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Symmetry conditions on conformally flat Riemannian manifolds

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Abstract. We describe the classification results concerning conformally flat Riemannian manifolds satisfying some symmetry conditions, namely, local symmetry, semi-symmetry and pseudo-symmetry.

1. Introduction

As it is well-known, a Riemannian manifold (M,g) is said to be *(locally) conformally flat* if for any point $p \in M$ there exist a neighborhood U of p and a positive smooth function $f : U \to \mathbb{R}$ such that fg is a flat metric. Conformally flat manifolds represent a classical field of research in Riemannian geometry. In particular, many authors were involved in the study of homogeneity and symmetry conditions on a conformally flat manifold. A starting point in this investigation field is the well-known result of P. Ryan [R], who provided the complete classification of conformally flat locally symmetric spaces. They turn out to be covered isometrically by either a space of constant sectional curvature, namely, $\mathbb{R}^n$, $S^n(k)$ or $\mathbb{H}^n(-k)$, or by one of the following Riemannian products: $\mathbb{R} \times S^{n-1}(k)$, $\mathbb{R} \times \mathbb{H}^{n-1}(-k)$, $S^p(k) \times \mathbb{H}^{n-p}(-k)$. We refer to Section 3 for more details.

A first possibility to extend this result is by replacing local symmetry by homogeneity conditions. H. Takagi [T2] proved that *a locally homogeneous conformally flat Riemannian manifold* (M,g) *is locally symmetric*. So, M must be one of the spaces given in Ryan's classification. Indeed, in the proof of his result, Takagi only used local homogeneity to provide that (M,g) has constant Ricci eigenvalues. This property still holds under the weaker assumption that the conformally flat manifold is curvature homogeneous. Therefore, as already remarked in [CV], *a conformally flat curvature homogeneous space is locally symmetric.*

Another possibility to extend Ryan's result is by assuming some weaker symmetry conditions and semi-symmetric spaces represent a well-known and natural generalization of locally symmetric spaces. A *semi-symmetric space* is a Riemannian manifold (M,g) such that its curvature tensor R satisfies the condition

$$R(X,Y) \cdot R = 0, \tag{1}$$

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for all vector fields X, Y on M, where $R(X,Y)$ acts as a derivation on R [Sz]. This is equivalent to requiring that the curvature tensor R_p of (M,g) at a point $p \in M$ is the same as the curvature tensor of a symmetric space (which may change with the point p). Therefore, locally symmetric spaces are semi-symmetric, but in any dimension greater than two there exist examples of semi-symmetric spaces which are not locally symmetric (the first example was given by H. Takagi in [T1]. We can refer to [BKV] for a survey).

In Section 4, we show that a conformally flat semi-symmetric space is either locally symmetric, or locally isometric to a *(semi-symmetric) real cone*. So, real cones turn out to be the only conformally flat semi-symmetric spaces which are not locally symmetric.

Even in the broader class of pseudo-symmetric spaces of constant type, real cones provide the only conformally flat not locally symmetric examples (see Section 5). A *pseudo-symmetric space of constant type* is a Riemannian manifold (M,g) whose curvature tensor R satisfies

$$R(X,Y)\cdot R = \tilde{c}(X\wedge Y)\cdot R, \tag{2}$$

for all vector fields X and Y on M, where $X \wedge Y$ is defined by

$$(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y$$

and $\tilde{c}$ is a real constant [D], [KS]. It is evident by this definition that pseudo-symmetric spaces of constant type generalize the semi-symmetric ones.

2. Preliminaries

Let (M,g) be a Riemannian manifold of dimension $n > 2$ and R its curvature tensor, taken with the sign convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, for all vector fields X, Y on M, where ∇ denotes the Levi Civita connection of M. By ϱ, Q and τ we denote respectively the Ricci tensor, the Ricci operator associated to ϱ through g and the scalar curvature of M. Let p be a point of M and $\{e_1,..,e_n\}$ an orthonormal basis of the tangent space T_pM. The components of R and ϱ with respect to $\{e_1,..,e_n\}$ are denoted respectively by R_{ijkh} and ϱ_{ik}. As is well-known, for a conformally flat space we have

$$\begin{aligned} R_{ijkh} &= \frac{1}{n-2}(g_{ih}\varrho_{jk} + g_{jk}\varrho_{ih} - g_{ik}\varrho_{jh} - g_{jh}\varrho_{ik}) + \\ &\quad -\frac{\tau}{(n-1)(n-2)}(g_{ih}g_{jk} - g_{ik}g_{jh}). \end{aligned} \tag{3}$$

Moreover, (3) characterizes conformally flat Riemannian manifolds of dimension $n \geqslant 4$, while it is trivially satisfied by any three-dimensional manifold. Conversely, the condition

$$\nabla_i\varrho_{jk} - \nabla_j\varrho_{ik} = \frac{1}{2(n-1)}(g_{jk}\nabla_i\tau - g_{ik}\nabla_j\tau), \tag{4}$$

which characterizes three-dimensional conformally flat spaces, is trivially satisfied by conformally flat spaces of dimension greater than three.

We now recall some basic facts about semi-symmetric spaces. We start by the following

DEFINITION 1. The *nullity vector space* of the curvature tensor at a point p of a Riemannian manifold (M, g) is given by

$$E_{0p} = \{X \in T_pM \mid R(X,Y)Z = 0 \text{ for all } Y, Z \in T_pM\}.$$

The *index of nullity at* p is the number $\nu(p) = \dim E_{0p}$. The *index of conullity at* p is the number $u(p) = \dim M - v(p)$.

By means of the index of nullity and the index of conullity, Szabó [Sz] classified locally irreducible semi-symmetric spaces, proving that such a space must be locally isometric to one of the following spaces:

(i) a symmetric space when $\nu(p) = 0$ and $u(p) > 2$ at each point p, or

(ii) a real cone when $\nu(p) = 1$ and $u(p) = n - 1 > 2$ at each point p, or

(iii) a Kählerian cone when $\nu(p) = 2$ and $u(p) = n - 2 > 2$ at each point p, or

(iv) a Riemannian manifold foliated by Euclidean leaves of codimension two when $\nu(p) = n - 2$ and $u(p) = 2$ at each point p of a dense open subset U of M.

The local structure of a semi-symmetric space (M, g) is described by the following

THEOREM 1. **[Sz].** *There exists an open dense subset U of M such that around every point of U the manifold is locally isometric to a Riemannian product of type*

$$\mathbb{R}^k \times M_1 \times \dots \times M_r, \tag{5}$$

where $k \geqslant 0$, $r \geqslant 0$ and each M_i is either a symmetric space, a two-dimensional manifold, a real cone, a Kählerian cone or a Riemannian space foliated by Euclidean leaves of codimension two. The decomposition (5) may vary in the different connected components U_α of U, while it is constant on each U_α. For more details and references we refer to [BKV].

We now give a short description of *real cones*. They will turn out to be the only examples of conformally flat not locally symmetric spaces inside the classes of semi-symmetric and pseudo-symmetric spaces of constant type.

Consider a Riemannian manifold $(\bar{M}, \bar{g})$. Let $\mu(t)$ be the unique solution of the differential equation $\frac{d\mu}{dt} = -\mu^2$ with initial condition $\mu(0) = \mu_0 > 0$, that is, $\mu(t) = (t + (1/\mu_0))^{-1}$. Put $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > -1/\mu_0\}$ and on the product manifold $\mathbb{R}_+ \times \bar{M}$ consider the Riemannian metric

$$g = dx^0 \otimes dx^0 + \mu(x^0)^{-2}\pi^* g,$$

where x^0 is the natural coordinate on $\mathbb{R}_+$ and $\pi : \mathbb{R}_+ \times \bar{M} \to \bar{M}$ the projection on the second factor. The manifold $(\mathbb{R}_+ \times \bar{M}, g)$ is called a *Riemannian cone over* $(\bar{M}, \bar{g})$. Let $T = \partial/\partial x^0$ denote the unit vector field tangent to $\mathbb{R}_+$ in $\mathbb{R}_+ \times \bar{M}$. The curvature tensor of $M = \mathbb{R}_+ \times \bar{M}$ is described by (see [BKV])

$$\begin{aligned} R(X,Y)Z &= g(B_0(Y), Z)B_0(X) - g(B_0(X), Z)B_0(Y) + \\ &\quad + (\pi^* \bar{R})(X,Y)Z, \end{aligned} \tag{6}$$

for all tangent vectors X, Y, Z to $\bar{M}$, where $B_0(X) := \nabla_X T = \mu(X - g(X,T)T)$.

Any semi-symmetric real cone $(M = \mathbb{R}_+ \times \bar{M}, g)$ is locally isometric to some maximal cone $M_c(\tilde{M}, \mu_0)$, where $(\tilde{M}, \tilde{g})$ is a real space form of constant curvature c [BKV]. Note that at any point of M, $T \in E_0$. If M is locally irreducible and $c \neq 1$, then at each point of M the index of nullity is equal to one and the index of conullity coincides with the dimension of $\bar{M}$. We include the case when $\dim\bar{M} = 2$. Such spaces do not appear explicitly in Szabó's classification, since they are also Riemannian manifolds foliated by Euclidean leaves of codimension two.

At any point p of a semi-symmetric real cone M, fix an orthonormal basis of tangent vectors $\{e_0, e_1, .., e_r\}$, with $e_0 = T_p$ and $e_1, .., e_r$ tangent to the real space form $(\tilde{M}^r, \tilde{g})$ $(r = n-1)$. Then, using (6) to compute the components of the curvature tensor, we get

$$\begin{cases} R_{ijkh} = 0 \quad \text{if } 0 \in \{i,j,k,h\}, \\ \\ R_{ijkh} = \mu^2(c-1)(\delta_{ik}\delta_{jh} - \delta_{jk}\delta_{ih}) \quad \text{otherwise.} \end{cases} \tag{7}$$

Computing the Ricci components and the scalar curvature of M starting from (7), it is easy to check that (3) is satisfied and, if $\dim M \geqslant 4$, this implies that M is conformally flat. If $\dim M = 3$, one can check that (4) holds and so, M is conformally flat also in this case. Therefore, a real semi-symmetric cone M is a conformally flat (semi-symmetric) Riemannian manifold, with scalar curvature $\tau = r(r-1)(c-1)\mu^2$. Note that τ cannot be constant, as μ depends on t and so, M is never locally symmetric.

3. Conformally flat locally symmetric spaces

Locally symmetric spaces are a well known field of interest in Riemannian Geometry. A Riemannian manifold (M, g) is *locally symmetric* if all its (local) geodesic symmetries, that is, its geodesic reflections with respect to all points, are isometries. This important geometric property is reflected by a very nice curvature property: (M, g) is locally symmetric if and only if its curvature tensor R is *parallel*, that is, $\nabla R = 0$.

If (M, g) is conformally flat, its curvature tensor can be completely expressed in terms of the Ricci tensor and the scalar curvature. Therefore, a conformally flat space (M, g) is locally symmetric if and only if it is Ricci-parallel ($\nabla\varrho = 0$ or, equivalently, $\nabla Q = 0$).

In order to classify conformally flat Riemannian manifolds with constant scalar curvature and semi-definite positive Ricci tensor, P. Ryan used (and proved) the following result:

THEOREM 2. **[R]** *let M be an n-dimensional conformally flat space with parallel Ricci tensor. Then, M has as its simply connected Riemannian covering one of the following spaces:*

$$\mathbb{R}^n, S^n(k), \mathbb{H}^n(-k), \mathbb{R} \times S^{n-1}(k), \mathbb{R} \times \mathbb{H}^{n-1}(-k), S^p(k) \times \mathbb{H}^{n-p}(-k),$$

where $S^n(k)$ denotes a Euclidean n-sphere of constant curvature $k > 0$, and $\mathbb{H}^n(-k)$ an n-dimensional simply connected, connected space of constant curvature $-k < 0$.

We now recall here the basic ideas of the proof of Theorem 2. Assume that (M, g) is Ricci-parallel. Then, the Ricci eigenvalues of M, namely, ϱ_i, $i = 1, .., n$, must be constant in value and multiplicity. In fact, the corresponding eigenspaces are parallel distributions and M is locally isometric to thei Riemannian product of their integral manifolds. If all the Ricci eigenvectors coincide, from (3) it follows that M has constant sectional curvature and so, it is locally isometric to either $\mathbb{R}^n$, $S^n(k)$ or $\mathbb{H}^n(-k)$.

Next, if $\varrho_i \neq \varrho_j$ are two distinct Ricci eigenvalues and e_i, e_j two corresponding eigenvectors, then from (3) we get

$$0 = R_{ijij} = \frac{1}{n-2}(\varrho_i + \varrho_j) - \frac{\tau}{(n-1)(n-2)},$$

that is,

$$\varrho_i + \varrho_j = \frac{\tau}{n-1}. \tag{8}$$

Since (8) holds whenever we consider two distinct Ricci eigenvalues, M can only have two distinct Ricci eigenvalues, say, $\lambda \neq \mu$, of multiplicity p and $n-p$, respectively. In particular, the scalar curvature is given by

$$\tau = p\lambda + (n-p)\mu. \tag{9}$$

Comparing (8) and (9), we get

$$(n-p-1)\lambda + (p-1)\mu = 0. \tag{10}$$

The integral manifolds of the distributions T_λ and T_μ are proved to have constant sectional curvature [R]. More precisely, taking into account (8)-(10), the simply connected Riemannian covering of M must be one of the following spaces:

(i) $S^p(k) \times \mathbb{H}^{n-p}(-k)$, with $k = \frac{\lambda}{p-1} = -\frac{\mu}{n-p-1}$, if $1 < p < n-1$, or

(ii) $\mathbb{R} \times S^{n-1}(k)$ or $\mathbb{R} \times \mathbb{H}^{n-1}(-k)$, with $k = \frac{\mu}{n-2}$ (respectively, $k = \frac{\lambda}{n-2}$), if $p = 1$ (respectively, $p = n-1$),

and this ends the proof.

4. Conformally flat semi-symmetric spaces

Using the special properties of the curvature of a conformally flat Riemannian manifold (in particular, the existence of an orthonormal basis of Ricci eigenvectors for the tangent space at any point), it is easy to prove the following rigidity result.

THEOREM 3. *Let* (M, g) *be a Riemannian manifold satisfying* (3)*, of dimension* $n \geqslant 3$ (that is, either $\dim M = 3$ or M is conformally flat)*. Then, at each point* p *of* M*, the index of nullity is either* $\nu(p) = 0, 1$ *or* n*.*

We can refer to [C1] for the details of the proof. Theorem 3 restricts the research of conformally flat semi-symmetric spaces to the ones having index of nullity equal to 0, 1 or n. The most interesting case is the one of a semi-symmetric space having nullity index equal to 1, since if the nullity index is constant and equal to 0 (respectively, to n), then the space is locally symmetric (respectively, flat).

We can now prove the following

THEOREM 4. *A conformally flat Riemannian manifold* (M, g) *(of dimension* $n > 2$*) is semi-symmetric if and only if it is locally symmetric or locally isometric to a (semi-symmetric) real cone.*

Proof. The "if" part is trivial, since a Riemannian manifold which is locally symmetric or locally isometric to a real cone is clearly semi-symmetric. We now prove the "only if" part.

A locally reducible conformally flat Riemannian manifold is always locally symmetric, as was proved by Yau in [Y]. So, we can assume M is locally irreducible. According to Szabó's classification, M must be locally isometric to one of the spaces (i)-(iv) listed before Theorem 1. Since M is locally irreducible, M cannot be flat. So, let p be a point of M with $R_p \neq 0$. Theorem 3 then implies that either $\nu(p) = 0$ or 1. This excludes the case (iii), while case (iv) is only possible when $n = 3$. If $\nu(p) = 0$, then M is locally isometric to an irreducible symmetric space and the conclusion follows. In order to complete the proof, we should prove that a three-dimensional conformally flat Riemannian manifold of conullity two is isometric to a real cone. Nevertheless, a more general classification result has been already obtained by N. Hashimoto and M. Sekizawa [HS]. Taking into account the definition of semi-symmetric real cones, their result can be rewritten in the following way:

Theorem 5. **[HS]** *A three-dimensional conformally flat pseudo-symmetric space of constant type is either locally symmetric or it is locally isometric to a semi-symmetric real cone.* □

5. Conformally flat pseudo-symmetric spaces

This section is devoted to the proof of the following result, which extends both Theorems 4 and 5. For all the details we refer to [C2].

Theorem 6. *A conformally flat pseudo-symmetric space of constant type (of dimension $n > 2$) is either locally symmetric or it is locally isometric to a semi-symmetric real cone.*

Proof. Let (M, g) be a conformally flat pseudo-symmetric space of constant type, with constant $\tilde{c}$. Taking into account Theorem 5, we can restrict ourselves to the case when the dimension of M is $n \geqslant 4$. Our purpose is to show that necessarily $\tilde{c} = 0$. Thus, (M, g) must be conformally flat and semi-symmetric and the conclusion follows from Theorem 4.

Let p be a point of M and $\{e_i\}$ an orthonormal basis of the tangent space T_pM, made by eigenvectors of the Ricci operator. Taking into account (3), the curvature components R_{ijkh} with respect to $\{e_i\}$ vanish, whenever at least three indeces are distinct.

Consider three arbitrary indeces i, j and k such that $i \neq j \neq k \neq i$. From (2), we have

$$(R(e_i, e_j) \cdot R)(e_k, e_i, e_j) = \tilde{c}((e_i \wedge e_j) \cdot R)(e_k, e_i, e_j),$$

Since the curvature components vanish when at least three indeces are distinct, after some standard calculations we get

$$(R_{ijij} + \tilde{c})(R_{jkjk} - R_{ikik}) = 0 \text{ whenever i, j and k are all distinct.} \tag{11}$$

Next, let W be a dense open subset of M, such that the multiplicities of the Ricci eigenvalues remain constant on a connected neighborhood V of any point $p \in W$. According to Theorem 3, at each point of M the nullity index is either 0, 1 or n. If $p \in W$ and V is a connected neighborhood of p, where the Ricci eigenvalues have constant multiplicities, then the nullity index will be $\nu(q) = 0, 1$ or n for all $q \in V$. So, have to deal with three different cases, according to the three different possible values of ν on V. If, in all these cases, we can conclude that $\tilde{c} = 0$ on V, then $\tilde{c} = 0$ on M, since it is a constant, and this will end the proof.

a) If $\nu = n$ on V, then V is flat. In particular, V is semi-symmetric and so, $\tilde{c} = 0$.

b) If $\nu = 1$ on V, let q be a point of V and e_1 a unit vector of the nullity space E_{0q}. So, $\varrho(e_1, \cdot) = 0$ and we can consider an orthonormal basis $\{e_i\}$ of T_qM of Ricci eigenvectors, including e_1. Taking $i = 1$ in (11), since $R_{1j1j} = 0$ for all j, we then get

$$\tilde{c}R_{jkjk} = 0 \quad \text{for all } j \neq k > 1. \tag{12}$$

Since $\nu(q) = 1$, e_j and e_k can never belong to the nullity space when $j, k > 1$. So, R_{jkjk} can not identically vanish for all j and k and (12) implies that $\tilde{c} = 0$.

c) If $\nu = 0$ on V, we first note that there exist some indeces $i \neq j$ such that $R_{ijij} \neq -\tilde{c}$ (otherwise, V will turn out to have constant sectional curvature $-\tilde{c}$ and $\tilde{c} = 0$. Thus, V should be flat, against the assumption that $\nu = 0$).

Without loss of generality, we can assume $R_{1212} \neq \tilde{c}$. Then, applying (11) with $j = 1$ and $k = 2$, we get

$$(R_{1i1i} + \tilde{c})(R_{2i2i} + \tilde{c}) = 0 \quad \text{for all } i > 2.$$

In other words, either $R_{1i1i} = -\tilde{c}$ or $R_{2i2i} = -\tilde{c}$, for all $i > 2$.

Next, we start from (11) and we take the sum over $k \neq i, j$, obtaining

$$(R_{ijij} + \tilde{c})(\varrho_i - \varrho_j) = 0 \quad \text{for all } i \neq j. \tag{13}$$

In particular, from (13) it follows that $\varrho_i = \varrho_j$ if and only if $R_{ijij} \neq -\tilde{c}$.

Fix an index $k > 2$. As we already remarked, either $R_{1k1k} = -\tilde{c}$ or $R_{2k2k} = -\tilde{c}$. After some more calculations, it is possible to prove (see [C2]) that if $R_{1k1k} = -\tilde{c}$ (respectively, $R_{2k2k} = -\tilde{c}$), then $\varrho_k = \varrho_2$ (respectively, $\varrho_k = \varrho_1$). Therefore, for all $k > 2$, either $\varrho_k = \varrho_1$ or $\varrho_k = \varrho_2$. Note that $\varrho_1 \neq \varrho_2$, otherwise, by (3), V should have constant sectional curvature and this can not occur, as we already noted. Thus, on V we have two distinct Ricci eigenvalues $\varrho_1 \neq \varrho_2$. By reordering the Ricci eigenvectors of the orthonormal basis $\{e_i\}$, we can assume, without loss of generality, that the Ricci eigenvalues in V are

$$\varrho_1 = \ldots = \varrho_r = a \neq b = \varrho_{r+1} = \ldots = \varrho_n, \tag{14}$$

for some integer r, greather than 1 and lesser than n. The scalar curvature τ in V is then given by

$$\tau = ra + (n - r)b. \tag{15}$$

We now get a different expression for τ. Using (14) in (3), one can easily obtain

$$R_{ijij} = -\frac{a + b}{n - 2} + \frac{\tau}{(n - 1)(n - 2)} \quad \text{if } i \leqslant r \text{ and } j > r \text{ or conversely.} \tag{16}$$

If $i \leqslant r$ and $j > r$, then $\varrho_i \neq \varrho_j$ So, by (11) it follows $R_{ijij} = -\tilde{c}$ and (16) implies

$$\tau = (n - 1)(a + b) - (n - 1)(n - 2)\tilde{c}. \tag{17}$$

We differentiate both (15) and (17) with respect to e_i, for any $i = 1,..,r$. We respectively get

$$e_i(\tau) = re_i(a) + (n-r)e_i(b) \tag{18}$$

and

$$e_i(\tau) = (n-1)(e_i(a) + e_i(b)). \tag{19}$$

Finally, since (M,g) is conformally flat, (4) holds. We apply (4) taking $k = j$, with $i \neq j \leqslant r$. Since $\{e_i\}$ is an orthonormal basis of eigenvectors of the Ricci operator, we obtain

$$\nabla_i \varrho_{jj} = e_i(\varrho_{jj}) = e_i(a) \quad \text{and} \quad \nabla_j \varrho_{ij} = 0$$

and so, by (4),

$$e_i(\tau) = 2(n-1)e_i(a) \quad \text{for all } i \leqslant r. \tag{20}$$

Comparing the expressions of $e_i(\tau)$ given by (18), (19) and (20), it is easy to show that $e_i(a) = e_i(b) = 0$ for all $i \leqslant r$. In a similar way one can prove that $e_i(a) = e_i(b) = 0$ also holds for all $i > r$. Hence, a and b are constant on V. Therefore, V is conformally flat and curvature homogeneous (by (3)). So, V is locally symmetric [CV]. In particular, $\tilde{c} = 0$ and this completes the proof. □

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