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The gravitational field of the Gödel spacetime

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Abstract. The purpose of this paper is to study a gravitational field of the Gödel spacetime. The type of the gravitational field is determined by the characteristic of λ-tensor.

In 1949, the mathematician Kurt Gödel published a solution of Einstein's equation for a dust matter model with a cosmological constant. That solution is completly homogeneous and has the property that there is a closed timelike curve through each point.

Definition. Let $x = (x^0, x^1, x^2, x^3)$ be standard coordinats of $\mathbb{R}^4$ and $\omega \in \mathbb{R}^+/\{0\}$. The spacetime

$$\left(\mathbb{R}^4, -dx^0dx^0 + dx^1dx^1 - \frac{1}{2}e^{2\sqrt{2}\omega x^1}dx^2dx^2 + dx^3dx^3 - 2e^{\sqrt{2}\omega x^1}dx^0dx^2\right)$$

is called the Gödel spacetime ([2]).

The metric tensor of signature $(1,3)$ has the form

$$g_{ab}(x) = \begin{pmatrix} -1 & 0 & -e^{\sqrt{2}\omega x^1} & 0 \\ 0 & 1 & 0 & 0 \\ -e^{\sqrt{2}\omega x^1} & 0 & -\frac{1}{2}e^{2\sqrt{2}\omega x^1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $a, b = 0, 1, 2, 3$. Since the metric tensor depends on only one variable this is a simple task to calculate the Christoffel symbols. Here we note that the value of the Christoffel symbols are

$$\begin{aligned}
\Gamma^0_{01} &= \Gamma^0_{10} = \sqrt{2}\omega, \\
\Gamma^2_{01} &= \Gamma^2_{10} = -\sqrt{2}\omega e^{-\sqrt{2}\omega x^1}, \\
\Gamma^1_{02} &= \Gamma^1_{20} = \frac{1}{\sqrt{2}}\omega e^{\sqrt{2}\omega x^1}, \\
\Gamma^0_{12} &= \Gamma^0_{21} = \frac{1}{\sqrt{2}}\omega e^{\sqrt{2}\omega x^1}, \\
\Gamma^1_{22} &= \frac{1}{\sqrt{2}}\omega e^{2\sqrt{2}\omega x^1},
\end{aligned}$$

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where it is understood that all other Christoffel symbols vanish. Thus components, different from zero, of the Riemann tensor R^c_{abd} are

$$
\begin{aligned}
R^0_{101}(x) &= \omega^2,\\
R^0_{110}(x) &= -\omega^2,\\
R^1_{001}(x) &= \omega^2,\\
R^1_{010}(x) &= -\omega^2,\\
R^1_{021}(x) &= \omega^2 e^{\sqrt{2}\omega x^1},\\
R^1_{012}(x) &= -\omega^2 e^{\sqrt{2}\omega x^1},\\
R^1_{201}(x) &= \omega^2 e^{\sqrt{2}\omega x^1},\\
R^1_{210}(x) &= -\omega^2 e^{\sqrt{2}\omega x^1},\\
R^0_{121}(x) &= 2\omega^2 e^{\sqrt{2}\omega x^1},\\
R^0_{112}(x) &= -2\omega^2 e^{\sqrt{2}\omega x^1},\\
R^2_{112}(x) &= \omega^2,\\
R^2_{121}(x) &= -\omega^2,\\
R^1_{221}(x) &= \frac{3}{2}\omega^2 e^{2\sqrt{2}\omega x^1},\\
R^1_{212}(x) &= -\frac{3}{2}\omega^2 e^{2\sqrt{2}\omega x^1},\\
R^0_{020}(x) &= \omega^2 e^{\sqrt{2}\omega x^1},\\
R^0_{002}(x) &= -\omega^2 e^{\sqrt{2}\omega x^1},\\
R^0_{220}(x) &= \frac{1}{2}\omega^2 e^{2\sqrt{2}\omega x^1},\\
R^0_{202}(x) &= -\frac{1}{2}\omega^2 e^{2\sqrt{2}\omega x^1},\\
R^2_{002}(x) &= \omega^2,\\
R^2_{020}(x) &= -\omega^2,\\
R^2_{202}(x) &= \omega^2 e^{\sqrt{2}\omega x^1},\\
R^2_{220}(x) &= -\omega^2 e^{\sqrt{2}\omega x^1}.
\end{aligned}
$$

Hence and from the formula $R_{acbd} = g_{cl}R^l_{abd}$ we have components, different from zero, of the Riemann tensor R_{acbd}

$$
\begin{aligned}
&R_{1010}(x) = -R_{1001}(x) = -R_{0110}(x) = \omega^2,\\
&R_{1012}(x) = -R_{1021}(x) = -R_{0112}(x) = R_{1210}(x) = \omega^2 e^{\sqrt{2}\omega x^1},\\
&R_{1212}(x) = -R_{1221}(x) = -R_{2112}(x) = \frac{3}{2}\omega^2 e^{2\sqrt{2}\omega x^1},\\
&R_{2020}(x) = -R_{2002}(x) = -R_{0220}(x) = \frac{1}{2}\omega^2 e^{2\sqrt{2}\omega x^1}.
\end{aligned}
$$

Let us consider spacetime (M, g) as a four dimensional real manifold with a metic tensor of signature $(1, 3)$. Thus at each point $x \in M$ there exists a basic of the space

T_x^*M in which the metric tensor g has a diagonal form

$$\eta_{ab} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}.$$

It is convinient to introduce a 6-dimensional formalism in the pseudo-Eucliden space $\mathbb{R}^6$. The rule changing to the 6-dimensional formalism is the following (see [3]).

$$\begin{array}{rcccccc} ab: & 23 & 31 & 12 & 10 & 20 & 30 \\ A: & 1 & 2 & 3 & 4 & 5 & 6. \end{array}$$

Let us notice that the tensor

$$\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} = \eta_{abcd} \to \eta_{AB} = \begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ 0 & & & & & -1 \end{pmatrix}$$

has the signature $(3,3)$.

Now, we introduce the metric tensor defined by

$$g_{ac}g_{bd} - g_{ad}g_{bc} = g_{abcd} \to g_{AB},$$

where g_{ab} are components a metric tensor at an arbitrary point of the Gödel spacetime and collective indices are the skew-symmetric pairs $ab \to A$, $cd \to B$. The tensor g_{AB} $(A, B = 1, \dots, 6)$ is symmetric and non-singular.

Thus tensors g_{AB} and R_{AB} are

$$g_{AB}(x) = \begin{pmatrix} -\frac{1}{2}e^{2\sqrt{2}\omega x^1} & 0 & 0 & 0 & 0 & e^{\sqrt{2}\omega x^1} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}e^{2\sqrt{2}\omega x^1} & -e^{\sqrt{2}\omega x^1} & 0 & 0 \\ 0 & 0 & -e^{\sqrt{2}\omega x^1} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}e^{2\sqrt{2}\omega x^1} & 0 \\ e^{\sqrt{2}\omega x^1} & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$R_{AB}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2}\omega^2 e^{2\sqrt{2}\omega x^1} & \omega^2 e^{\sqrt{2}\omega x^1} & 0 & 0 \\ 0 & 0 & \omega^2 e^{\sqrt{2}\omega x^1} & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\omega^2 e^{2\sqrt{2}\omega x^1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, it only remains to find canonical form of the λ-tensor $R_{AB} - \lambda g_{AB}$. The canonical forms of symmetric λ-tensors have been given in paper ([1]).

In our case the determinant λ-matrix $R_{AB}(x) - \lambda g_{AB}(x)$ is equal to zero if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 0$ or $\lambda_4 = \lambda_5 = -\omega^2$ or $\lambda_6 = \omega^2$, so we have

$$g_{A'B'} = \begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ 0 & & & & & -1 \end{pmatrix},$$

$$R_{A'B'} = \begin{pmatrix} 0 & & & & & 0 \\ & 0 & & & & \\ & & 0 & & & \\ & & & \omega^2 & & \\ & & & & \omega^2 & \\ 0 & & & & & -\omega^2 \end{pmatrix}.$$

It is the type $G_1[(111)(11)1]$ in the Segre symbols (see[1]).

Let us still notice that the Ricci tensor has the form

$$R_{ab}(x) = \begin{pmatrix} -2\omega^2 & 0 & -2\omega^2 e^{\sqrt{2}\omega x^1} & 0 \\ 0 & 0 & 0 & 0 \\ -2\omega^2 e^{\sqrt{2}\omega x^1} & 0 & -2\omega^2 e^{2\sqrt{2}\omega x^1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the Ricci scalar is

$$R = R^a{}_a(x) = 2\omega^2.$$

Furthermore

$$\begin{aligned} R_{ab}(x) - \frac{1}{2}Rg_{ab}(x) &= \begin{pmatrix} -\omega^2 & 0 & -\omega^2 e^{\sqrt{2}\omega x^1} & 0 \\ 0 & -\omega^2 & 0 & 0 \\ -\omega^2 e^{\sqrt{2}\omega x^1} & 0 & -\frac{3}{2}\omega^2 e^{2\sqrt{2}\omega x^1} & 0 \\ 0 & 0 & 0 & -\omega^2 \end{pmatrix} \\ &= \omega^2 \left(-2 \begin{pmatrix} 1 & 0 & e^{\sqrt{2}\omega x^1} & 0 \\ 0 & 0 & 0 & 0 \\ e^{\sqrt{2}\omega x^1} & 0 & e^{2\sqrt{2}\omega x^1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \\ &- \begin{pmatrix} -1 & 0 & -e^{\sqrt{2}\omega x^1} & 0 \\ 0 & 1 & 0 & 0 \\ -e^{\sqrt{2}\omega x^1} & 0 & -\frac{1}{2}e^{2\sqrt{2}\omega x^1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \omega^2(-2t_{ab}(x) - g_{ab}(x) \end{aligned}$$

where

$$t_{ab}(x) = \begin{pmatrix} 1 & 0 & e^{\sqrt{2}\omega x^1} & 0 \\ 0 & 0 & 0 & 0 \\ e^{\sqrt{2}\omega x^1} & 0 & e^{2\sqrt{2}\omega x^1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is a tensor such that $\nabla_a(-2\omega^2 t_{ab}(x)) = 0$. Here, the symbol ∇_a stands for the (Christoffel) covariant derivative.

Thus we have the following

Proposition. *The Gödel solution is a Lorentzian spacetime and satisfies*

$$R_{ab} - \frac{1}{2} R g_{ab} + \omega^2 g_{ab} = -8\pi \left(\frac{\omega^2}{4\pi} \right) t_{ab}.$$

It corresponds therefore to a dust solution with positive cosmological constant.

References

[1] W. Borgiel, Classification of Gravitational Fields, Acta Cosmologica 22 (1996) (1) 47–71.
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