, *we may interprete the extra coordiante of $\hat{H}$ as the fibre of the determinant bundle of H.*

7. Regularized volume form and regularized volume of $\hat{S}^\infty$

Let $\mathbb{R}^n$ be the subspace of H (or $\hat{H}$) spanned by $e_1, \ldots, e_n$, $\hat{\mathbb{R}}^{n-1}$ the subspace of $\mathbb{R}^n$ defined by $\varphi = 0, \pi$ by the polar coordiante of $\mathbb{R}^n$.

By uisng polar coordinate of $\mathbb{R}^n$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(\frac{\partial^n}{\partial x_1 \cdots \partial x_n} |x_1|^{\mu_1^s} \cdots |x_n|^{\mu_n^s}\Big) f(x) d^n x \\
= &\int_0^\infty \int_{S^{n-1}} r^{\mu_1^s + \cdots + \mu_n^s - n} \sin^{\mu_2^s + \cdots + \mu_n^s - n+1} \theta_1 \cdots \sin^{\mu_{n-1}^s + \mu_n^s} \theta_{n-2} \times \\
&\times F_n(x;s) f(x) r^{n-1} \sin^{n-2}\theta_1 \cdots \sin\theta_{n-2} dr d\theta_1 \cdots d\theta_{n-2} d\varphi \\
= &\int_0^\infty r^{\mu_1^s + \cdots \mu_n^s - 1} dr \int_{S^{n-1}} F_n(x;s) f(x) \times \\
&\times \sin^{\mu_2 + \cdots + \mu_n^s - 1}\theta_1 d\theta_1 \cdots \sin^{\mu_{n-1}^s + \mu_n^s - 1}\theta_{n-2} d\theta_{n-2} d\varphi. \\
&F_n(x;s) = \mathcal{F}_n(|\cos\theta_1|, \ldots, |\cos\theta_{n-2}|, |\cos\varphi|, |\sin\varphi|; s), \\
&\mathcal{F}_n(x;s) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} x_1^{\mu_1^s} \cdots x_n^{\mu_n^s}.
\end{aligned}$$

We set $\lim_{n\to\infty} F_n(x;s)_{\varphi=0,\pi} = F(\theta_1, \theta_2, \ldots; s)$. Then $\theta_1, \theta_2, \ldots$ should be independent as variables in $F(\theta_1, \theta_2, \ldots)$. Hence we get

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\hat{\mathbb{R}}^{n-1}} \Big(\frac{\partial^n}{\partial x_1 \cdots \partial x_n} |x_1|^{\mu_1^s} \cdots |x_n|^{\mu_n^s}\Big) f(x) d^{n-1} x|_{s=0} \\
= &\int_0^\infty r^{\nu-1} dr \int_{\hat{S}^\infty} F(\theta_1, \theta_2, \ldots; s) f(x) \prod_{n=1}^\infty \sin^{\nu-n-1}\theta_n d\theta_n|_{s=0}. \qquad (32)
\end{aligned}$$

Since $\lim_{n\to\infty} \mathcal{F}_n(x;s)|_{s=0} = 1$, as the element of $W^1(\mathbb{R}^\infty)$, $F(\theta_1, \theta_2, \ldots)|_{s=0}$ is equal to 1, if $f \in W^1(\mathbb{R}^\infty)$. Hence we may claim

Theorem 2. *Regularized volume form $d^\infty x$ of $\hat{H}$ is given by*

$$d^\infty x = r^{\nu-1} dr d\omega, \quad d\omega = \prod_{n=1}^\infty \sin^{\nu-n-1}\theta_n d\theta_n. \qquad (33)$$

Definition 3. *We say $d\omega$ to be the regularixed volume form of $\hat{S}^\infty$.*

Note. $d\omega$ depends on ν. Denoting $d\omega = d\omega(\nu)$, we have

$$d\omega(\mu) = t_\infty^{\mu-\nu} d\omega(\nu) \qquad (34)$$

Note. $d\omega$ has singularities on S^∞, the unit sphere of H. Hence we can not obtain regularized volume form on S^∞ from $d\omega$.

We also note that there exist eigenfunctions of regualrized spherical Laplacian (cf. [16]) which cancel the singularities of $d\omega$ ([3]).

Since we have

$$\lim_{n\to\infty} \int_{\hat{\mathbb{R}}^{\ltimes - \Bbbk}} e^{-\pi\|x\|^2} d^{n-1} x = \int_{\hat{H}} e^{-\pi\|x\|^2} d^\infty x = 1, \qquad (35)$$

in one hand, and

$$\int_{\hat{H}} e^{-\pi\|x\|^2} d^\infty x = \int_0^\infty e^{-\pi r^2} dr \int_{\hat{S}^\infty} F(\theta_1, \theta_2, \ldots; s) d\omega|_{s=0}, \qquad (36)$$

on the other hand, we may claim

THEOREM 3. *Regularized volume* $\mathrm{vol}(\hat{S}^\infty)$ *of* $\hat{S}^\infty$ *by the regularized volume form* $d\omega = d\omega(\nu)$ *is*

$$\mathrm{vol}(\hat{S}^\infty) = \frac{2\pi^{\nu/2}}{\Gamma(\frac{\nu}{2})}. \tag{37}$$

By (37) and $d^\infty x = r^{\nu-1}d;d\omega$, the regularized volume of $\hat{S}^\infty_r$ is given by

$$\mathrm{vol}(\hat{S}^\infty_r) = r^{\nu-1}\frac{2\pi^{\nu/2}}{\Gamma(\frac{\nu}{2})}. \tag{38}$$

Note. ν may take negative value. In this case, it is convenient to rewrite

$$\frac{2\pi^{\nu/2}}{\Gamma(\frac{\nu}{2})} = -\frac{\sin(\frac{|\nu|\pi}{2})}{\pi^{\frac{|\nu|}{2}+1}r^{|\nu|+1}}\Gamma(\frac{|\nu|}{2}+1). \tag{39}$$

Hence $\mathrm{vol}(\hat{S}^\infty)$ is 0, if ν is a negative even number, negative if $\nu \equiv 1, \mathrm{mod}.4$.

Existences of regularized volume form and finite non-zero regularized volume of $\hat{S}^\infty$ suggest existence of *de Rham type cohomology with infinite degree elements* for $\hat{S}^\infty$ and other infinite dimensional manifolds (cf.[7]). Since regularized volume of $\hat{S}^\infty$ depends on ν and vanishes if ν is a negative even number, this cohomology must be *not* topological invariant if exists. These will be next problems.

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