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Explicit construction of maximal surfaces with singularities in complete flat 3-manifolds.

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Abstract. It is known that, up to a finite covering, any complete flat 3-Lorentzian manifold containing a complete maximal surface with a finite set of singular points is $\mathbb{L}^3$ or its quotient under a group generated by 1 or 2 translations. In this paper we construct families of (embedded) maximal surfaces in each one of these three spaces.

1. Introduction

A maximal surface in a Lorentzian manifold is a spacelike surface with zero mean curvature. Besides of their mathematical interest these surfaces and more generally those having constant mean curvature have a significant importance in classical Relativity. More information on this aspect can be found for instance in [**9**].

Maximal surfaces in $\mathbb{L}^3$ share some properties with minimal surfaces in the Euclidean space $\mathbb{R}^3$. Both families arise as solutions of variational problems: local maxima (minima) for the area functional in the Lorentzian (Euclidean) case. Like minimal surfaces in $\mathbb{R}^3$, the Gauss map is conformal and they admit a Weierstrass representation in terms of meromorphic data (cf.[**6**]).

When the ambient space is the Minkowski space $\mathbb{L}^3$, one of the most important results is the proof of a Bernstein-type theorem for maximal surfaces in $\mathbb{L}^3$. Calabi [**1**] proved that a complete maximal surface in $\mathbb{L}^3$ is necessarily a spacelike plane. Therefore, problems of interest should deal with surfaces having non empty boundary or having a certain type of singularities. In this line, Estudillo and Romero in [**2**], and Umehara and Yamada in [**11**] have proved some results on the global behavior of maximal immersed surfaces in $\mathbb{L}^3$ with analytical curves of singularities. From a different point of view Klyachin [**7**] has studied entire maximal graphs with compact singular set. On the other hand, in [**5**] and [**4**] the authors study those maximal surfaces having isolated singularities in $\mathbb{L}^3$ and its quotients. For several reasons, isolated singularities of maximal graphs in $\mathbb{L}^3$ are specially interesting. They are called *conelike* singularities, and correspond to points where the Gauss curvature blows up, the Gauss map has no well defined limit and the surface is asymptotic to a half light cone.

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In [**6**], Kobayashi showed that the unique complete maximal surface in $\mathbb{L}^3$ with one conelike singularity is the Lorentzian catenoid (see Figure 1). Recently, López, López and Souam [**8**] have given a new family of complete maximal graphs with two singular points (see Figure 2).

A consequence of Calabi's theorem is the systematic arranging of groups $G \subset \mathrm{Iso}(\mathbb{L}^3)$ for which $\mathbb{L}^3/G$ contains a complete maximal surface $\mathcal{S}$. For topological considerations, it is natural to assume that the fundamental group of the surface is isomorphic to G. Under these conditions, it turns out that G contains a maximal free abelian normal subgroup G_0 of spacelike translations of rank one or two, and the quotient G/G_0 is either trivial, $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover, $\mathcal{S}$ must be a flat cylinder, torus, Möbius strip or Klein bottle.

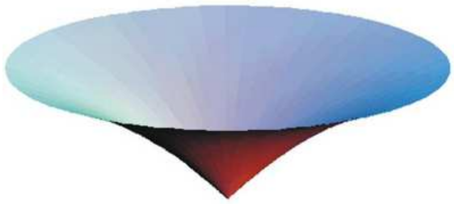

FIGURE 1. The Lorentzian catenoid

In this work we focus our attention on complete embedded maximal surfaces with a finite number of singular points in a complete flat Lorentzian 3-manifold or equivalently, in a quotient of $\mathbb{L}^3$ under a group of isometries G acting properly and freely (see [**12**]). These surfaces lift to entire maximal graphs in $\mathbb{L}^3$ with isolated singularities which are invariant under G.

In [**4**] we show that any quotient of $\mathbb{L}^3$ admitting such a surface must be finitely covered by $\mathbb{L}^3/G_0$ where $G_0 = \{\mathrm{Id}\}$, $\langle T\rangle$ or $\langle T_1, T_2\rangle$, where T, T_1 and T_2 are spacelike translations. In the second case the lifted surface in $\mathbb{L}^3$ is said to be *singly periodic* and in the third one the lifted surface is called *doubly periodic*.

In this paper we deal with the Weierstrass representation of maximal surfaces to construct explicitly examples of these surfaces. Concretely, we construct families of embedded maximal surfaces with singularities in $\mathbb{L}^3/G$ having the following properties:

- $G = \{\mathrm{Id}\}$. For any $n \geqslant 1$ we give two families, Γ_n^C and Γ_n^P, characterized as the unique maximal surfaces with $n+1$ singularities in $\mathbb{L}^3$ and whose group of symmetries has $2(n+1)$ elements (Figures 5 and 6).
- $G = \langle T\rangle$. For any $n \geqslant 1$ we give a family of surfaces with $2n$ singular points such that the symmetry group of the lifted surface in $\mathbb{L}^3$ contains a symmetry with respect to a vertical plane and a sliding symmetry[1] with respect to a horizontal plane (Figure 8).
- $G = \langle T_1, T_2\rangle$. The surfaces have two singular points and the symmetry group of the lifted surface contains the symmetry with respect to the plane $\{x_1 = 0\}$ and a sliding symmetry with repsect to $\{x_2 = 0\}$ (Figure 9).

[1]that is, a symmetry with respect to a plane followed by a translation in this plane.

In particular, the case $n = 1$ in the family Γ_n^P leads to the surfaces due to López, López and Souam mentioned above.

FIGURE 2. A Riemann type maximal surface obtained by López, López and Souam

2. Background

Throughout this paper $\mathbb{L}^3$ will denote the three dimensional Lorentz-Minkowski space, that is $\mathbb{L}^3 = (\mathbb{R}^3, \langle , \rangle)$ where $\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + x_2y_2 - x_3y_3$.

A vector $v \in \mathbb{L}^3 \setminus \{0\}$ is said to be spacelike, lightlike or timelike provided $\langle v, v \rangle$ is positive, null or negative respectively. The vector $\mathbf{0}$ is spacelike by definition. Likewise, we say that a plane is spacelike, lightlike or timelike if the restriction of the metric is definite positive, degenerate or non degenerate and indefinite.

Let $\mathfrak{N}$ be a Lorentzian 3-manifold. A differentiable immersion $X : M \to \mathfrak{N}$ from a surface M is said to be spacelike if the tangent plane at any point is spacelike, that is to say, the induced metric on M is Riemannian.

A maximal immersion is a spacelike immersion whose mean curvature vanishes. Using isothermal parameters, any (orientable) maximal surface can be endowed with a conformal structure and therefore it becomes in a Riemann surface.

We denote by σ the stereographic projection of the sphere of radius -1,

$$\sigma : \overline{\mathbb{C}} \setminus \{|z| = 1\} \longrightarrow \mathbb{H}^2 := \{x \in \mathbb{L}^3 \ : \ \langle x, x \rangle = -1\}$$

$$\sigma(z) = \left(\frac{2\mathrm{Im}(z)}{|z|^2 - 1}, \frac{-2\mathrm{Re}(z)}{|z|^2 - 1}, \frac{|z|^2 + 1}{|z|^2 - 1} \right), \qquad \sigma(\infty) = (0, 0, 1)$$

THEOREM 1 (Weierstrass representation in $\mathbb{L}^3$ [**6**]). *Let $X : M \to \mathbb{L}^3$ be a conformal maximal immersion of a Riemann surface and let $N : M \to \mathbb{H}^2$ be its Gauss map. Then $g \stackrel{\text{def}}{=} \sigma^{-1} \circ N$ is meromorphic. Moreover, there exists a holomorphic 1-form φ_3 defined on M such that*

(i) the 1-forms given by $\varphi_1 = \frac{i}{2}(\frac{1}{g} - g)\varphi_3$ and $\varphi_2 = \frac{-1}{2}(\frac{1}{g} + g)\varphi_3$ are holomorphic and $\varphi_1, \varphi_2, \varphi_3$ have non common zeroes.

(ii) $|g| \neq 1$ on M.

(iii) For any closed curve γ in M we have $Re \int_\gamma (\varphi_1, \varphi_2, \varphi_3) = 0$

(iv) Up to a translation, the immersion is given by $X = Re \int_{P_0} (\varphi_1, \varphi_2, \varphi_3)$, where $P_0 \in M$.

Conversely, given a meromorphic function g and a holomorphic 1-form φ_3 on M such that (i), (ii) and (iii) are satisfied, (iv) defines a conformal maximal immersion of M in $\mathbb{L}^3$. Moreover, the induced metric on M is given by

$$ds^2 = |\varphi_1|^2 + |\varphi_2|^2 - |\varphi_3|^2 = \frac{1}{4}\big(\frac{1}{|g|} - |g|\big)^2 |\varphi_3|^2 \tag{1}$$

Remark 1 (Weierstrass data in translational spaces). *The Weierstrass data* (g, φ_3) *of a maximal surface in* $\mathbb{L}^3$ *are invariant by translations. Therefore, if* G *is a translational group acting properly and freely on* $\mathbb{L}^3$*, maximal surfaces in the quotient space* $\mathbb{L}^3/G$ *also have a Weierstrass representation as above except that the condition* (iii) *now becomes in*

(iii') *for any closed curve* γ *in* M *we have that the translation of vector given by* $v = Re \int_\gamma (\varphi_1, \varphi_2, \varphi_3)$ *is in the group* G

Finally, we introduce the following notation.

For any meromorphic function or 1-form ψ on a Riemann surface with zeroes $P_1, \ldots, P_k$ of orders $m_1, \ldots, m_k$ respectively and with poles at $Q_1, \ldots, Q_l$ of orders $n_1, \ldots, n_l$, we put

$$(\psi) \stackrel{\text{def}}{=} \frac{P_1^{m_1} \cdot \ldots \cdot P_k^{m_k}}{Q_1^{n_1} \cdot \ldots \cdot Q_l^{n_l}}$$

(ψ) is called the *divisor associated to* ψ. For more details on divisors and Riemann surfaces see for example [**3**].

If $\Phi = (\varphi_1, \varphi_2, \varphi_3)$ are three meromorphic 1-forms on a Riemann surface we say that Φ has a pole at a point P if at least one of the three 1-forms φ_1, φ_2 and φ_3 has a pole at P. The order of Φ at a pole P is defined as the maximum of the orders of φ_1, φ_2 and φ_3 at P.

3. Maximal surfaces with isolated singularities

Definition 1. *Let* $\mathfrak{N}$ *be a Lorentzian 3-manifold,* $X : M \to \mathfrak{N}$ *a topological embedding of a smooth surface and* $F \subset M$ *a closed discrete subset. We say that* X *is a maximal embedding with singular set* F *if* $X|_{M \setminus F}$ *is a maximal embedding and the induced metric on* $M \setminus F$ *converges to zero at any point of* F.

In this case we say that $S = X(M)$ *is a maximal surface in* $\mathfrak{N}$ *with singularities at* $X(F)$.

A maximal embedding with singularities $X : M \to \mathfrak{N}$ *is* complete *if any divergent curve in* M *has infinite length.*

Taking into account the expression of the metric in terms of the Weierstrass data given in Equation (1), it follows that isolated singularities in a maximal surface with Weierstrass data (g, φ_3) appear at the zeroes of φ_3 that are neither zeroes nor poles of g and at the points for which $|g| = 1$.

If in addition the surface is embedded, it turns out that the singularities must be of the last type and the 1-form φ_3 does not vanish at the singularity. We refer to [**4**] for a good setting. If the ambient space is $\mathbb{L}^3$ the surface is asymptotic around an embedded singularity to the top or the bottom component of the light cone. For this reason they are called *conelike singularities.*

Theorem 2 (Global behaviour [**4**]). *Let* $\mathfrak{N}$ *be a complete flat Lorentzian 3-manifold and* $X : M \to \mathfrak{N}$ *a complete embedded maximal surface with finite set of singularities* F.

Then $\mathfrak{N}$ *is finitely covered by* $\mathbb{L}^3$ *or its quotient under a translational group of rank 1 or 2. Moreover,*

(a) If $\mathfrak{N} = \mathbb{L}^3$, $M \setminus F$ *is conformally equivalent* $\mathcal{M}_0 := \mathbb{C} \setminus \cup_{p \in F} D_p$*, where* D_p *are pairwise disjoint closed discs. At* $z = \infty$ *the surface is asymptotic to either half catenoid or a plane.*

(b) *If $\mathfrak{N} = \mathbb{L}^3/\langle T\rangle$, $M \setminus F$ is conformally equivalent $\mathcal{M}_0 := \mathbb{C}^* \setminus \cup_{p\in F} D_p$, where D_p are pairwise disjoint closed discs. At $z = 0, \infty$ the surface is asymptotic to spacelike flat half cylinders.*

(c) *If $\mathfrak{N} = \mathbb{L}^3/\langle T_1, T_2\rangle$, $M \setminus F$ is conformally equivalent $\mathcal{M}_0 := \mathbb{T} \setminus \cup_{p\in F} D_p$, where $\mathbb{T}$ is a flat torus and D_p are pairwise disjoint closed discs. In this case the surface is compact.*

Moreover, in all these cases the associated conformal reparameterization, $X_0 : \mathcal{M}_0 \to \mathfrak{N}$, extends analytically to the Riemann surface with boundary $\mathcal{M} := \mathcal{M}_0 \cup_{p\in F} \partial(D_p)$, by putting $X_0(\partial(D_p)) = X(p)$.

In the above setting, let $\mathcal{M}^*$ be the mirror surface of the circular domain $\mathcal{M}$ (see figure 3). We denote by $\mathcal{N}$ the double surface obtained by gluing $\mathcal{M}$ and $\mathcal{M}^*$ along their boundaries (see [**10**] for more details). If the number of singular points of the surface is $n + 1$ it follows that $\mathcal{N}$ is biholomorphic to

(a) A compact Riemann surface $\overline{\mathcal{N}}$ of genus n minus two points if $\mathfrak{N} = \mathbb{L}^3$,
(b) A compact Riemann surface $\overline{\mathcal{N}}$ of genus n minus four points if $\mathfrak{N} = \mathbb{L}^3/\langle T\rangle$,
(c) A compact Riemann surface $\overline{\mathcal{N}}$ of genus $n + 2$ if $\mathfrak{N} = \mathbb{L}^3/\langle T_1, T_2\rangle$.

Finally, we label $J : \mathcal{N} \to \mathcal{N}$ as the mirror involution mapping a point in $\mathcal{M}$ into its mirror image and vice versa. Note that J extends to an antiholomorphic involution on $\overline{\mathcal{N}}$ and its fixed point set coincides with $\partial(\mathcal{M}) \equiv \partial(\mathcal{M}^*)$.

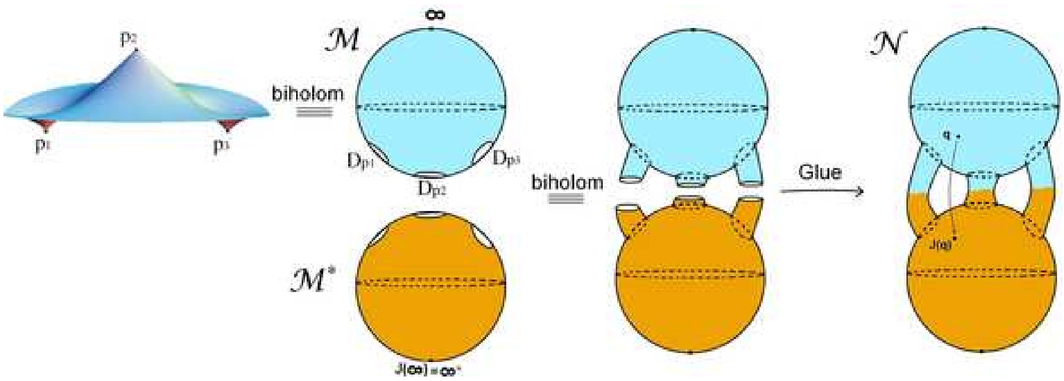

FIGURE 3. The double surface associated to a maximal surface with singularities in $\mathbb{L}^3$

We use this double surface and the mirror involution to give a description of complete maximal surfaces with a finite number of singularities.

THEOREM 3 (Representation [**5**]). *Let $\overline{\mathcal{N}}$ be a compact Riemann surface and $J : \overline{\mathcal{N}} \to \overline{\mathcal{N}}$ an antiholomorphic involution such that the fixed point set of J consists of $n+1$ pairwise disjoint analytic Jordan curves γ_j, $j = 0, 1, \ldots, n$, and $\overline{\mathcal{N}} \setminus \bigcup_{j=0}^n \gamma_j$ has two connected components, Ω and $J(\Omega)$. Let also (g, φ_3) be meromorphic data on $\overline{\mathcal{N}}$ satisfying $|g| < 1$ in Ω, $g \circ J = 1/\overline{g}$ and $J^*(\varphi_3) = -\overline{\varphi_3}$.*

Suppose that one of the following conditions holds:

(a) **:** *$\overline{\mathcal{N}}$ has genus n,*
: *Ω is topologically equivalent (and so conformally) to $\overline{\mathbb{C}}$ minus $n + 1$ pairwise disjoint closed discs,*
: *Φ is holomorphic and without zeroes in $\overline{\mathcal{N}} \setminus \{p_\infty, J(p_\infty)\}$, $p_\infty \in \Omega$, and Φ has poles of order at most 2 at p_∞ and $J(p_\infty)$.*

(b) : $\overline{\mathcal{N}}$ *has genus* n,
: Ω *is topologically equivalent (and so conformally) to* $\overline{\mathbb{C}}$ *minus* $n+1$ *pairwise disjoint closed discs,*
: Φ *is holomorphic and without zeroes in* $\overline{\mathcal{N}} \backslash \{p_\infty, q_\infty, J(p_\infty), J(q_\infty)\}$, $p_\infty, q_\infty \in \Omega$, *and at these four points* Φ *has simple poles.*

(c) : $\overline{\mathcal{N}}$ *has genus* $n+2$,
: Ω *is topologically equivalent (and so conformally) to a flat torus* $\mathbb{T}$ *minus* $n+1$ *pairwise disjoint closed discs,*
: Φ *is holomorphic in* $\overline{\mathcal{N}}$.

Define the map $X : \Omega_0 \to \mathbb{L}^3/G$, $X(p) := Re \int_{p_0}^{p} \Phi$ *where in each one of the above cases (a) (b) and (c),* Ω_0 *and the group of isometries* G *are given by*

(a) $\Omega_0 = \overline{\Omega} \setminus \{p_\infty\}$ *and* $G = \{Id\}$,
(b) $\Omega_0 = \overline{\Omega} \setminus \{p_\infty, q_\infty\}$ *and* G *is generated by the translation of vector* $v = Re \int_\gamma \Phi$ *where* γ *is a closed curve winding once one of the two ends, namely* p_∞ *or* q_∞.
(c) $\Omega_0 = \overline{\Omega}$ *and* G *is the group generated by the two translations of vectors* $v_i = Re \int_{\gamma_i} \Phi$, $i = 1, 2$, *where* γ_i *are the two curves generating the first homology group of the torus* $\mathbb{T}$.

Then, X *is a well defined maximal immersion and* $S = X(\Omega_0)$ *is a complete embedded maximal surface with* $n+1$ *singularities corresponding to the points* $q_j := X(\gamma_j)$, $j = 0, 1, \ldots, n$.

Conversely, any complete embedded maximal surface with a finite number of singularities in $\mathbb{L}^3$ *or* $\mathbb{L}^3/G$, *where* G *is a translational group of rank 1 or 2, can be recovered in this way.*

REMARK 2. *If the ambient space is* $\mathbb{L}^3$, *up to a Lorentzian isometry we can suppose that the limit normal vector at the end* $z = \infty$ *is vertical. With this normalization, it is easy to show that the end is asymptotic to a horizontal plane if and only if* φ_3 *is regular at the end.*

4. Construction of the examples

4.1. Examples in $\mathbb{L}^3$: the families Γ_n^C and Γ_n^P.. For any $n \in \mathbb{N}$, $n \geqslant 1$, and $a \in]0, 1[$, we label $\Sigma_{n,a}$ as the Riemann surface associated to the function $w = \sqrt[n+1]{\frac{z(az-1)}{z-a}}$, that is to say,

$$\Sigma_{n,a} := \{(z, w) \in \overline{\mathbb{C}}^2 \ : \ w^{n+1} = \frac{z(az-1)}{z-a}\}$$

This surface can be thought of as a $n+1$ branched covering of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Indeed, consider $n+1$ copies of $\overline{\mathbb{C}}$, and label these copies as **sheets** $\mathbf{1}, \ldots, \boldsymbol{n+1}$. We "cut" each sheet along two curves, one of them joining 0 and a and the other one joining $1/a$ and ∞, and we assume these two cuts does not intersect each other. Each cut has two banks: a N-bank and a S-bank. A model of the surface $\Sigma_{n,a}$ can be constructed by identifying the N-bank of a cut in the sheet $\boldsymbol{j}$ with the corresponding S-bank in the sheet $\boldsymbol{j+1}$, and the N-bank of the sheet $\boldsymbol{n+1}$ with the S-bank of the sheet $\mathbf{1}$. It is not hard to convince oneself that this surface $\Sigma_{n,a}$ is a compact surface of genus n (see figure 4).

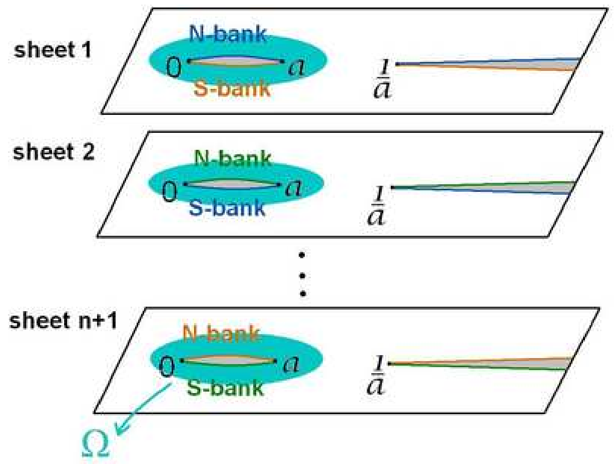

FIGURE 4. A model for $\Sigma_{n,a}$.

Consider the antiholomorphic involution $J = J_{n,a} : \Sigma_{n,a} \to \Sigma_{n,a}$ given by $J(z,w) = (1/\overline{z}, 1/\overline{w})$. The fixed points of J are the $n+1$ Jordan curves $\gamma_1, \ldots, \gamma_{n+1}$, corresponding to the $n+1$ branches of w on $|z| = 1$ (i.e., in the above model, the curves with $|z| = 1$ in each sheet $\mathbf{1}, \ldots, \boldsymbol{n+1}$). It is not hard to see that $\Sigma_{n,a} \backslash \cup_{j=1}^{n+1} \gamma_j$ has two connected components, namely $\Omega_{n,a} = \{(z,w) \in \Sigma_{n,a} \ : \ |z| < 1\}$ and $J(\Omega_{n,a}) = \{(z,w) \in \Sigma_{n,a} \ : \ |z| > 1\}$.

Denote by $z, w : \Sigma_{n,a} \to \overline{\mathbb{C}}$ the two canonical projections. Recall that the associated divisor for w and for the 1-form dz are

$$(w) = \frac{0 \cdot (1/a)}{\infty \cdot a} \quad (dz) = \frac{0^n \cdot (1/a)^n \cdot a^n}{\infty^{n+2}}$$

where we are identifying $0 \equiv (0,0)$, $1/a \equiv (1/a, 0)$, $a \equiv (a, \infty)$ and $\infty \equiv (\infty, \infty)$.

4.1.1. *Family Γ_n^C: surfaces with catenoidal end.* For each $A \in \mathbb{R}^*$ we define the following meromorphic data on $\Sigma_{n,a}$,

$$g_{n,a} = \frac{z-a}{az-1} w \quad \text{and} \quad \varphi_3^{n,a,A} = A \frac{dz}{z} \tag{2}$$

It is not hard to check that the associated divisors are

$$(g_{n,a}) = \frac{(1/a)^n \cdot 0}{a^n \cdot \infty} \quad \text{and} \quad (\varphi_3^{n,a,A}) = \frac{(1/a)^n \cdot a^n}{0 \cdot \infty}$$

and therefore $\big(\overline{\mathcal{N}} = \Sigma_{n,a}, J, (g_{n,a}, \varphi_3^{n,a,A})\big)$ satisfy the hypothesis of Theorem 3 (a) and thus, the map

$$X_{n,a,A} : \overline{\Omega}_{n,a} \setminus \{0\} \longrightarrow \mathbb{L}^3 \qquad X_{n,a,A} = \operatorname{Re} \int_a \Phi_{n,a,A},$$

where $\Phi_{n,a,A} = (\varphi_1^{n,a,A}, \varphi_2^{n,a,A}, \varphi_3^{n,a,A})$ is defined as in Theorem 1, gives a complete maximal surface with $n+1$ singularities asymptotic to a half catenoid (see Remark 2).

We denote by Γ_n^C this family of surfaces,

$$\Gamma_n^C = \{X_{n,a,A}(\overline{\Omega_{n,a}} \setminus \{0\}) \ : \ a \in]0,1[, \ A \in \mathbb{R}^*\}$$

FIGURE 5. Examples of surfaces in Γ_n^C for $n=1$ and $n=2$

The surfaces in Γ_n^C are invariant under a vertical rotation. Indeed, consider the holomorphic map $S_1 : \Sigma_{n,a} \to \Sigma_{n,a}$, $S_1(z,w) = (z, e^{\frac{2\pi i}{n+1}} w)$. From Equation (2) it is straightforward to conclude that the Weierstrass data (g, φ_3) of any surface in Γ_n^C satisfy

$$g \circ S_1 = e^{\frac{2\pi i}{n+1}} g \quad \text{and} \quad S_1^*(\varphi_3) = \varphi_3$$

Since $T(a) = a$, the associated maximal immersion

$$X = (X_1, X_2, X_3) = \mathrm{Re} \int_a (\varphi_1, \varphi_2, \varphi_3)$$

satisfy the following symmetry

$$\begin{aligned} X \circ S_1 &= \mathrm{Re} \int_a \big(S_1^*(\varphi_1), S_1^*(\varphi_2), S_1^*(\varphi_3)\big) = \\ &= \big(\cos(\frac{2\pi i}{n+1})\, X_1 - \mathrm{sen}(\frac{2\pi i}{n+1})\, X_2, \mathrm{sen}(\frac{2\pi i}{n+1})\, X_1 + \cos(\frac{2\pi i}{n+1})\, X_2, X_3\big) \end{aligned}$$

That is, the surface is invariant under the rotation about the x_3-axis of angle $\frac{2\pi}{n+1}$.

Likewise, if we consider the antiholomorphic involution given by $S_2(z,w) = (\overline{z}, \overline{w})$, it follows that $g \circ S_2 = \overline{g}$ and $S_2^*(\varphi_3) = \overline{\varphi_3}$ and as a consequence the surface is also invariant by the symmetry with respect to the plane $\{x_1 = 0\}$.

4.1.2. *Family Γ_n^P: surfaces with a planar end.* Assume $n = 2k+1$ is odd. Consider the following data on $\Sigma_{n,a}$:

$$g_{n,a} = \frac{z}{w^k} \quad \text{and} \quad \varphi_3^{n,a,A} = iA \frac{dz}{w^{k+1}(z-a)} \tag{3}$$

The associated divisors are

$$(g_{n,a}) = \frac{a^k \cdot 0^{k+2}}{(1/a)^k \cdot \infty^{k+2}} \quad \text{and} \quad (\varphi_3^{n,a,A}) = a^k \cdot (1/a)^k \cdot 0^k \cdot \infty^k$$

and therefore, Theorem 3 gives that the map $Y = Y_{n,a,A} : \overline{\Omega_{n,a}} \setminus \{0\} \to \mathbb{L}^3$, $Y_{n,a,A} = \mathrm{Re} \int_a \Phi_{n,a,A}$ provides a maximal surface with $n+1$ conelike singularities asymptotic to a plane (see Remark 2).

Denote by Γ_n^P, $n = 2k+1$, the set of all these surfaces,

$$\Gamma_n^P = \{Y_{n,a,A}(\overline{\Omega_{n,a}} \setminus \{0\}) \ : \ a \in]0,1[, \ A \in \mathbb{R}^*\}$$

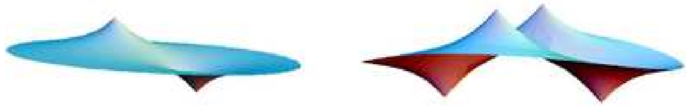

FIGURE 6. Examples of surfaces in Γ_n^P for $n=1$ and $n=3$

The surfaces in Γ_n^P are invariant under a vertical rotation followed by a horizontal symmetry. To see this consider the map S_1 defined above, $S_1(z,w) = (z, e^{\frac{2\pi i}{n+1}} w)$. Then, for any Weierstrass data of a maximal surface in Γ_n^P, (g, φ_3), given as in Equation (3), we have

$$g \circ S_1 = e^{\frac{2\pi i k}{n+1}} g \quad \text{and} \quad S_1^*(\varphi_3) = -\varphi_3$$

Therefore, reasoning as before we have that the surface is invariant under a vertical rotation of angle $\frac{2\pi k}{n+1}$ followed by a symmetry with respect to the origin, or equivalently, a vertical rotation of angle $\frac{2\pi}{n+1}$ followed by a horizontal symmetry.

In addition, $S_2(z,w) = (\overline{z}, \overline{w})$ induces a symmetry with respect to the x_1−axis in the surface.

REMARK 3. *We have seen that the group of symmetries of the surfaces in* Γ_n^C *and* Γ_n^P *contains at least* $2n+2$ *elements. Using an uniqueness result due to Klyachin* [**7**] *it can be proved that, the symmetry group of any complete maximal surface with* $n+1$ *singularities in* $\mathbb{L}^3$ *has at most* $2n+2$ *elements, and any surface having exactly* $2n+2$ *symmetries must lie (up to Lorentzian isometry) in one of these two families.*

4.2. Examples in $\mathbb{L}^3/\langle T \rangle$. For any $n \in \mathbb{N}$ odd, and $\alpha = (a_1, \ldots, a_{n+1}) \in (\mathbb{R}^+)^{n+1}$, $a_j < a_{j+1}$, label Δ_α as the compact genus n Riemann surface associated to the function $\sqrt{\prod_{j=1}^{n+1}(z^2 - a_j^2)}$,

$$\Delta_\alpha = \{(z,w) \in \overline{\mathbb{C}}^2 \ : \ w^2 = \prod_{j=1}^{n+1}(z^2 - a_j^2)\}$$

As in the case of the surfaces $\Sigma_{n,a}$, we can see the surface Δ_α as a branched covering of the Riemann sphere $\overline{\mathbb{C}}$. In this case the number of sheets is two and we cut along curves joining a_j with a_{j+1} and $-a_{j+1}$ with $-a_j$, $j = 1, \ldots, n$, (see Figure 7).

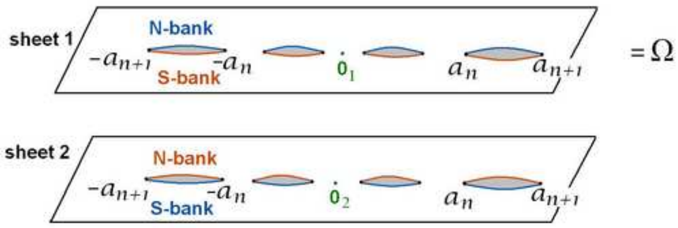

FIGURE 7. A model for the surface Δ_α

Let $J = J_\alpha : \Delta_\alpha \to \Delta_\alpha$ the antiholomorphic involution given by $J(z,w) = (\overline{z}, -\overline{w})$. The fixed points of J are the Jordan curves $\gamma_j^1 = \{(z,w) \in \Delta_\alpha \ : \ z \in [a_j, a_{j+1}]\}$ and $\gamma_j^2 = \{(z,w) \in \Delta_\alpha \ : \ z \in [-a_{j+1}, -a_j]\}$, $j = 1, \ldots, n+1$. Moreover, $\Delta_\alpha \setminus \cup_{i,j}\gamma_j^i$ has two connected components, namely Ω_α and $J_\alpha(\Omega_\alpha)$, each one of them corresponding to a single-valued branch of w (that is, in the above model, each one of the two sheets).

For simplicity, we label $\infty_1 \in \Omega_\alpha$ and $\infty_2 \in J(\Omega_\alpha)$ the two points in Δ_α with $z = \infty$. Likewise, $0_1 \in \Omega_\alpha$ and $0_2 \in J(\Omega_\alpha)$ are the two points with $z = 0$.

For any $A \in \mathbb{R}^*$, consider the following meromorphic data on Δ_α,

$$g_\alpha = \frac{w - \prod_{j=1}^{n+1}(z-a_j)}{w + \prod_{j=1}^{n+1}(z-a_j)} \quad \text{and} \quad \varphi_3^{\alpha,A} = A\frac{\prod_{j=1}^{n+1}(z+a_j) - \prod_{j=1}^{n+1}(z-a_j)}{z\,w}dz \tag{4}$$

As $a_j \in \mathbb{R}$, it is straightforward that

$$g \circ J = 1/\overline{g} \quad \text{and} \quad J^*(\varphi_3^{\alpha,A}) = -\overline{\varphi_3^{\alpha,A}} \tag{5}$$

Observe that

$$(w) = \frac{a_1 \cdot \ldots a_{n+1} \cdot (-a_1) \cdot \ldots \cdot (-a_{n+1})}{(\infty_1)^{n+1} \cdot (\infty_2)^{n+1}}$$

$$\text{and} \quad (dz) = \frac{a_1 \cdot \ldots a_{n+1} \cdot (-a_1) \cdot \ldots \cdot (-a_{n+1})}{(\infty_1)^2 \cdot (\infty_2)^2}$$

where we identify $a_j \equiv (a_j, 0)$ and $-a_j \equiv (-a_j, 0)$.

Taking into account that n is odd and Equation (5), it is not hard to see that the divisors for the Weierstrass data defined above are

$$(g_\alpha) = \frac{\infty_1 \cdot 0_1 \cdot q_1 \cdot \ldots \cdot q_{n-1}}{\infty_2 \cdot 0_2 \cdot J(q_1) \cdot \ldots \cdot J(q_{n-1})} \quad \text{and} \quad (\varphi_3^{\alpha,A}) = q_1 \cdot \ldots \cdot q_{n-1} \cdot J(q_1) \cdot \ldots \cdot J(q_{n-1})$$

where $q_1, \ldots, q_{n-1} \in \Omega_\alpha$, not necessarily distinct.

Moreover, if we compute the residue at 0_1 of $\Phi_{\alpha,A} = (\varphi_1^{\alpha,A}, \varphi_2^{\alpha,A}, \varphi_3^{\alpha,A})$, where $\varphi_1^{\alpha,A}$ and $\varphi_2^{\alpha,A}$ are given as in Theorem 1, we get

$$\text{Residue}_{0_1}(\Phi_{\alpha,A}) = (i\lambda, \lambda, 0) \quad \text{for some} \lambda \in \mathbb{R}^*$$

Hence, if we label T as the translation of vector

$$v = \text{Re}\big(2\pi i \,\text{Residue}_{0_1}(\varphi_1^{\alpha,A}, \varphi_2^{\alpha,A}, \varphi_3^{\alpha,A})\big) = (-2\pi\lambda, 0, 0)$$

and define

$$X = X_{\alpha,A} : \overline{\Omega} \setminus \{\infty_1, 0_1\} \longrightarrow \mathbb{L}^3/\langle T \rangle, \quad X = \text{Re}\int_{a_1} (\varphi_1^{\alpha,A}, \varphi_2^{\alpha,A}, \varphi_3^{\alpha,A}) \tag{6}$$

we obtain a complete maximal surface with $n+1$ singularities and with ends asymptotic to half horizontal cylinders (see figure 8 for an example with $n = 1$).

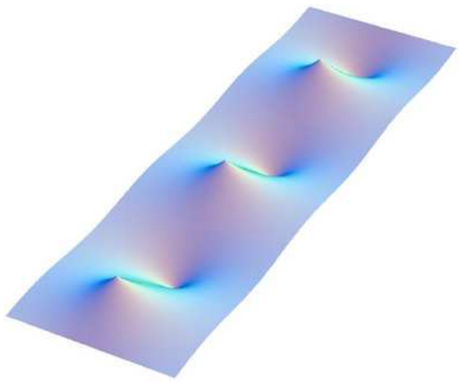

FIGURE 8. The lifted singly periodic surface of a maximal surface in $\mathbb{L}^3/\langle T \rangle$ with two singularities

To see the symmetries of the maximal surfaces constructed above, consider the antiholomorphic involution $S_1(z,w) = (\overline{z}, \overline{w})$ on Δ_α. Observe that S_1 fixes the points $\pm a_1, \ldots, \pm a_{n+1}$ and thus, for any g and φ_3 defined as in Equation (4), we have

$$g \circ S_1 = \overline{g} \quad \text{and} \quad S_1^*(\varphi_3) = \overline{\varphi_3}$$

As a consequence, the map $X = (X_1, X_2, X_3)$ given in Equation (6) satisfies that $X \circ S_2 = (-X_1, X_2, X_3)$, and so the resulting surface has all its singular points in $\{x_1 = 0\} \subset \mathbb{L}^3/\langle T\rangle$ (remember that T is a translation in the x_1-axis), and is invariant by the symmetry with respect to this plane.

Secondly, consider the holomorphic map $S_2(z,w) = (-z, w)$. Reasoning as above, from $g \circ S_2 = -g$ and $S_2^*(\varphi_3) = -\varphi_3$ (recall that n is odd) we infer that

$$\begin{aligned} X \circ S_2 &= (X_1, X_2, -X_3) + \mathrm{Re} \int_{-a_1}^{a_1} \Phi = \\ &= (X_1, X_2, -X_3) + v/2 \end{aligned}$$

That is to say, the surface is invariant under the symmetry with respect to $\{x_3 = 0\}$ followed by a translation of vector $v/2$.

4.3. Examples in $\mathbb{L}^3/\langle T_1, T_2\rangle$. Consider $a, b \in]-1, 1[$, $a \notin \{\pm b\}$, and label $\Upsilon_{a,b}$ as the compact genus 3 Riemann surface associated to the function $\sqrt{\frac{(z^2-a^2)(z^2-b^2)}{(a^2z^2-1)(b^2z^2-1)}}$,

$$\Upsilon_{a,b} = \{(z,w) \in \overline{\mathbb{C}}^2 \ : \ w^2 = \frac{(z^2-a^2)(z^2-b^2)}{(a^2z^2-1)(b^2z^2-1)}\}$$

Let $J = J_{a,b} : \Upsilon_{a,b} \to \Upsilon_{a,b}$, $J(z,w) = (1/\overline{z}, 1/\overline{w})$. Label γ_1 and γ_2 as the two curves in $\Upsilon_{a,b}$ corresponding to $|z| = 1$, these curves are the fixed points of J and $\Upsilon_{a,b} \setminus \cup_{i=1,2}\gamma_i = \Omega_{a,b} \cup J(\Omega_{a,b})$, where $\Omega_{a,b} = \{(z,w) \in \Upsilon_{a,b} \ : \ |z| < 1\}$.

As in the previous examples, write $z^{-1}(0) = \{0_1, 0_2\}$ and $z^{-1}(\infty) = \{\infty_1, \infty_2\}$, $0_1, \infty_1 \in \Omega_{a,b}$. Finally, define

$$g_{a,b} = z \quad \text{and} \quad \varphi_3^{a,b,A} = A\frac{z dz}{w(a^2z^2-1)(b^2z^2-1)} \tag{7}$$

where $A \in \mathbb{R}^*$. Since the associated divisors for these data are given by

$$(g_{a,b}) = \frac{0_1 \cdot 0_2}{\infty_1 \cdot \infty_2} \quad \text{and} \quad (\varphi_3^{a,b,A}) = 0_1 \cdot 0_2 \cdot \infty_1 \cdot \infty_2$$

it follows from Theorem 3 that

$$X = X_{a,b,A} : \overline{\Omega_{a,b}} \to \mathbb{L}^3/G, \qquad X = \mathrm{Re} \int_a \Phi_{a,b,A},$$

is a complete maximal surface with two singularities in $\mathbb{L}^3/G$, where G is the group generated by the two translations of vectors

$$v_1 = \mathrm{Re}(2\int_{-a}^{a} \Phi_{a,b,A}) = (0, \lambda, 0) \quad \text{and} \quad v_2 = \mathrm{Re}(2\int_a^b \Phi_{a,b,A}) = (\mu, 0, 0)$$

Moreover, the antiholomorphic maps $S_1(z,w) = (-\overline{z}, \overline{w})$ and $S_2(z,w) = (\overline{z}, \overline{w})$ induce symmetries of the surface, namely, $\tilde{S}_1(x_1, x_2, x_3) = (x_1, -x_2, x_3) + v_1/2$ and $\tilde{S}_2(x_1, x_2, x_3) = (-x_1, x_2, x_3)$.

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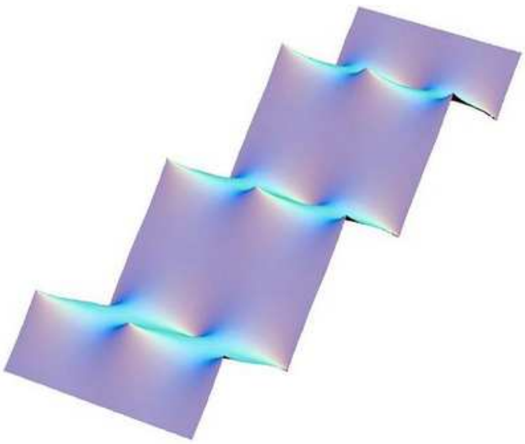

FIGURE 9. Lifted doubly periodic surface of a maximal surface in $\mathbb{L}^3/\langle T_1, T_2\rangle$ with two singularities.

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