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Canonical $f(5,1)$-structures on homogeneous k-symmetric spaces

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Abstract. We present a wide collection of invariant $f(5,1)$-structures (i.e. $F^5 + F = 0$) on homogeneous manifolds. More exactly, all canonical $f(5,1)$-structures on homogeneous k-symmetric spaces are completely described. The particular cases of 3- and 4-symmetric spaces are considered in more detail.

1. Introduction

An important role among homogeneous manifolds of Lie groups is played by the homogeneous Φ-spaces, that is, the homogeneous spaces generated by automorphisms Φ of Lie groups (see [**14**], [**16**], [**6**], [**11**], [**3**]). In particular, homogeneous k-symmetric spaces ($\Phi^k = id$) form the most significant class of homogeneous Φ-spaces. Specifically, any of these spaces possesses the commutative algebra $\mathcal{A}(\theta)$ of all canonical affinor structures including remarkable classical structures such as almost complex structures J, almost product structures P, and, more generally, f-structures of K.Yano ($f^3 + f = 0$), h-structures ($h^3 - h = 0$) (see [**3**], [**1**]). On the other hand, there exist many other types of affinor structures, which are similar to above mentioned and fairly interesting for studying (see, as an example, [**8**], [**9**], [**4**], [**5**], [**12**]). However, up to now there are no natural classes of invariant examples for these and many other non-classical structures.

The main goal of the paper is to present a wide collection of invariant $f(5,1)$-structures (i.e. affinor structures F satisfying the condition $F^5 + F = 0$ [**8**]). More exactly, we completely describe all canonical $f(5,1)$-structures on homogeneous k-symmetric spaces.

The paper is organized as follows.

In Section 2, we collect some basic notions and results about homogeneous regular Φ-spaces and canonical affinor structures on them. In particular, the full algebraic description of all canonical structures of classical type (almost complex, almost product, f-structures, h-structures) is formulated. The particular cases of homogeneous 3- and 4-symmetric spaces are also included.

In Section 3, we prove the theorem about all canonical $f(5,1)$-structures on arbitrary homogeneous k-symmetric space. More precisely, the number N of canonical

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$f(5,1)$-structures is calculated as well as the corresponding formulae for these structures are indicated.

Finally, in Section 4, we particularize the general results for homogeneous 3- and 4-symmetric spaces. The relation of canonical $f(5,1)$-structures with the classical canonical structures on these spaces is also included. As an accessory observation, we can conclude that the assertion about the rank of an $f(5,1)$-structure in [**8**] (and, certainly, for more general situation [**4**]) is wrong.

2. Canonical structures of classical type on regular Φ-spaces

We here formulate in a brief form some basic definitions and results related to regular Φ-spaces and canonical affinor structures of classical type on them. More detailed information can be found in [**3**], [**16**], [**11**], [**6**], [**14**], [**15**].

Let G be a connected Lie group, Φ its (analytic) automorphism. Denote by G^Φ the subgroup of all fixed points of Φ and G^Φ_o the identity component of G^Φ. Suppose a closed subgroup H of G satisfies the condition

$$G^\Phi_o \subset H \subset G^\Phi.$$

Then G/H is called a *homogeneous Φ-space.*

Homogeneous Φ-spaces include homogeneous symmetric spaces ($\Phi^2 = id$) and, more general, *homogeneous Φ-spaces of order k* ($\Phi^k = id$) or, in other terminology, *homogeneous k-symmetric spaces* (see [**11**]). Note that there exist homogeneous Φ-spaces, which are not reductive. That is why so-called regular Φ-spaces first introduced by N.A.Stepanov [**14**] are of fundamental importance.

Let G/H be a homogeneous Φ-space, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras for G and H, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$. Consider the linear operator $A = \varphi - id$ and the Fitting decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with respect to A, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ denote $0-$ and 1-component of the decomposition respectively. It is clear that $\mathfrak{h} = Ker\ A$ and $\mathfrak{h} \subset \mathfrak{g}_0$.

Definition 1 ([**14**], [**3**], [**6**]) . A homogeneous Φ-space G/H is called a *regular Φ-space* if one of the following equivalent conditions is satisfied:

(1) $\mathfrak{h} = \mathfrak{g}_0$;
(2) $\mathfrak{g} = \mathfrak{h} \oplus A\mathfrak{g}$;
(3) The restriction of the operator A to $A\mathfrak{g}$ is non-singular;
(4) $A^2X = 0 \Longrightarrow AX = 0$ for all $X \in \mathfrak{g}$.
(5) The matrix of automorphism φ can be represented in the form $\begin{pmatrix} E & 0 \\ 0 & B \end{pmatrix}$, where the matrix B does not admit the eigenvalue 1.

We recall two basic facts:

THEOREM 1. ([**14**]).

- *Any homogeneous Φ-space of order k ($\Phi^k = id$) is a regular Φ-space.*
- *Any regular Φ-space is reductive. More exactly, the Fitting decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},\ \ \mathfrak{m} = A\mathfrak{g} \tag{1}$$

is a reductive one.

Decomposition (1) is the *canonical reductive decomposition* corresponding to a regular Φ-space G/H, and $\mathfrak{m}$ is the *canonical reductive complement.*

Decomposition (1) is obviously φ-invariant. Denote by θ the restriction of φ to $\mathfrak{m}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$. It is important to note that the operator θ commutes with every element of the linear isotropy group $Ad(H)$ (see [**14**]).

An *affinor structure* on a manifold is known to be a tensor field of type $(1,1)$. Suppose F is an invariant affinor structure on a homogeneous manifold G/H. Then F is completely determined by its value F_o at the point o, where F_o is invariant with respect to $Ad(H)$. For simplicity, we will denote by the same manner both any invariant structure on G/H and its value at o throughout the rest of the paper.

Definition 2 ([**3**]) . An invariant affinor structure F on a regular Φ-space G/H is called *canonical* if its value at the point $o = H$ is a polynomial in θ.

Denote by $\mathcal{A}(\theta)$ the set of all canonical affinor structures on a regular Φ-space G/H. It is easy to see that $\mathcal{A}(\theta)$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on G/H. Moreover,

$$dim\ \mathcal{A}(\theta) = deg\ \nu\ \leqslant\ dim\ G/H,$$

where ν is the minimal polynomial of the operator θ. It is easy to see that the algebra $\mathcal{A}(\theta)$ for any symmetric Φ-space ($\Phi^2 = id$) consists of scalar structures only, i.e. it is isomorphic to $\mathbb{R}$. However, this algebra is much more essential for homogeneous k-symmetric spaces ($k \geqslant 3$). We note that the algebra $\mathcal{A}(\theta)$ for these spaces was completely described in [**2**].

It should be mentioned that all canonical structures are invariant with respect to the "symmetries" of G/H, which are generated by the automorphism Φ (see [**14**]).

The most remarkable example of canonical structures is the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ on a homogeneous 3-symmetric space (see [**15**], [**16**], [**7**]). It turns out that it is not an exception. In other words, the algebra $\mathcal{A}(\theta)$ contains many affinor structures of classical type.

In this Section, we will concentrate on the following affinor structures of classical type:

almost complex structures J ($J^2 = -id$);
almost product structures P ($P^2 = id$);
f*-structures* ($f^3 + f = 0$) [**17**];
f-structures of hyperbolic type or, briefly, h*-structures* ($h^3 - h = 0$) [**10**].

Clearly, f-structures and h-structures are generalizations of structures J and P respectively.

All the canonical structures of classical type on regular Φ-spaces were completely described [**1**],[**3**]. In particular, for homogeneous k-symmetric spaces, precise computational formulae were indicated. For future reference we select here some results.

Denote by $\tilde{s}$ (respectively, s) the number of all irreducible factors (respectively, all irreducible quadratic factors) over $\mathbb{R}$ of a minimal polynomial ν.

THEOREM 2. *([**3**],[**1**]) Let G/H be a regular Φ-space.*

(1) The algebra $\mathcal{A}(\theta)$ contains precisely $2^{\tilde{s}}$ structures P.
(2) G/H admits a canonical structure J if and only if $s = \tilde{s}$. In this case $\mathcal{A}(\theta)$ contains 2^s different structures J.
(3) G/H admits a canonical f-structure if and only if $s \neq 0$. In this case $\mathcal{A}(\theta)$ contains $3^s - 1$ different f-structures. Suppose $s = \tilde{s}$. Then 2^s f-structures are almost complex and the remaining $3^s - 2^s - 1$ have non-trivial kernels.

(4) *The algebra* $\mathcal{A}(\theta)$ *contains* $3^{\tilde{s}}$ *different* h*-structures. All these structures form a (commutative) semigroup in* $\mathcal{A}(\theta)$ *and include a subgroup of order* $2^{\tilde{s}}$ *of canonical structures* P.

Further, let G/H be a homogeneous k-symmetric space. Then $\tilde{s} = s+1$ if $-1 \in spec\,\theta$, and $\tilde{s} = s$ in the opposite case. We indicate explicit formulae enabling us to compute all canonical f-structures and h-structures. We shall also use the notation

$$u = \begin{cases} n & \text{if} \quad k = 2n+1 \\ n-1 & \text{if} \quad k = 2n \end{cases}.$$

THEOREM 3. *([**3**],[**1**]) Let* G/H *be a homogeneous* Φ*-space of order* k.

(1) *All non-trivial canonical* f*-structures on* G/H *can be given by the operators*

$$f = \frac{2}{k}\sum_{m=1}^{u}\left(\sum_{j=1}^{u}\zeta_j \sin\frac{2\pi mj}{k}\right)\left(\theta^m - \theta^{k-m}\right),$$

where $\zeta_j \in \{-1;0;1\}$, $j = 1,2,\ldots,u$, *and not all coefficients* ζ_j *are zero. Moreover, the polynomials* f *define canonical structures* J *iff all* $\zeta_j \in \{-1;1\}$.

(2) *All canonical* h*-structures on* G/H *can be given by the polynomials* $h = \sum\limits_{m=0}^{k-1} a_m\theta^m$, *where:*

(a) *if* $k = 2n+1$, *then*

$$a_m = a_{k-m} = \frac{2}{k}\sum_{j=1}^{u}\xi_j \cos\frac{2\pi mj}{k};$$

(b) *if* $k = 2n$, *then*

$$a_m = a_{k-m} = \frac{1}{k}\left(2\sum_{j=1}^{u}\xi_j \cos\frac{2\pi mj}{k} + (-1)^m\xi_n\right)$$

Here the numbers ξ_j, $j = 1,2,\ldots,u$, *take their values from the set* $\{-1;0;1\}$ *and the polynomials* h *define canonical structures* P *iff all* $\xi_j \in \{-1;1\}$.

We now particularize the results for homogeneous Φ-spaces of orders 3 and 4 only for future reference.

COROLLARY 1. *([**3**],[**1**]) Let* G/H *be a homogeneous* Φ*-space of order* 3. *There are (up to sign) only the following canonical structures of classical type on* G/H*:*

$$J = \frac{1}{\sqrt{3}}(\theta - \theta^2),\ P = id.$$

We note that the existence of the structure J and its properties are well known (see [**15**],[**16**],[**7**]).

COROLLARY 2. *([**3**],[**1**]) On a homogeneous* Φ*-space of order* 4 *there are (up to sign) the following canonical classical structures:*

$$P = \theta^2,\ f = \frac{1}{2}(\theta - \theta^3),\ h_1 = \frac{1}{2}(id - \theta^2),\ h_2 = \frac{1}{2}(id + \theta^2).$$

The operators h_1 *and* h_2 *form a pair of complementary projectors:* $h_1 + h_2 = id$, $h_1^2 = h_1$, $h_2^2 = h_2$. *Moreover, the following conditions are equivalent:*

(1) $-1 \notin spec\,\theta$;
(2) the structure P *is trivial* ($P = -id$);
(3) the f*-structure is an almost complex structure;*
(4) the structure h_1 *is trivial* ($h_1 = id$);
(5) the structure h_2 *is null.*

We note that the procedure of describing all canonical structures on homogeneous Φ-spaces of finite order is constructive (see [**3**], §4 and §5). It should be mentioned that in particular case of homogeneous Φ-spaces of any odd order $k = 2n + 1$ the method of constructing invariant almost complex structures was described in [**11**]. It can be easily shown that all these structures are canonical in the above sense.

3. The description of canonical $f(5,1)$-structures for homogeneous k-symmetric spaces

We recall that an affinor structure F on a smooth manifold M is called $f(5,1)$-*structure* if it satisfies the condition $F^5 + F = 0,\ F \neq 0$ (see [**8**]).

Here we explicitly characterize all canonical $f(5,1)$-structures for arbitrary homogeneous k-symmetric spaces.

We also recall the notation which is convenient for the presentation that follows:

$$u = \begin{cases} n & \text{if} \quad k = 2n+1 \\ n-1 & \text{if} \quad k = 2n \end{cases} .$$

THEOREM 4. *Let* G/H *be a homogeneous* k*-symmetric space* ($k \geqslant 3$), s *the number of pairs of conjugate* k*th roots of unity, which are included into the spectrum* $spec\,\theta$ *(apart from* -1*, which can also be an eigenvalue). Then* G/H *admits* $5^s - 1$ *non-trivial canonical* $f(5,1)$*-structures. Moreover, all these* $f(5,1)$*-structures can be given by the operators* $F(\theta) = \sum\limits_{m=0}^{k-1} a_m \theta^m$, *where*

$$a_m = \frac{\sqrt{2}}{k} \sum_{j=1}^{u} (\xi_j \cos\frac{2\pi m j}{k} + \eta_j \sin\frac{2\pi m j}{k}).$$

Here $(\xi_j, \eta_j) \in \{(1,1); (1,-1); (-1,1); (-1,-1); (0,0)\}$, $j = 1, 2, \ldots, u$, *and not all pairs* (ξ_j, η_j) *are trivial.*

PROOF. To prove the theorem we modify for this case the general approach from the paper ([**3**], §4 and §5).

Any canonical structure F can be represented as follows:

$$F(\theta) = \sum_{m=0}^{k-1} a_m \theta^m, \quad a_m \in \mathbb{R}. \tag{2}$$

From the identity $\theta^{k-1} + \theta^{k-2} + \ldots + \theta + id = 0$ it follows, that without loss of generality we can impose the following "normalizing" condition on the coefficients of the polynomial F:

$$a_0 + a_1 + \ldots + a_{k-1} = 0. \tag{3}$$

In the case under consideration we can reduce the matrix $[\theta]$ of the endomorphism θ over $\mathbb{C}$ to the following form

$$[\theta] \sim diag\{\varepsilon E_1, \varepsilon^2 E_2, \ldots, \varepsilon^{k-1} E_{k-1}\},$$

where $\varepsilon = \cos\frac{2\pi}{k} + i\sin\frac{2\pi}{k}$ and the orders of the identity matrices E_j and E_{k-j}, $j = 1, 2, \ldots, k-1$ are the same (in particular, they can be of zero order). Hence the matrix $[F(\theta)]$ can be reduced over $\mathbb{C}$ to the form

$$diag\{(\sum_{m=0}^{k-1} a_m\varepsilon^m)E_1, (\sum_{m=0}^{k-1} a_m\varepsilon^{2m})E_2, \ldots, (\sum_{m=0}^{k-1} a_m\varepsilon^{(k-1)m})E_{k-1}\}. \tag{4}$$

It follows from (4) that the defining condition $F(\theta)^5 + F(\theta) = 0$ for a canonical $f(5,1)$-structure is satisfied if and only if

$$[F(\theta)] \sim diag\{\mu_1E_1, \mu_2E_2, \ldots, \mu_{k-1}E_{k-1}\}, \tag{5}$$

where the numbers μ_j, $j = 1, 2, \ldots, k-1$, take their values from the set $\{\pm z; \pm\bar{z}; 0\}$; here $z = \frac{1}{\sqrt{2}}(1+i)$ (i is the imaginary unit). Besides, the condition $F(\theta) \neq 0$ implies that all the collections $(\mu_1, \mu_2, \ldots, \mu_{k-1})$ include non-zero elements.

Thus, taking into account (4) and (5), the problem of determining the polynomials F reduces to finding real solutions of the emerging systems of linear equations with complex coefficients with respect to the unknowns $a_0, a_1, \ldots, a_{k-1}$. We solve this problem in two steps.

Step 1 *(the case of maximal spectrum).* Suppose that the spectrum *spec* θ is maximal, i.e. *spec* $\theta = \{\varepsilon, \varepsilon^2, \ldots, \varepsilon^{k-1}\}$. Then, in view of (3), we arrive at the following systems of linear equations determining a_m, $m = 0, 1, \ldots, k-1$:

$$\left\{\begin{array}{llllllll} a_0 & +a_1 & +a_2+ & \ldots+ & a_{k-1} & = & 0, \\ a_0 & +a_1\varepsilon & +a_2\varepsilon^2+ & \ldots+ & a_{k-1}\varepsilon^{k-1} & = & \mu_1, \\ a_0 & +a_1\varepsilon^2 & +a_2\varepsilon^4+ & \ldots+ & a_{k-1}\varepsilon^{2(k-1)} & = & \mu_2, \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_0 & +a_1\varepsilon^{k-1} & +a_2\varepsilon^{2(k-1)}+ & \ldots+ & a_{k-1}\varepsilon^{(k-1)^2} & = & \mu_{k-1}, \end{array}\right. \tag{6}$$

The matrix of these systems is the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \ldots & \varepsilon^{k-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \ldots & \varepsilon^{2(k-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \varepsilon^{k-1} & \varepsilon^{2(k-1)} & \ldots & \varepsilon^{(k-1)^2} \end{pmatrix}.$$

Introducing the notation $\bar{a} = (a_0, a_1, \ldots, a_{k-1})$ and $\bar{\mu} = (\mu_0, \mu_1, \ldots, \mu_{k-1})$, we can write the system (6) as follows

$$\bar{a}\, V = \bar{\mu}. \tag{7}$$

Since the matrix V is non-singular, each of the systems (7) has a unique solution

$$\bar{a} = \bar{\mu}\, V^{-1}, \tag{8}$$

where we can explicitly write down V^{-1}:

$$V^{-1} = \frac{1}{k}\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \varepsilon^{-1} & \varepsilon^{-2} & \ldots & \varepsilon^{-(k-1)} \\ 1 & \varepsilon^{-2} & \varepsilon^{-4} & \ldots & \varepsilon^{-2(k-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \varepsilon^{-(k-1)} & \varepsilon^{-2(k-1)} & \ldots & \varepsilon^{-(k-1)^2} \end{pmatrix} := \frac{1}{k}\begin{pmatrix} 1 & 1 \\ 1 & \mathcal{E} \end{pmatrix}.$$

Hence we have reduced the problem to that of finding collections $\bar{a}$ with real elements $a_0, a_1, \ldots, a_{k-1}$ among $5^{k-1}-1$ collections, whose elements are in general complex numbers.

Using some properties of the matrix $\mathcal{E}$ (see [**3**], §4), we can conveniently write it as follows:

$$\mathcal{E} = \begin{pmatrix} \varepsilon^{k-1} & \varepsilon^{k-2} & \ldots & \varepsilon^{k-m} & \ldots & \varepsilon^2 & \varepsilon \\ \varepsilon^{k-2} & \varepsilon^{k-4} & \ldots & \varepsilon^{k-2m} & \ldots & \varepsilon^4 & \varepsilon^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon^2 & \varepsilon^4 & \ldots & \varepsilon^{2m} & \ldots & \varepsilon^{k-4} & \varepsilon^{k-2} \\ \varepsilon & \varepsilon^2 & \ldots & \varepsilon^m & \ldots & \varepsilon^{k-2} & \varepsilon^{k-1} \end{pmatrix}. \tag{9}$$

Therefore, using the procedure (8) for finding the collection $\bar{a}$ and in view of the form (9) of $\mathcal{E}$, we can obtain by a direct calculation

$$a_m = \frac{1}{k}(\mu_1\varepsilon^{k-m} + \mu_2\varepsilon^{k-2m} + \ldots + \mu_{k-2}\varepsilon^{2m} + \mu_{k-1}\varepsilon^m), \tag{10}$$

where $m = 0, 1, \ldots, k-1$.

An elementary analysis shows that a sufficient condition for all numbers a_m to be real is that all the pairs (μ_j, μ_{k-j}), $j = 1, 2, \ldots, k-1$, are those of complex conjugate numbers:

$$\mu_{k-j} = \bar{\mu}_j, \quad j = 1, 2, \ldots, k-1. \tag{11}$$

We can also show that condition (11) is necessary.

The form of such collections $\bar{\mu}$ satisfying condition (11) depends on the parity of k; namely,

$$\begin{aligned} &\text{if } k = 2n+1, \text{then } \bar{\mu} = (0, \mu_1, \ldots, \mu_n, \bar{\mu}_n, \ldots, \bar{\mu}_1); \\ &\text{if } k = 2n, \text{then } \bar{\mu} = (0, \mu_1, \ldots, \mu_{n-1}, 0, \bar{\mu}_{n-1}, \ldots, \bar{\mu}_1), \end{aligned} \tag{12}$$

where, as before, $\mu_j \in \{\pm z, \pm\bar{z}, 0\}$, $j = 1, 2, \ldots, u$.

Varying all the collections $\bar{\mu}$ in (12), we obtain respectively all canonical $f(5,1)$-structures of different possible ranks. Using combinatorial arguments, the number N of distinct non-trivial canonical $f(5,1)$-structures can be exactly calculated:

$$N = \sum_{j=1}^{u} C_u^j 4^j = \sum_{j=0}^{u} C_u^j 4^j - 1 = 5^u - 1.$$

We also note that in this case $u = s$, therefore there is a unique way to represent the solution $F(\theta)$ by polynomials F satisfying condition (3).

We are also able to calculate the coefficients of the polynomials $F(\theta)$. For the purpose, one can detail formula (10) for the collections $\bar{\mu}$ from (12):

$$a_m = \frac{1}{k}\sum_{j=1}^{u}(\mu_j\varepsilon^{-jm} + \bar{\mu}_j\varepsilon^{jm}), \quad m = 0, 1, \ldots, k-1.$$

Further, all the numbers μ_j belonging to the set $\{\pm z, \pm\bar{z}, 0\}$ can be written in the form $\mu_j = \frac{1}{\sqrt{2}}(\xi_j + i\eta_j)$, where (ξ_j, η_j) take their values from the set

$$\{(1,1); (1,-1); (-1,1); (-1,-1); (0,0)\}, \ j = 1, 2, \ldots, u.$$

Thus we get

$$\mu_j \varepsilon^{-jm} + \bar{\mu}_j \varepsilon^{jm} = \sqrt{2}(\xi_j \cos \frac{2\pi mj}{k} + \eta_j \sin \frac{2\pi mj}{k}).$$

Finally, we obtain

$$a_m = \frac{\sqrt{2}}{k} \sum_{j=1}^{u} (\xi_j \cos \frac{2\pi mj}{k} + \eta_j \sin \frac{2\pi mj}{k}) \tag{13}$$

for all $m = 0, 1, \ldots, k-1$.

This completes the proof of the theorem in the case of maximal spectrum.

Step 2 *(the case of arbitrary spectrum).* Assume now that the spectrum of the operator θ contains s pairs of conjugate kth roots of unity (apart from -1), where $s \leqslant [\frac{k}{2}]$. Then the number of points in the spectrum of θ is $2s$ if -1 is outside the spectrum and it is $2s+1$ if -1 is in the spectrum (the latter situation is possible only for $k = 2n$). Thus, the matrix $[\theta]$ of θ can be reduced over $\mathbb{C}$ to one of the respective forms:

(a) $[\theta] \sim diag\{\varepsilon^{b_1} E_1, \ldots, \varepsilon^{b_s} E_s, \varepsilon^{-b_s} E_s \ldots, \varepsilon^{-b_1} E_1\}$ or
(b) $[\theta] \sim diag\{\varepsilon^{b_1} E_1, \ldots, \varepsilon^{b_s} E_s, -E_{s+1}, \varepsilon^{-b_s} E_s \ldots, \varepsilon^{-b_1} E_1\}$.

Here $\varepsilon = \cos \frac{2\pi}{k} + i \sin \frac{2\pi}{k}$ is a primitive kth root of unity, E_j are identity matrices of the appropriate orders, and positive integers b_j satisfy the condition $1 \leqslant b_1 < b_2 < \ldots < b_s \leqslant [\frac{k}{2}]$.

As before, we shall consider the polynomials F having the form (2) and satisfying the additional condition (3). Imposing the condition $F(\theta)^5 + F(\theta) = 0$ and reasoning in a similar way we arrive at the systems of linear equations

$$\bar{a}\, \tilde{V} = \tilde{\mu}. \tag{14}$$

Here the matrix $\tilde{V}$ is rectangular and includes the corresponding columns of the matrix V, and the collection $\tilde{\mu}$ has the form $\tilde{\mu} = (0, \mu_1, \mu_2, \ldots, \mu_l)$, where the value of l is $2s$ or $2s+1$ in cases (a) and (b), respectively, and the μ_j, as before, belong to the set $\{\pm z; \pm \bar{z}; 0\}$, where $z = \frac{1}{\sqrt{2}}(1+i)$.

It is not difficult to see that there exist real solutions $\bar{a}$ of (14) provided that the collection $\tilde{\mu}$ has the following form, respectively:

(a) $\tilde{\mu} = (0, \mu_1, \ldots, \mu_s, \bar{\mu}_s, \ldots, \bar{\mu}_1)$ or
(b) $\tilde{\mu} = (0, \mu_1, \ldots, \mu_s, 0, \bar{\mu}_s, \ldots, \bar{\mu}_1)$.

Indeed, if $\tilde{\mu}$ is as above, then it suffices to "extend" system (14) to a square system (7) by considering collection $\bar{\mu}$ of the form (12) "completing" the fixed collection $\tilde{\mu}$. Then solving the system (7), we obtain, as was shown above, a real collection $\bar{a}$ satisfying also the system (14). We note that there can be various ways to complete the collection $\tilde{\mu}$ to a corresponding collection $\bar{\mu}$, therefore the representation of linear operators $F(\theta)$ in terms of the polynomials F is not unique.

At the same time, using this method we obtain all distinct solutions $F(\theta)$ of the equation $F^5 + F = 0$ in the algebra $\mathcal{A}(\theta)$. The number N of these non-trivial solutions is equal to that of distinct collections in cases (a) or (b) above. It means that

$$N = \sum_{j=1}^{s} C_s^j 4^j = \sum_{j=0}^{s} C_s^j 4^j - 1 = 5^s - 1.$$

Thus, we have reduced the case of operator θ with arbitrary spectrum to that of maximal spectrum. Consequently, we can again find the coefficients of the polynomials $F(\theta)$ by formula (13) used in the first step of our arguments. Hence Theorem 4 is completely proved. $\square$

REMARK 1. *The above procedure for finding the polynomials F highlights the factors essential for the coincidence of the corresponding linear operators $F(\theta)$. Namely, if distinct polynomials F_1 and F_2 are determined by distinct collections $\bar{\mu}_1$ and $\bar{\mu}_2$ of the form (12), then $F_1(\theta) = F_2(\theta)$ if and only if $\bar{\mu}_1$ and $\bar{\mu}_2$ are generated by the same collection $\tilde{\mu}$ of the form (a) or (b) above. In other words, the polynomials F_1 and F_2 must coincide by modulo ν: $F_1 - F_2 \equiv 0 (mod\, \nu)$.*

REMARK 2. *The assertions about the number N of all canonical $f(5,1)$-structures on a homogeneous k-symmetric space as well as the formulae for these structures can be obtained in the essentially different way. More exactly, it can be deduced from the precise description for the algebraic structure of the algebra $\mathcal{A}(\theta)$ for arbitrary homogeneous k-symmetric space (see* **[2]***) and the formulae for canonical f-structures and h-structures in the case (see Theorem 3).*

4. Canonical $f(5,1)$-structures on 3- and 4-symmetric spaces

In this Section, we particularize the results just obtained for homogeneous k-symmetric spaces of smaller orders ($k = 3$ and $k = 4$). It should be noted that there are no fundamental obstructions to considering cases of higher orders k.

4.1. Homogeneous 3-symmetric spaces. In this case, the spectrum of the operator θ consists of two elements, i.e. it is maximal: $spec\ \theta = (\varepsilon, \varepsilon^2)$, where $\varepsilon = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$. Clearly, $s = 1$, and it follows from Theorem 4 that there are 4 canonical $f(5,1)$-structures on G/H. For finding the corresponding formulae we make use of this theorem.

We have $u = 1$. It is evident that, up to sign, there are only two possibilities:

1) $(\xi_1, \eta_1) = (1, 1)$. By Theorem 4 the polynomial representing this $f(5,1)$-structure is

$F_1(\theta) = \frac{\sqrt{2}}{3}(id + \frac{\sqrt{3}-1}{2}\theta - \frac{\sqrt{3}+1}{2}\theta^2)$.

In view of the identity $\theta^2 + \theta + id = 0$ and the formula for the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ (see Corollary 1), we can get

$F_1(\theta) = \frac{1}{\sqrt{6}}(\sqrt{3}\, id + \theta - \theta^2) = \frac{1}{\sqrt{2}}(id + J)$.

2) $(\xi_1, \eta_1) = (1, -1)$. In this case, using the same arguments we obtain

$F_2(\theta) = \frac{1}{\sqrt{6}}(\sqrt{3}\, id - \theta + \theta^2) = \frac{1}{\sqrt{2}}(id - J)$.

Besides, by direct calculations we can establish the following relations:

$F_1(\theta)^2 = -F_2(\theta)^2 = J$; $\quad F_1(\theta)\, F_2(\theta) = id$;

$F_1(\theta) + F_2(\theta) = \sqrt{2}\, id$; $\quad F_1(\theta) - F_2(\theta) = \sqrt{2}\, J$.

In particular, it follows that

$rank\, F_1(\theta) = rank\, F_2(\theta) = rank\, J = dim\, G/H$.

To sum up, we obtain the following result.

THEOREM 5. *Any homogeneous 3-symmetric space G/H admits (up to sign) two canonical $f(5,1)$-structures F_1 and F_2 which can be represented in the form*

$$F_1(\theta) = \frac{1}{\sqrt{6}}(\sqrt{3}\, id + \theta - \theta^2) = \frac{1}{\sqrt{2}}(id + J) = \frac{1}{\sqrt{2}}(P + J);$$

$$F_2(\theta) = \frac{1}{\sqrt{6}}(\sqrt{3}\,id - \theta + \theta^2) = \frac{1}{\sqrt{2}}(id - J) = \frac{1}{\sqrt{2}}(P - J).$$

Here J and P are the canonical structures on G/H (see Corollary 1). In addition, the following relations are satisfied:
$F_1^2 = -F_2^2 = J;\quad F_1\,F_2 = id;\quad F_1 + F_2 = \sqrt{2}\,id;\quad F_1 - F_2 = \sqrt{2}\,J\,.$
In particular, $rank\ F_1 = rank\ F_2 = rank\ J = dim\ G/H$.

Remark 3. *It is well known* [**17**] *that the rank r of any f-structure is even: $r = 2m$. On "analogy" it was "proved" in* [**8**] *that for any $f(5,1)$-structure $rank\ f(5,1) = 4m$. However, Theorem 5 implies, in particular, that such an assertion is wrong. Hence, more general "result" (see* [**4**]*) is also erroneous.*

Indeed, by Theorem 5 the rank of the canonical $f(5,1)$-structures on any homogeneous 3-symmetric space coincides with its dimension. There exist many homogeneous 3-symmetric spaces whose dimensions are not divisible by 4 (see, for example, [**16**], [**7**], [**11**]). For instance, the spheres $S^2 = SO(3)/SO(2)$ and $S^6 = G_2/SU(3)$ can be considered as homogeneous 3-symmetric spaces (see [**16**], [**7**], [**11**], [**13**]).

No doubt, there are counterexamples of a very simple geometrical character. Specifically, on the plane $M = \mathbb{R}^2$ we put F to be a right (or left) rotation to the angle $\frac{\pi}{4}$ of tangent vectors at any point. Then $F^4 = -id$, i.e. $F^5 + F = 0$, and $rank\,F = 2$.

4.2. Homogeneous 4-symmetric spaces. For the spectrum of θ there are the following possibilities: $spec\,\theta = \{\varepsilon, \varepsilon^2, \varepsilon^3\} = \{i, -1, -i\}$ or $spec\,\theta = \{\varepsilon, \varepsilon^3\} = \{i, -i\}$. In both cases we have $u = s = 1$. By Theorem 4, the search for non-trivial canonical $f(5,1)$-structures brings us to two (up to sign) pairs (ξ_1, η_1). Consider these pairs:

1) $(\xi_1, \eta_1) = (1, 1)$. Using Theorem 4, we get
$F_1(\theta) = \frac{\sqrt{2}}{4}(id + \theta - \theta^2 - \theta^3)$.

Taking into account the identity $\theta^3 + \theta^2 + \theta + id = 0$ and the formulae for the canonical structures $f = \frac{1}{2}(\theta - \theta^3)$ and $h_1 = \frac{1}{2}(id - \theta^2)$ for a homogeneous 4-symmetric space (see Corollary 2) , we obtain
$F_1(\theta) = \frac{1}{\sqrt{2}}(id + \theta) = \frac{1}{\sqrt{2}}(h_1 + f)$.

2) $(\xi_1, \eta_1) = (1, -1)$. In this case, using the same arguments we obtain
$F_2(\theta) = \frac{\sqrt{2}}{4}(id - \theta - \theta^2 + \theta^3) = \frac{1}{\sqrt{2}}(id + \theta^3) = \frac{1}{\sqrt{2}}(h_1 - f)$.

By direct calculations, we can get the following relations for these canonical $f(5,1)$-structures:
$F_1(\theta)^2 = -F_2(\theta)^2 = f;\ \ F_1(\theta)\,F_2(\theta) = -\theta\, f;$
$F_1(\theta) + F_2(\theta) = \sqrt{2}\,h_1;\ \ F_1(\theta) - F_2(\theta) = \sqrt{2}\,f.$

It implies, in particular, that $rank\ F_1 = rank\ F_2 = rank\ f$. Besides, both $f(5,1)$-structures are non-degenerate if and only if $-1 \notin spec\,\theta$.

These facts can be summarized as follows.

Theorem 6. *Any homogeneous 4-symmetric space G/H admits (up to sign) two canonical $f(5,1)$-structures F_1 and F_2. They can be represented in the form*

$$F_1(\theta) = \frac{1}{\sqrt{2}}(id + \theta) = \frac{1}{\sqrt{2}}(h_1 + f),$$

$$F_2(\theta) = \frac{1}{\sqrt{2}}(id + \theta^3) = \frac{1}{\sqrt{2}}(h_1 - f),$$

where f and h_1 are the canonical structures on G/H (see Corollary 2). In addition, the following relations are satisfied:

$$F_1^2 = -F_2^2 = f; \quad F_1 F_2 = -\theta f; \quad F_1 + F_2 = \sqrt{2} h_1; \quad F_1 - F_2 = \sqrt{2} f.$$

In particular, rank F_1 = rank F_2 = rank f. Besides, both $f(5,1)$-structures F_1 and F_2 are non-degenerate if and only if $-1 \notin$ spec θ.

References

[1] V.V.Balashchenko, Canonical f-structures of hyperbolic type on regular Φ-spaces, *Uspekhi Mat. Nauk*, 1998, V.53, no.4, 213-214; English transl. in *Russian Math. Surveys*, 1998, V.53, no.4, 861-863.

[2] V.V.Balashchenko, The algebra of canonical affinor structures on homogeneous k-symmetric spaces, *Differential Geometry and Its Applications*, Proc. Conf., Opava (Czech Republic), August 27-31, 2001, Silesian University, Opava, 2001, 3-13.

[3] V.V. Balashchenko and N.A. Stepanov, Canonical affinor structures of classical type on regular Φ-spaces, *Matematicheskii Sbornik*, 1995, V.186, no.11, 3-34; English transl. in *Sbornik: Mathematics*, 1995, V.186, no.11, 1551-1580.

[4] D. Demetropoulou-Psomopoulou and F. Gouli-Andreou, On necessary and sufficient conditions for an n-dimensional manifold to admit a tensor field $f \neq 0$ of type $(1,1)$ satisfying $f^{2\nu+3}+f=0$, *Tensor*, 1985, V.42, 245-251.

[5] D. Demetropoulou-Psomopoulou, On integrability conditions of a structure f satisfying $f^{2\nu+3}+f=0$, *Tensor*, 1985, V.42, 252-257.

[6] A.S. Fedenko, *Spaces with symmetries*, Belarusian State University, Minsk, 1977 (Russian).

[7] A. Gray, Riemannian manifolds with geodesic symmetries of order 3, *J. Diff. Geom.*, 1972, V.7, no.3-4, 343-369.

[8] F. Gouli-Andreou, On a structure defined by a tensor field f satisfying $f^5+f=0$, *Tensor*, 1982, V.36, 79-84, 180-184.

[9] F. Gouli-Andreou, On integrability conditions of a structure f satisfying $f^5+f=0$, *Tensor*, 1983, V.40, 27-31.

[10] V.F. Kirichenko, Methods of generalized Hermitian geometry in the theory of almost contact manifolds, (Russian) *Itogi Nauki i Tekhniki: Probl. Geom.* V.18, VINITI, Moscow, 1986, 25-71; English transl. in *J. Soviet Math.* 1988, V.42, no.5.

[11] O. Kowalski, *Generalized symmetric spaces*, LN in Math., V.805, Berlin, Heidelberg, New York: Springer-Verlag, 1980.

[12] A.L. Petrakis, Structures hermitiennes subordonnees a une structure f qui satisfait $f^{2\nu+3}+f=0$, *Tensor*, 1984, V.41, 119-127.

[13] C.U. Sanchez, Regular s-structure on spheres, *Indiana Univ. Math. J.*, 1988, V.37, no.1, 165-180.

[14] N.A. Stepanov, Basic facts of the theory of φ-spaces, (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.*, 1967, no.3, 88-95; English transl. in *Soviet Math. (Iz. VUZ)*, 1967, V.11, no.3.

[15] N.A. Stepanov, Homogeneous 3-cyclic spaces, (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.*, 1967, no.12, 65-74; English transl. in *Soviet Math. (Iz. VUZ)*, 1967, V.11, no.12.

[16] J.A. Wolf and A. Gray, Homogeneous spaces defined by Lie group automorphisms, *J. Diff. Geom.*, 1968, V.2, no.1-2, 77-159.

[17] K. Yano, On a structure defined by a tensor field f of type $(1,1)$ satisfying $f^3+f=0$, *Tensor*, 1963, V.14, 99-109.

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