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Almost conformally Einstein manifolds and obstructions

A. Rod Gover

Abstract. A Riemannian or pseudo-Riemannian (or conformal) structure is conformally Einstein if and only if there is a suitably generic parallel section of a certain vector bundle – the so-called standard conformal tractor bundle. We show that this characterisation leads to a systematic approach to constructing obstructions to conformally Einstein metrics. Relaxing the requirement that the parallel tractor field be generic gives a natural generalisation of the Einstein equations.

1. Introduction

In these partly expository notes we review some recent results concerning obstructions for metrics (on manifolds of dimension $n \geqslant 3$) to be conformally Einstein and discuss the relationship between these and a generalisation of the Einstein condition. Given an initial metric we show that the equations for a conformally related metric to be Einstein, when prolonged and written as a first order system, give a conformally invariant connection on a certain vector bundle. This is the so-called (standard conformal) tractor connection and bundle respectively. A solution of the original equations, that is an Einstein metric, is then equivalent to a suitably generic parallel section of the tractor bundle. Relaxing the condition that the parallel tractor be generic then leads to an obvious generalisation of the Einstein equations. These generalised structures, viz. Riemannian or pseudo-Riemannian manifolds equipped with a parallel standard tractor, are termed almost Einstein structures. A closely related term is almost conformally Einstein: we say a Riemannian or pseudo-Riemannian manifold is almost conformally Einstein if on an open dense subset it is conformally related to Einstein metric. In sections 2 and 3 we include some new (although elementary) analysis of the zero set of the scaling function on almost Einstein spaces, that is the set where the Einstein metric is singular. There is no attempt to be complete in this treatment. Rather it is intended to merely point out some of the most obvious properties of parallel tractors.

The problem of finding necessary and sufficient conditions for a Riemannian or pseudo-Riemannian manifold to be locally conformally related to an Einstein metric has been studied for almost 100 years. Early results date back to the work of Brinkman [**2, 3**] and Schouten [**19**]. Substantial progress was made by Szekeres in the 1963 [**20**] and then Kozameh, Newman and Tod (KNT) [**16**] in 1980's. One approach is to seek invariants, polynomial in the Riemannian curvature and its covariant derivatives, that give a sharp obstruction to conformally Einstein metrics

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in the sense that they vanish if and only if the metric concerned is conformally related to an Einstein metric. For example in dimension 3 it is well known that this problem is solved by the Cotton tensor, which is a certain tensor part of the first covariant derivative of the Ricci tensor. This tensor is also a sharp obstruction to local conformal flatness. So locally 3-manifolds are conformally Einstein if and only if they are conformally flat. In [**16**] KNT described conformal invariants that gave sharp obstructions on 4 manifolds given a restriction that the class of metrics to be considered is suitably generic. One component of the KNT system is the Bach tensor. For manifolds of higher even dimension there is a natural analogue of the Bach tensor; this is the Fefferman-Graham obstruction tensor. This trace-free, divergence-free (density valued) 2-tensor is of considerable current interest, in part due to its relationship to Branson's Q-curvature [**15**]. In the original work of Fefferman and Graham this tensor $\mathcal{B}$ arose as an obstruction to their ambient metric construction [**8**]. They observed that for conformally Einstein manifolds the ambient construction works to all orders and hence $\mathcal{B}$ vanishes identically (see [**15**] where the equivalent argument in terms of the Poincaré metric is given explicitly).

One point of this article is to illustrate that the tractor bundle and the relationship of parallel tractors to Einstein metrics leads to a a uniform treatment of these obstructions. We review here aspects of the recent work [**12**] of the author with Nurowski where it is observed that many of the classical obstructions to conformally Einstein metrics, including the KNT invariants, can be recovered via the integrability conditions for a parallel tractor. In fact the treatment via tractors leads to new obstructions and in particular a system of invariants that gives a sharp obstruction to conformally Einstein metrics for the class of metrics that are weakly generic (this means that, viewed as a bundle map $TM \to \otimes^3 TM$, the Weyl curvature is injective). This is a broader class of metrics than treated previously and in particular extends considerably the recent results of Listing [**17**]. Following [**14**] we then observe that essentially the same ideas lead to a simple and direct proof that the Fefferman-Graham tensor $\mathcal{B}$ vanishes for conformally Einstein metrics. Aside from these new results for Riemannian and pseudo-Riemannian geometry a point that should be made is that the ideas here for the construction and study of obstructions via parallel tractor fields adapt easily to other equations and to other structures. An obvious example is projective structures where one could follow these ideas to give obstructions to Ricci flat scales. There is also evidence that the related ideas will have a useful role in CR geometry [**6**], [**11**].

Of course, by continuity, conformal invariants which vanish for conformally Einstein metrics also vanish for almost conformally Einstein metrics. Thus in any case where such invariants give a sharp obstruction the equations they determine are naturally viewed as the equations for almost conformally Einstein metrics.

2. Einstein metrics and the tractor bundle

Let M be a smooth manifold, of dimension $n \geqslant 3$, equipped with a Riemannian or pseudo-Riemannian metric g_{ab}. Here and throughout we employ Penrose's abstract index notation [**18**] and indices should be assumed abstract unless otherwise indicated. We write $\mathcal{E}^a$ to denote the space of smooth sections of the tangent bundle on M, and $\mathcal{E}_a$ for the space of smooth sections of the cotangent bundle. (In fact we will often use the same symbols for the corresponding bundles, and also in other situations we will often use the same symbol for a given bundle and its space of smooth sections, since the meaning will be clear by context.) We write $\mathcal{E}$ for the space of smooth functions and all tensors considered will be assumed smooth without further comment. An index which appears twice, once raised and once lowered, indicates a contraction. The metric g_{ab} and its inverse g^{ab} enable the identification of $\mathcal{E}^a$ and $\mathcal{E}_a$ and we indicate this by raising and lowering indices in the usual way.

Recall that the Levi-Civita connection ∇_a is the unique torsion free connection preserving the metric g_{ab}. The Riemann curvature tensor $R_{ab}{}^c{}_d$ is given by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = R_{ab}{}^c{}_d V^d \quad \text{where} \quad V^c \in \mathcal{E}^c.$$

This can be decomposed into the totally trace-free *Weyl curvature* C_{abcd} and the symmetric *Schouten tensor* P_{ab} according to

$$R_{abcd} = C_{abcd} + 2g_{c[a}\mathsf{P}_{b]d} + 2g_{d[b}\mathsf{P}_{a]c}.$$

Thus P_{ab} is a trace modification of the Ricci tensor $\mathrm{Ric}_{ab} = R_{ca}{}^c{}_b$:

$$R_{ab} = (n-2)\mathsf{P}_{ab} + \mathsf{J}\mathsf{g}_{\mathsf{ab}}, \qquad \mathsf{J} := \mathsf{P}^{\mathsf{a}}{}_{\mathsf{a}}.$$

Recall that a metric is *Einstein* if the Ricci tensor is pure trace. Equivalently this means the Schouten tensor is pure trace,

$$\mathsf{P}_{ab} - \frac{1}{n}\mathsf{J}\mathsf{g}_{\mathsf{ab}} = 0.$$

Given a (pseudo-)Riemannian metric g we are interested in the question of whether g is *conformally Einstein*. That is whether there is some metric $\widehat{g}$, conformally related to g

$$\widehat{g} = e^{2\omega} g \quad \omega \in \mathcal{E}$$

which is Einstein. We write $\widehat{\mathsf{P}}_{ab} - \frac{1}{n}\widehat{\mathsf{J}}\widehat{g}_{ab} = 0$ where the hatted quantities refer to the metric $\widehat{g}$. From the formula for the Levi-Civita connection in terms of the metric we can calculate the conformal variation of the Schouten tensor. From this we obtain that, in terms of the metric g_{ab}, the condition for $\widehat{g}_{ab}$ to be Einstein is

$$\mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{n} T g_{ab} = 0, \tag{1}$$

where

$$\Upsilon := d\omega$$

and $T = \mathsf{J} - \nabla^{\mathsf{a}}\Upsilon_{\mathsf{a}} + \Upsilon^{\mathsf{a}}\Upsilon_{\mathsf{a}}$.

As an equation on the function ω, the system (1) is clearly overdetermined and we do not expect solutions in general. We will see over the following pages that there is systematic way to construct obstructions. First note that the system (1) can be linearised by the change of variables $\omega \mapsto \sigma := e^{-\omega}$. The resulting equivalent equation on the positive function σ is

$$\mathrm{TF}(\nabla_a \nabla_b + \mathsf{P}_{ab})\sigma = 0,$$

where "TF" indicates that we take the trace-free part. Next we replace this equation with the equivalent first order system:

$$\nabla_a \sigma - \mu_a = 0, \quad \text{and} \quad \nabla_a \mu_b + \mathsf{P}_{ab}\sigma + g_{ab}\rho = 0, \tag{2}$$

where μ_a is a 1-form field and ρ a function. Differentiating the second of these again and contracting yields an equation on ρ:

$$\nabla_a \rho - \mathsf{P}_{ab}\mu^b = 0.$$

The system has closed up linearly. Thus the original equation is equivalent to a connection and the equation for a parallel section of this. Let $\mathbb{I} := (\sigma, \mu_b, \rho) \in \mathcal{E} \oplus \mathcal{E}_b \oplus \mathcal{E}$ then

$$(1) \Leftrightarrow \nabla \mathbb{I} = 0,$$

with the qualification that σ is a positive function, and where

$$\nabla_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab}\rho + \mathsf{P}_{ab}\sigma \\ \nabla_a \rho - \mathsf{P}_{ab}\mu^b \end{pmatrix}. \tag{3}$$

Note that the formula gives a sum of the trivial extension of the Levi-Civita connection with a bundle endomorphism of $\mathcal{E} \oplus \mathcal{E}_b \oplus \mathcal{E}$ and so this is a connection on $\mathcal{E} \oplus \mathcal{E}_b \oplus \mathcal{E}$. Through context, no confusion should arise from the use of the symbol ∇ to denote this new connection as well as the Levi-Civita connection.

We can be more precise concerning the relationship between (1) and the prolonged system $\nabla \mathbb{I} = 0$. Note that if $\mathbb{I}$ is parallel then we recover the system (2). From the first of these and a trace of the second we have, respectively, $\mu_a = \nabla_a \sigma$ and $\rho = -\frac{1}{n}(\Delta + \mathsf{J})\sigma$ (here $\Delta = \nabla^a \nabla_a$). Thus we can say that there is a 1-1 correspondence between scales σ such that $\widehat{g} = \sigma^{-2} g$ is Einstein and sections $\mathbb{I} = (\sigma, \mu_a, \rho)$ of the bundle $\mathcal{E} \oplus \mathcal{E}_a \oplus \mathcal{E}$ with σ nowhere vanishing (note we do not need σ positive for this statement). The mapping from Einstein scales to parallel sections of $\mathcal{E} \oplus \mathcal{E}_b \oplus \mathcal{E}$ is given by $\sigma \mapsto \frac{1}{n} D\sigma$ where $D\sigma := (n\sigma, n\nabla_a \sigma, -(\Delta + \mathsf{J})\sigma)$.

A main point that we wish to come to is that from the connection and its curvature it is easy to construct obstructions to the equation (1) for a metric to be conformally Einstein. Before we do this let us observe the conformal invariance of the connection. Of course the question of whether there is a metric, conformally related to g, that is Einstein is tautologically an issue of conformal structure.

Recall that a *conformal structure* on M of signature (p, q) is an equivalence class $[g]$ of pseudo–Riemannian metrics of signature (p, q) on M, with two metrics being equivalent if and only if one is obtained from the other by multiplication with a positive smooth function. Equivalently a conformal structure is a smooth ray subbundle $\mathcal{Q} \subset S^2 T^* M$ whose fibre over x consists of the values of g_x for all metrics g in the conformal class. In this picture a metric in the conformal class is a section of $\mathcal{Q}$.

We can view $\mathcal{Q}$ as a principal bundle $\pi : \mathcal{Q} \to M$ with structure group $\mathbb{R}_+$, and so there are natural line bundles on $(M, [g])$ induced from the irreducible representations of $\mathbb{R}_+$. For $w \in \mathbb{R}$, we write $\mathcal{E}[w]$ for the line bundle induced from the representation of weight $-w/2$ on $\mathbb{R}$ (that is, $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \mathrm{End}(\mathbb{R})$). Clearly the fibres inherit an ordering from $\mathbb{R}$. A section of $\mathcal{E}[w]$ corresponds to a real-valued function f on $\mathcal{Q}$ with the homogeneity property $f(x, \Omega^2 g) = \Omega^w f(x, g)$, where Ω is a positive function on M, $x \in M$, and g is a metric from the conformal class $[g]$. Given a vector bundle $\mathcal{V}$ or section space thereof we write $\mathcal{V}[w]$ as a shorthand for $\mathcal{V} \otimes \mathcal{E}[w]$.

There is a tautological function $\boldsymbol{g}$ on $\mathcal{Q}$ taking values in $S^2 T^* M$, namely the function which assigns to the point $(x, g_x) \in \mathcal{Q}$ the metric g_x at x. This is homogeneous of degree 2, since $\boldsymbol{g}(x, s^2 g_x) = s^2 g_x$ and so $\boldsymbol{g}$ is equivalent to a section of $\mathcal{E}_{(ab)}[2]$ that we denote by the same symbol and term the *conformal metric.* (Note $(a \cdots b)$ means the symmetric part over the enclosed indices.) Then a metric g from the conformal class is determined by a non-vanishing section σ of $\mathcal{E}[1]$ (a so-called *conformal scale*) via the equation $g = \sigma^{-2}\boldsymbol{g}$. The conformal metric $\boldsymbol{g}_{ab}$ has an inverse $\boldsymbol{g}^{ab}$ in $\mathcal{E}^{(ab)}[-2]$.

By using density-valued tensor fields and $\boldsymbol{g}_{ab}$ and its inverse to raise and lower indices and so forth we can avoid carrying around conformal factors in calculations. Otherwise the calculations are almost formally identical to the calculations where one instead picks a "reference metric". For example in these terms the conformal scale $\sigma \in \mathcal{E}[1]$ gives an Einstein metric if and only if it solves the equation

$$\mathrm{TF}(\nabla_a\nabla_b + \mathsf{P}_{ab})\sigma = 0. \tag{4}$$

Here the Schouten tensor P and the Levi-Civita connection ∇ are constructed from any metric g from the conformal class and ∇ is extended to act on densities (and density valued tensors) through the trivialisation of the density bundles given by the choice g. It is easily verified explicitly that this equation is *conformally invariant.* Indeed the left-hand-side is conformally invariant; if $\widehat{g}$ is another metric from the conformal class then we have $\mathrm{TF}(\widehat{\nabla}_a\widehat{\nabla}_b+\widehat{\mathsf{P}}_{ab})\sigma = \mathrm{TF}(\nabla_a\nabla_b+\mathsf{P}_{ab})\sigma$. Thus if we begin with (4), in our argument above, then the corresponding connection (3) is conformally invariant. This is the so-called (conformal) tractor connection **[7, 21, 1]** (which is an induced connection equivalent to the normal conformal Cartan connection – see **[5]**). Note that the connection really acts on the (standard) *tractor bundle* $\mathcal{T}$ (or $\mathcal{E}^A$ in abstract index notation) which may be defined as the quotient of $J^2\mathcal{E}[1]$ by the image of $\mathcal{E}_{(ab)_0}[1]$ in $J^2\mathcal{E}[1]$ through the jet exact sequence at 2-jets. Here $\mathcal{E}_{(ab)_0}[1]$ means the trace-free symmetric part of $\mathcal{E}_{ab}\otimes\mathcal{E}[1]$. By construction then there is a canonical surjection $\mathcal{T} \to J^1\mathcal{E}[1]$. Composing on the left with the jet projection $J^1\mathcal{E}[1] \to \mathcal{E}[1]$ we obtain a canonical map $X_A : \mathcal{E}^A \to \mathcal{E}[1]$. The invariant version of the operator D above (that we again denote by D) is just the composition involving the universal 2-jet operator $j^2 : \mathcal{E}[1] \to J^2\mathcal{E}[1]$ followed by the canonical projection $J^2\mathcal{E}[1] \to \mathcal{T}$. Via a choice of metric g, and the Levi-Civita connection it determines, we may use the formula given for ($1/n$ times) D above, viz $\sigma \mapsto (\sigma, \nabla_a\sigma, -\frac{1}{n}(\Delta+\mathsf{J})\sigma)$ to split the tractor bundle $[\mathcal{E}^A]_g = \mathcal{E}[1]\oplus\mathcal{E}_a[1]\oplus\mathcal{E}[-1]$. In the discussions that follow we will often use the splitting determined by some choice of metric without explicit comment. In terms of such a splitting the conformally invariant tractor connection is then exactly given by the formula (3). This connection preserves a conformally invariant signature $(p+1, q+1)$ *tractor metric* h which, in terms of the splitting mentioned, is given by $h(V,V) = 2\sigma\rho + \boldsymbol{g}^{ab}\mu_a\mu_b$, if $[V]_g = (\sigma, \mu_a, \rho)$. For further details on the tractor calculus from this point of view see **[4]**. The relationship between parallel tractors and conformally Einstein metrics, while implicit in **[1]**, was probably first observed and treated in detail by Paul Gauduchon in **[9]**. Let us summarise our observations and results as a theorem (cf. theorem 3.1 in **[12]**).

THEOREM 1. *On a conformal manifold $(M,[g])$ there is a 1-1 correspondence between conformal scales $\sigma \in \mathcal{E}[1]$, such that $g = \sigma^{-2}\boldsymbol{g}$ is Einstein, and parallel standard tractors $\mathbb{I}$ with the property that $X_A\mathbb{I}^A$ is nowhere vanishing. The mapping from Einstein scales to parallel tractors is given by $\sigma \mapsto \frac{1}{n}D_A\sigma$ while the inverse is $\mathbb{I}^A \mapsto X_A\mathbb{I}^A$.*

From this theorem it is natural to investigate the interpretation of parallel tractors in general, i.e. without assuming $X_A\mathbb{I}^A$ is non-vanishing. If $\nabla_a\mathbb{I}_B = 0$ at $x \in M$, then at x we have $\mathbb{I}_B = \frac{1}{n}D_B\sigma$. So for parallel $\mathbb{I}^A$, it follows from the definition of D that $X_A\mathbb{I}^A$ vanishes on an open set if and only if $\mathbb{I}^A$ vanishes on the same open set. So suppose $\mathbb{I}^A \neq 0$ is parallel (and hence non-vanishing). Then $\sigma := X_A \mathbb{I}^A$ may vanish at points or on suitable closed sets, but the 2-jet of σ is non-vanishing. In particular, σ is non-vanishing on an open dense set. From this observation and the Theorem, it follows that a manifold admits a parallel tractor if and only if there is a section σ of $\mathcal{E}[1]$ such that on some open dense subset $\sigma^{-2}\boldsymbol{g}$ is Einstein. Since, for a tractor $\mathbb{I}$, $\sigma = X_A \mathbb{I}^A$ determines a metric by $g = \sigma^{-2}\boldsymbol{g}$ let us call any point where σ vanishes a point of scale singularity.

Let us say a metric g (or a conformal structure $[g]$) on a manifold M is *almost conformally Einstein* if it is conformally Einstein on an open dense subset of M. The Poincaré metrics of **[8]** are examples of such metrics and it is straightforward to construct other examples **[11]**. This is a natural generalisation of the Einstein condition since any natural conformally invariant tensor which vanishes for conformally Einstein metrics also vanishes, by dint of continuity, for almost conformally Einstein metrics. The requirement that a metric admit a parallel standard tractor (which is ostensibly a stronger requirement than being almost conformally Einstein) is an especially natural condition from two points of view. Firstly although it is a conformal condition each parallel tractor determines an actual Einstein metric on an open dense subset of the manifold. Secondly the equation of parallel transport for the tractor field gives a canonical extension of the Einstein equations through any scale singularities. Hence a Riemannian or pseudo-Riemannian structure (or a conformal structure $[g]$) equipped with a parallel standard tractor will be termed an *almost Einstein* structure.

Let us make a few simple observations concerning almost conformally Einstein structures. Suppose that $(M, [g])$ is an almost conformally Einstein manifold such that the (maximal) conformally Einstein subset is connected. Then it has a parallel tractor on a connected open dense subset of M and thus (extending this by parallel transport) a parallel tractor on M. For example Fefferman and Graham's Poincaré metrics admit a parallel tractor and so are almost Einstein. Note that when we have a parallel tractor $\mathbb{I}$, then the constant function $-\frac{n}{2}\mathbb{I}^A\mathbb{I}_A$ extends the scalar curvature J across points of scale singularity.

Since $\mathbb{I}$ parallel implies $\mathbb{I} = \frac{1}{n} D\sigma$ for some density of weight 1, an obvious question is whether, at any point x, we can have $j^1_x\sigma = 0$. In fact this can happen since on conformally flat structures the tractor connection is flat; we may set $\mathbb{I} = (0,0,1)$ at some point x and then extend (locally at least) to a parallel field by parallel transport. We will see below (see the comment following Theorem 4) that if $\mathbb{I} = \frac{1}{n}D\sigma$ (non-trivial) is parallel and $j^1_x\sigma = 0$ then the Weyl curvature must vanish at x. On top of this there there are other severe restrictions.

Proposition 1. *If $\mathbb{I} = \frac{1}{n}D\sigma \neq 0$ is parallel and $j^1_x\sigma = 0$ then $\sigma \neq f_1 f_2 \tilde{\sigma}$ where $\tilde{\sigma}$ is a section of $\mathcal{E}[1]$, and f_1 and f_2 are functions which vanish at x.*

If $\mathbb{I} = \frac{1}{n}D\sigma \neq 0$ is parallel then $j^1\sigma$ can only vanish at isolated points. In Riemannian signature if $\mathbb{I} = \frac{1}{n}D\sigma \neq 0$ is parallel and $j^1_x\sigma = 0$ then there is a neighbourhood of x such that, in this neighbourhood, σ is non-vanishing away from x.

Proof. For the first statement suppose that $\mathbb{I} = \frac{1}{n}D\sigma \neq 0$ is parallel and $j^1_x\sigma = 0$. Then (without loss of generality, by a constant rescaling we may assume) $\mathbb{I}(x) = (0,0,-1)$. Thus from (3) we have $\nabla_a\nabla_b\sigma(x) = g_{ab}(x)$ (whereas $\nabla_a\nabla_b(f_1f_2\tilde{\sigma})$ has rank at most 2 at x).

For the second claim suppose that $\mathbb{I} = \frac{1}{n}D\sigma \neq 0$ is parallel and $j^1\sigma$ vanishes along a curve $x(t)$ through x. Then if t^a is the tangent field we have $t^a\nabla_a\nabla_b\sigma = 0$ along the curve and in particular at x. But at x, as observed, we have $\nabla_a\nabla_b\sigma = g_{ab}$ and so $t^a\nabla_a\nabla_b\sigma = t^a g_{ab} \neq 0$ and we have a contradiction.

Finally let us pick a metric g from the conformal class and with this trivialise the density bundles. If $\mathbb{I} = \frac{1}{n}D\sigma \neq 0$ is parallel and $j^1_x\sigma$ vanishes then $\nabla_a\nabla_b\sigma(x) = g_{ab}(x)$ and so, in terms of Riemann normal coordinates based at x, the first non-vanishing term in the Taylor series for σ (based at x) is $g_{ij}x^ix^j$. □

3. Obstructions

There is an obvious integrability condition for the existence of a parallel tractor:

$$\nabla_a\mathbb{I}^C = 0 \Rightarrow [\nabla_a, \nabla_b]\mathbb{I}^C = \Omega_{ab}{}^C{}_D\mathbb{I}^D = 0$$

where $\Omega_{ab}{}^C{}_D$ is the curvature of the tractor connection. So $\Omega_{ab}{}^C{}_D\mathbb{I}^D = 0$ is a necessary condition for a metric to be conformal to Einstein. It is an elementary exercise to interpret this in terms of tensors. An easy calculation establishes that

$$\Omega_{ab}{}^C{}_D = \begin{pmatrix} 0 & 0 & 0 \\ A^c{}_{ab} & C_{ab}{}^c{}_d & 0 \\ 0 & -A_{dab} & 0 \end{pmatrix} \tag{5}$$

where $A_{abc} := 2\nabla_{[b}\mathsf{P}_{c]a}$ is the *Cotton tensor* for the metric g giving the splitting of the tractor bundles. Thus, if

$$\begin{pmatrix} \sigma \\ \mu^d \\ \rho \end{pmatrix}$$

gives $\mathbb{I}^D$ in the scale g then

$$\Omega_{ab}{}^C{}_D\mathbb{I}^D = 0 \Leftrightarrow \sigma A^c{}_{ab} + \mu^d C_{ab}{}^c{}_d = 0. \tag{6}$$

(For the implication $\Leftarrow$ note that $\sigma A^c{}_{ab} + \mu^d C_{ab}{}^c{}_d = 0$ implies $\mu^d A_{dab} = 0$ from the symmetries of the Weyl curvature.) Where σ is non-vanishing this is

$$\text{[C]} \qquad A_{dab} + K^c C_{cdab} = 0$$

with $K_c = -\sigma^{-1}\mu_c$. If we require in addition that K_c is exact then this is the C-space equation of [**20, 16**], the equation that is satisfied if and only if the Cotton tensor vanishes in some scale; $\Omega_{ab}{}^C{}_D\mathbb{I}^D = 0$ generalises this.

We can now differentiate to obtain more integrability conditions. $\Omega_{ab}{}^C{}_D\mathbb{I}^D = 0$ implies

$$0 = \mathbb{I}^D\nabla_e\Omega_{ab}{}^C{}_D + \Omega_{ab}{}^C{}_D\nabla_e\mathbb{I}^D$$

so if $\mathbb{I}^D$ is parallel then

$$\text{[D]} \qquad \mathbb{I}^D\nabla_e\Omega_{ab}{}^C{}_D = 0\ .$$

Obviously we could continue along these lines. More abstractly the situation is this. If $\mathbb{I}$ is parallel then it is obviously fixed by the conformal holonomy group for the tractor connection. Thus the infinitesimal holonomy group fixes $\mathbb{I}$ and the Lie algebra of this is generated by the tractor derivatives of the tractor curvature. In general calculating even the infinitesimal holonomy is not possible. We shall see that in a rather general setting it suffices, for our purposes, to stop at [D]. Note that the pair $(\Omega_{bc}{}^D{}_E, \nabla_a\Omega_{bc}{}^D{}_E)$ simply recovers the 1-jet of the curvature at each point, and so the pair is conformally invariant.

Contracting [D] via g^{ea} gives $\mathbb{I}^D\nabla^a\Omega_{ab}{}^C{}_D = 0$ and via a short calculation we may re-express this by the equation

$$[\mathrm{B}] \qquad B_{ab} + (n-4)K^d K^c C_{dabc} = 0,$$

provided $X_A\mathbb{I}^A$ is non-vanishing. Here

$$B_{ab} := \nabla^c A_{acb} + \mathsf{P}^{dc}C_{dacb}.$$

is the *Bach tensor* (which is conformally invariant in dimension 4). Since the pair

$$(\Omega_{bc}{}^D{}_E, \nabla_a\Omega_{bc}{}^D{}_E)$$

is conformally invariant it is clear that so is the system [C], [D] and hence so also the system [C], [B]. In dimension 4 the latter is exactly the system considered by Kozameh, Newman and Tod in [**16**]. In any dimension assuming the metric is sufficiently generic this generalisation of the KNT system is sufficient to determine whether or not a metric is conformally Einstein (see theorems 2.2 and 2.3 of [**12**]).

For the purposes of characterising conformally Einstein metrics or almost conformally Einstein metrics the system [C], [D] has more information. Let us say that a (pseudo-)Riemannian manifold is *weakly generic* if, at each $x \in M$, the only solution $V^d_x \in T_xM$ to

$$C_{abcd}V^d_x = 0 \ , \qquad \text{at } x \in M \ ,$$

is $V^d_x = 0$. We will say that a (pseudo-)Riemannian manifold is *almost weakly generic* if the only smooth vector field V^d solving

$$C_{abcd}V^d = 0 \ ,$$

is $V^d = 0$. The following is a generalisation of Theorem 3.4 of [**12**].

THEOREM 2. *A weakly generic conformal manifold is conformally Einstein if and only if there exists a non-vanishing tractor field $\mathbb{I}^A \in \mathcal{E}^A$ such that*

$$\begin{array}{rl} \mathbb{I}^E\Omega_{bcDE} = 0 & [\tilde{\mathrm{C}}] \\ \mathbb{I}^E\nabla_a\Omega_{bcDE} = 0 & [\tilde{\mathrm{D}}]. \end{array}$$

An almost weakly generic conformal manifold admits a parallel standard tractor if and only if there exists a non-vanishing tractor field $\mathbb{I}^A \in \mathcal{E}^A$ such that $[\tilde{\mathrm{C}}]$ *and* $[\tilde{\mathrm{D}}]$ *hold.*

PROOF. $\Rightarrow$: is clear for both statements.
$\Leftarrow$: We treat the first statement first. So let us assume the structure is weakly generic. Let (σ, μ^e, τ) be the components of $\mathbb{I}^E$. Following [**13**] we write Y^E and $Z^E{}_e$ for the injectors (determined by the choice metric giving the splitting) into the first two tractor slots. Since $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ we have

$$X^A : \mathcal{E}[-1] \to \mathcal{E}^A \quad Z^{Aa} : \mathcal{E}_a[1] \to \mathcal{E}^A \quad Y^A : \mathcal{E}[1] \to \mathcal{E}^A$$

so that we may write $\mathbb{I}^E = Y^E\sigma + Z^{Ed}\mu_d + X^E\tau$. Suppose that $X_A\mathbb{I}^A = \sigma$ vanishes at some point x. Then from (6) we have $\mu^d C_{abcd} = 0$ at x and so, since the conformal class is weakly generic, $\mu^d(x) = 0$. Thus $\mathbb{I}^E = \tau X^E$, at x, and $[\tilde{\mathrm{D}}]$ gives $X^E\nabla_a\Omega_{bcDE} = 0$ at x. But from (3) $\nabla_a X^E = Z^E{}_a$ and from (5) $X^E\Omega_{bcDE} = 0$, and so $Z_D{}^d C_{bcda} - X_D A_{abc} = Z^E{}_a\Omega_{bcDE} = 0$ at x. But this means $C_{bcda}(x) = 0$ which contradicts the assumption that the conformal class is weakly generic. So $X_A\mathbb{I}^A$ is non-vanishing.

Differentiating [$\tilde{\mathrm{C}}$] and using [$\tilde{\mathrm{D}}$] gives

$$\Omega_{ab}{}^{C}{}_{D}\nabla_e \mathbb{I}^D = 0.$$

Since g weakly generic $\Omega_{bc}{}^{D}{}_{E}$ must have rank at least n as a map $\Omega_{bc}{}^{D}{}_{E} : \mathcal{E}^{bc}{}_{D} \to \mathcal{E}_E$. On the other hand from [$\tilde{\mathrm{C}}$] and the formula for $\Omega_{bc}{}^{D}{}_{E}$ we have that $\mathbb{I}^E$ and (the linearly independent) $X^E = (0,0,1)$ are orthogonal to its range. So

$$\nabla_a \mathbb{I}^E = \alpha_a \mathbb{I}^E + \beta_a X^E,$$

for some 1-forms α_a and β_a. Differentiating again and alternating leads to $\nabla_{[a}\alpha_{b]} = 0$. So locally $\alpha = df$ (some function f) and $\tilde{\mathbb{I}}^E := e^{-f}\mathbb{I}^E$ satisfies

$$\nabla_a \tilde{\mathbb{I}}^E = \tilde{\beta}_a X^E, \quad \text{that is:}$$

$$\begin{aligned} &\nabla_a \tilde{\sigma} - \tilde{\mu}_a = 0 \\ &\nabla_a \tilde{\mu}_b + \boldsymbol{g}_{ab}\tilde{\tau} + \mathcal{V}_{ab}\tilde{\sigma} = 0\ . \end{aligned}$$

So, calculating in terms of the metric $g := \tilde{\sigma}^{-2}\boldsymbol{g}$, we have $\tilde{\mu}_a = \nabla_a\tilde{\sigma} = 0$ and $\mathcal{V}_{ab} + \boldsymbol{g}_{ab}\tilde{\tau}/\tilde{\sigma} = 0$. That is the metric g is Einstein and $\frac{1}{n}D_A\tilde{\sigma}$ is parallel.

Now the implication $\Leftarrow$ follows easily for the second statement. If the structure is almost weakly generic then it follows easily that it is weakly generic on an open dense subset of M. Thus we obtain a parallel tractor on an open dense subset which then extends by continuity. $\square$

We have the following consequence of the theorem above.

COROLLARY 1. *A weakly generic (almost weakly generic) pseudo-Riemannian or Riemannian metric g on an n-manifold is conformally Einstein (resp. admits a parallel standard tractor) if and only if the natural invariants*

$$\Omega_{abKD_1}\cdots\Omega_{cdLD_s}\nabla_e\Omega_{fgPD_{s+1}}\cdots\nabla_h\Omega_{k\ell QD_{n+2}},$$

for $s = 0, 1, \cdots, n+1$, all vanish identically. Here the sequentially labelled indices $D_1, \cdots, D_{n+2}$ are completely skewed over.

PROOF. The Theorem can clearly be rephrased to state that g is conformally Einstein (or, simply admits a parallel tractor, in the almost weakly generic case) if and only if the map

$$(\Omega_{bcDE}, \nabla_a\Omega_{bcDE}) : \mathcal{E}^{bcD} \oplus \mathcal{E}^{abcD} \to \mathcal{E}_E \tag{7}$$

given by

$$(V^{bcD}, W^{abcD}) \mapsto V^{bcD}\Omega_{bcDE} + W^{abcD}\nabla_a\Omega_{bcDE}$$

fails to have maximal rank at every point of M. But by elementary linear algebra this happens if and only if the induced alternating multi-linear map to $\Lambda^{n+2}(\mathcal{E}^E)$ vanishes. This is equivalent to the claim in the Corollary, since for any metric the tractor curvature satisfies $\Omega_{bcDE}X^E = 0$. $\square$

The power of Corollary 1 is that it gives sharp curvature obstructions for any signature and the idea clearly generalises in an obvious way to related problems on other structures.

We should point out that in the case of Riemannian signature there is a much simpler sharp obstruction to conformally Einstein metrics in the weakly generic setting. Let us write $L^a_b := C^{acde}C_{bcde}$ and $\tilde{L}^a_b$ for the tensor field which is the pointwise adjugate of L^a_b. $\tilde{L}^a_b$ is given by a formula which is a partial contraction polynomial (and homogeneous of degree $2n-2$) in the Weyl curvature and for any structure we have

$$\tilde{L}^a_b L^b_c = ||L||\delta^a_c,$$

where $||L||$ denotes the determinant of L^a_b. Now define

$$D^{acde} := -\tilde{L}^a_b C^{bcde}$$

Then D^{acde} is a natural conformal invariant and if [C] has a solution for K^e (which is necessary for the metric to be conformally Einstein) then by uniqueness the solution is $||L||^{-1}D^{edab}A_{dab}$. If K_e is exact then it is the Υ_e in (1) and so we have the following result [**12**].

Theorem 3. *The natural invariant*

$$G_{ab} := \textit{Trace-free}\big[||L||^2\mathsf{P}_{ab} - ||L||\nabla_a(D_{bcde}A^{cde}) + (\nabla_a||L||)(D_{bcde}A^{cde}) + D_{aijk}A^{ijk}D_{bcde}A^{cde}\big].$$

is a conformal invariant of weight $-8n$. A manifold with a weakly generic Riemannian metric g is conformally Einstein if and only if G_{ab} vanishes. The same is true on pseudo-Riemannian manifolds where the conformal invariant $||L||$ is non-vanishing.

We observe here that weakly generic almost conformally Einstein metrics do not admit parallel tractors with scale singularities. This observation motivates the consideration above of almost weakly generic metrics.

Theorem 4. *If a weakly generic conformal structure admits a non-zero parallel standard tractor field $\mathbb{I}^A$ then it is conformally Einstein and $X_A\mathbb{I}^A$ is an Einstein scale.*

Proof. If $\mathbb{I}^A$ is parallel then the system $[\tilde{\mathrm{C}}]$ and $[\tilde{\mathrm{D}}]$ holds. Early in the proof of Theorem 2 it is established that if these hold and the structure is weakly generic then $X_A\mathbb{I}^A$ is non-vanishing. $\square$

Finally a related observation concerning the analysis of points of scale singularities. If $\mathbb{I}_A$ is parallel then, recall, $\mathbb{I}_A = \frac{1}{n}D_A\sigma$ for some density of σ of weight 1. If at some point x we have $j^1_x\sigma$ then at x we have $\mathbb{I} = \tau X^A$ and so from $[\tilde{\mathrm{D}}]$ and arguing once again as in the proof of Theorem 2 we have $C_{abcd}(x) = 0$.

4. The Fefferman-Graham ambient obstruction tensor

Recall that the splitting of the tractor bundle given by a metric is determined by the formula $(\sigma, \nabla_a\sigma, -\frac{1}{n}(\Delta + \mathsf{J})\sigma)$ for the operator $\frac{1}{n}D$ on densities of weight 1. From this it is easily verified that, under a conformal transformation $g \mapsto \widehat{g} = e^{2\omega}g$, the injector $Z_A{}^a$ transforms to $Z_A{}^a + \Upsilon^a X_A$. With the convention that sequentially labelled subscript indices, e.g. A_0, A_1, and A_2, are implicitly skewed over, it follows immediately that

$$X_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c\Omega_{bc}{}^D{}_E,$$

is conformally invariant. The operator D extends to an operator (the so-called tractor-D operator) on densities and tractors of general weight w by the formula

$$D^AV := (n+2w-2)wY^AV + (n+2w-2)Z^{Aa}\nabla_aV - X^A\Box V,$$

where $\Box V := \Delta V + w\mathsf{J}V$ and in these formulae ∇ is the coupled Levi-Civita-tractor connection. Using the conformal transformation formula for $Z_A{}^a$ and the corresponding formula for Y^A it is easily verified directly that this is conformally invariant. Thus, by construction the W-tractor field

$$W_{A_1A_2}{}^D{}_E := \frac{3}{n-2} D^{A_0} X_{A_0} Z_{A_1}{}^b Z_{A_2}{}^c \Omega_{bc}{}^D{}_E,$$

is conformally invariant. It is readily verified that it has Weyl curvature type symmetries:

$$W_{ABCD} = W_{[AB][CD]}, \quad W_{[ABC]D} = 0, \quad \text{Trace-free},$$

etcetera.

Recall that if $\mathbb{I}^E$ is a parallel tractor then $\Omega_{bc}{}^D{}_E \mathbb{I}^E = 0$ and from this and the definition of the W-tractor we have

$$W_{A_1A_2}{}^D{}_E \mathbb{I}^E = 0, \tag{8}$$

since (viewed as an order 0 operator) $\mathbb{I}^E$ commutes with the tractor connection and, indeed, the tractor-D operator. So the existence of a tractor satisfying (8) is a necessary condition for a metric to admit a parallel tractor and, in particular, for a metric to be conformally Einstein. In fact in dimensions other than 4, the equation (8) is in fact exactly an alternative expression for the system [C], [B] (see [**12**]).

The situation is more interesting in dimension 4. Expanding W_{ABCE} out we obtain,

$$W_{ABCE} = \left(\begin{array}{ccccc} & & & 0 & \\ & 0 & & & 0 \\ (n-4)C_{abce} & & 0 & 0 & & 0 \\ & (n-4)A_{bce} & & & 0 \\ & & & B_{eb} & \end{array} \right)$$

where we have indicated components with respect to the composition series for tractors (of weight -2) with Weyl tensor type symmetries (see [**14**] for details). So in dimension 4 the only possibly non-vanishing component of the W-tractor is the Bach tensor. This is extracted by the tractor for a scale σ. For calculations it is more convenient to express the expansion in terms of the injectors/projectors X, Y and Z. In dimension n we have that W_{ABCE} is given by

$$\begin{aligned} &(n-4)\big(Z_A{}^a Z_B{}^b Z_C{}^c Z_E{}^e C_{abce} - 2Z_A{}^a Z_B{}^b X_{[C} Z_{E]}{}^e A_{eab} \\ &-2X_{[A} Z_{B]}{}^b Z_C{}^c Z_E{}^e A_{bce}\big) + 4X_{[A} Z_{B]}{}^b X_{[C} Z_{E]}{}^e B_{eb}. \end{aligned} \tag{9}$$

Let σ be a choice of conformal scale and $\mathbb{I}_E := \frac{1}{n} D_E \sigma$. Splitting the standard tractor bundle via the metric $g = \sigma^{-2}\boldsymbol{g}$ given by the scale σ (and noting that for the connection given by this scale we have $\nabla\sigma = 0$) we have $\mathbb{I}_E = (\sigma, 0, -\frac{1}{n}\mathsf{J}\sigma)$, or in other terms $\mathbb{I}_E = \sigma Y_E - \frac{1}{n}\sigma \mathsf{J} \mathsf{X}_\mathsf{E}$. Contracting this into the last display and setting $n = 4$ we obtain

$$W_{ABCE}\mathbb{I}^E = -2\sigma X_{[A} Z_{B]}{}^b Z_C{}^e B_{eb}.$$

(For the contraction note that from the formula for the tractor metric we have that X^E, Y^E are null, $X^E Y_E = 1$, and both are orthogonal to $Z_E{}^e$.) Now if σ is an Einstein scale (i.e. $g = \sigma^{-2}\boldsymbol{g}$ an Einstein metric) then this vanishes by (8). Of course there are easier ways to show that the Bach tensor is an obstruction to conformally Einstein metrics. The main point is that this generalises.

Note that the W-tractor has conformal weight -2. In dimension 6 this is $1-n/2$ and so exactly the weight on which the tractor coupled conformal wave operator $\Box$ (as defined earlier) acts invariantly. In dimension 6 a straightforward direct calculation shows that

$$\not\Box W_{ABCE} = \left(\begin{array}{ccccccc} & & & 0 & & & \\ & 0 & & & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & & & 0 & \\ & & & \mathcal{B}^{(6)}_{eb} & & & \end{array}\right)$$

where $\not\Box$ is the modification of $\Box$ given by

$$\not\Box = \Box + \frac{1}{4} W \sharp\sharp,$$

and $\sharp$ indicates the natural action of sections of $\mathrm{End}(\mathcal{E}^A)$ on tractors. (Note that W is a section in $\otimes^2(\mathrm{End}(\mathcal{E}^A))$ and that if an element of $\mathrm{End}(\mathcal{E}^A)$ is skew relative to the tractor metric h then its action by $\sharp$ obviously preserves the symmetry type of tractor bundles.) Evidently $\mathcal{B}^{(6)}$ is some conformal invariant and from the composition series concerned we know this is a trace-free symmetric density-valued 2-tensor of conformal weight -4. By essentially the same argument as for the Bach tensor above we see this an obstruction to conformally Einstein metrics. On the one hand for any scale σ contracting $\mathbb{I}^E := \frac{1}{n} D^E \sigma$ into $\not\Box W_{ABCE}$ extracts (the non-vanishing scale σ times) $\mathcal{B}^{(6)}_{eb}$. On the other hand if σ is an Einstein scale then $\mathbb{I}$ is parallel and so commutes with $\not\Box$. Thus from (8) $\mathcal{B}^{(6)}_{eb}$ necessarily vanishes for conformally Einstein metrics.

More generally we have the following (an adaption of part of Theorems 4.1 and 4.2 from [**14**]).

THEOREM 5. *Let M be a conformal manifold of dimension n even. There is a (conformally invariant) operator*

$$\not\Box_{n/2-2} : \otimes^2(\Lambda^2\mathcal{E}^A)[-2] \to \otimes^2(\Lambda^2\mathcal{E}^A)[2-n],$$

which may be expressed by a formula polynomial in $X, \nabla_A := Z_A{}^a \nabla_a, W, h,$ and h^{-1}, such that

$$\not\Box_{n/2-2} W_{A_1A_2B_1B_2} = K(n) X_{A_1} Z_{A_2}{}^a X_{B_1} Z_{B_2}{}^b \mathcal{B}^{(n)}_{ab},$$

Here $K(n)$ is a (known) non-zero constant depending on n. In the polynomial formula for the left-hand-side the free indices always appear on a W.

From this we have immediately the generalisation of the result for the Bach tensor and its dimension 6 analogue. Once again from an easy analysis of the composition series for the tractor bundle (with Weyl tensor symmetries) that $\not\Box_{n/2-2} W_{A_1A_2B_1B_2}$ takes values in we can conclude that $\mathcal{B}^{(n)}_{ab}$ is a trace-free symmetric density-valued 2-tensor of conformal weight $2-n$. By construction it is clearly natural (i.e. can be given by a formula polynomial in the conformal metric and its inverse and the Levi-Civita covariant derivatives of the Riemannian curvature).

COROLLARY 2. *The natural tensor $\mathcal{B}^{(n)}_{ab}$ is conformally invariant and vanishes for almost conformally Einstein metrics.*

PROOF. The conformal invariance is immediate from the conformal invariance of the operator $\not\Box_{n/2-2}$, the tractor field $W_{A_1A_2B_1B_2}$ and the result that $X_{A_1} Z_{A_2}{}^a : \mathcal{E}_a \to \mathcal{E}_{A_1A_2}$ is injective and conformally invariant.

For a scale σ let $\mathbb{I}_A := \frac{1}{n} D_A \sigma$. Then, in terms of the metric $g = \sigma^{-2}\boldsymbol{g}$, we have

$$\mathbb{I}^A = \sigma Y^A - \frac{1}{n} \mathsf{J}\sigma \mathsf{X}^{\mathsf{A}}$$

and so from the theorem (and the tractor metric) we have

$$\sigma^2 \mathcal{B}^{(n)}_{ab} = 4(K(n))^{-1} Z^{A_2}{}_a Z^{B_2}{}_b \mathbb{I}^{A_1} \mathbb{I}^{B_1} \not{\Box}_{n/2-2} W_{A_1 A_2 B_1 B_2}.$$

But if σ is an Einstein scale then $\mathbb{I}^A$ is parallel and so commutes with all terms in the polynomial expression for $\not{\Box}_{n/2-2} W_{A_1 A_2 B_1 B_2}$ and the free indices $A_1 A_2 B_1 B_2$ always appear on a W. Thus,

$$\mathbb{I}^{B_1} \not{\Box}_{n/2-2} W_{A_1 A_2 B_1 B_2} = 0 \quad \text{since} \quad \mathbb{I}^A W_{ABCD} = 0$$

and so from the previous display it follows that, in the scale σ,

$$\mathcal{B}^{(n)}_{ab} = 0.$$

Since $\mathcal{B}^{(n)}_{ab}$ is conformally invariant it must vanish for any metric which is conformally Einstein and then by continuity for any metric which is almost conformally Einstein. $\square$

The proof of Theorem 5 in [**14**] uses the conformal ambient metric construction of Fefferman-Graham and it follows easily from this proof that $\mathcal{B}^{(n)}$ is the obstruction to the ambient metric construction found by Fefferman-Graham.

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Department of Mathematics
The University of Auckland
Private Bag 92019
Auckland 1 New Zealand

e-mail: gover@math.auckland.ac.nz