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On linking of J–holomorphic curves with periodic orbits of a contact structure

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Abstract. This article is the revised and expanded version of a conference paper presented at the 9th International Conference on Differential Geometry and its Applications, Prague, Czech Republic, August 30th – September 3rd, 2004. The following is a summary of recent results of the author in collaboration with K. Wysocki, concerning topological classification of J–holomorphic curves, asymptotic to a given periodic orbit within a closed three–dimensional contact manifold.

1. Contact Structures on Convex hypersurfaces

For the purposes of this note it will be assumed that all manifolds M are of dimension three. A "contact structure" on M is then defined to be a form $\lambda \in C^\infty(M, T^*M)$, together with a rank–two distribution $\xi := ker(\lambda) \subset TM$, such that $d\lambda \mid_\xi$ is non-degenerate (equivalently, $\lambda \wedge d\lambda$ is a volume form on M). A theorem of Martinet [6] states that every closed (i.e., compact, without boundary), oriented three–manifold possesses at least one such structure. Moreover, the equation

$$d\lambda_p(\mathbf{v}, \mathbf{w}) = -\frac{1}{2}\lambda([\mathbf{v}, \mathbf{w}]) \ , \quad \mathbf{v}, \mathbf{w} \in \xi_p \ , \ p \in M \ ,$$

together with the non–degeneracy assumption implies that ξ is nowhere–integrable. The "Reeb vector field" X associated with λ is uniquely determined by the conditions $\lambda(X) \equiv 1$ and $\mathcal{L}_X\lambda = 0$ (ie., vanishing of the Lie derivative here implies that the contact form is preserved by the Reeb flow). A "periodic orbit" of X is then a smooth map $\gamma : \mathbb{S}^1 \to M$ such that

$$\dot{\gamma}(t) = X(\gamma(t)) \ ; \quad \gamma(t + \tau) = \gamma(t)$$

for some minimal $\tau > 0$ independent of t. It is a conjecture of Weinstein that every Reeb vector field on a closed contact three–manifold has a periodic orbit.

The structure determined by λ is said to be "pseudohermitian" when additionally equipped with an endomorphism $J \in C^\infty(M, \xi \otimes \xi^*)$ such that $J^2 = -1$ and $d\lambda_p(*, J_p...)$ is a positive–definite, symmetric quadratic form on ξ_p for all $p \in M$. The standard example of such a structure is given on $\mathbb{S}^3$, viewed as the real hypersurface of $\mathbb{C}^2 \sim \mathbb{R}^4$ defined by the equation $|z_1|^2 + |z_2|^2 = 1$. For $p \in \mathbb{S}^3$ let $\mathbf{v}_p$ denote the unit position vector of p in $\mathbb{R}^4$. If J_0 denotes the standard complex structure of $\mathbb{C}^2$ then $J_0 \cdot \mathbf{v}_p \in T_p\mathbb{S}^3$ allows us to define the covector field

$$\lambda_p := (..., J_0 \cdot \mathbf{v}_p) \ ,$$

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with respect to the standard Euclidean inner product, as a contact form on $\mathbb{S}^3$. The Reeb vector field is then simply given by $X(p) = J_0 \cdot \mathbf{v}_p$ for all $p \in \mathbb{S}^3$, and consequently periodic orbits correspond to complex one–dimensional subspaces of $\mathbb{C}^2$, i.e., $\mathcal{P} := span_{\mathbb{R}}\{\mathbf{v}_p, J_0 \cdot \mathbf{v}_p\} \cap \mathbb{S}^3$ is a circle corresponding to the unique periodic orbit of X passing through a given point p. The circle–fibration of $\mathbb{S}^3$ determined in this way is of course the Hopf fibration.

One generalisation of this construction is to consider other real convex hypersurfaces of $\mathbb{C}^2$, such as the ellipsoids

$$\mathbb{E}_{a,b} := \{(z_1, z_2) \in \mathbb{C}^2 \mid a|z_1|^2 + b|z_2|^2 = 1 \ ; \ (a,b) \in \mathbb{R}^2_+\} \ .$$

The restriction to $\mathbb{E}_{a,b}$ of λ as defined above determines a different contact structure on $T\mathbb{E}_{a,b}$, and in particular a Reeb vector field $X_p := A \cdot J_0 \cdot \mathbf{v}_p$ for all $p \in \mathbb{E}_{a,b}$, where

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

is viewed as an element of $\mathbf{GL}(2,\mathbb{C})$. Solutions of the equation $\dot{\gamma}(t) = X(\gamma(t))$, $\gamma(0) = p = (z_1, z_2)$, then correspond to maps $t \mapsto (z_1 e^{\mathbf{i}at}, z_2 e^{\mathbf{i}bt})$. Note that there are two periodic orbits corresponding to $z_1 = 0$ and $z_2 = 0$ separately. These are the *only* periodic orbits of the Reeb flow if $\frac{a}{b}$ is irrational, whereas $\mathbb{E}_{a,b}$ is foliated by periodic orbits if $\frac{a}{b} \in \mathbb{Q}$. Moreover the periodic orbit corresponding to $z_1 = 0$ has minimal period $\tau = \frac{2\pi}{b}$ and for $z_2 = 0$ it is $\frac{2\pi}{a}$, while $\tau = \frac{2\pi k}{a} = \frac{2\pi l}{b}$ for all other orbits with repsect to fixed relatively prime positive integers k, l (as references on Reeb dynamics and its relationship to the Weinstein conjecture, cf. e.g., [3], [4]).

Another type of explicit generalisation of the standard construction is to look at more general circle fibrations of $\mathbb{S}^3$.Given a four–dimensional real vector space V, these fibrations correspond to families of invariant planes (i.e., "complex lines") distinguished by linear endomorphisms J_0 determining standard complex structures on V, and are parametrised by $\mathbb{O}(4)/\mathbb{U}(2)$. The base space of each such fibration is a (Riemann) sphere inside the Grassmann manifold $\mathbb{G}r_2(V)$. Still more general fibrations correspond to families of planes ("line congruences") parametrised by compact surfaces $\Sigma \subset \mathbb{G}r_2(V)$. A line congruence is said to be *elliptic* if for all $P \in \Sigma$, there exists a 2–sphere $\mathbb{S}$ corresponding to some J_0 such that $T_P\Sigma = T_P\mathbb{S}$ inside $\mathbb{G}r_2(V)$, hence in particular Σ is itself diffeomorphic to a sphere. It is a theorem of Gluck and Warner [1] that the surfaces Σ of generalised Hopf fibrations are precisely the graphs of distance–decreasing smooth maps $f : \mathbb{S}_- \to \mathbb{S}_+$, with standard Hopf fibrations coresponding to constant maps. In [8], McKay also observed that they correspond to general elliptic line congruences and hence determine non–linear complex structures on V which "osculate" with linear structures at each $P \in \Sigma$. We apply this idea to the explicit construction of contact structures on $\mathbb{S}^3$ as follows (cf. [2]). Let $\mathcal{J}(V)$ denote the space of linear endomorphisms of V corresponding to linear complex structures, then it was shown in [8] that each elliptic line congruence Σ determines a map

$$J : \Sigma \to \mathcal{J}(V) \subset V \otimes V^*$$

such that for each plane $P \in \Sigma$, $J(P)$ is linear and is the "osculating" structure to Σ at P in the sense that both P and $P^\perp$ (with respect to a given inner product on V) are complex lines relative to $J(P)$. The family of planes determined by Σ describes a rank–two vector bundle $\pi : \mathcal{P} \to \Sigma$ such that the total space, corresponding to the incidence manifold $\mathcal{P} = \{(\mathbf{v}, P) \in V \times \Sigma \mid \mathbf{v} \in P\}$ also maps surjectively to

V. In fact, there exists $\sigma : \mathcal{P} \to V$ such that $\mathcal{P} \setminus \sigma^{-1}(0) \cong V \setminus \{0\}$. Hence define $\varphi := \pi \circ \sigma : V \setminus \{0\} \to \Sigma$, so that $\varphi^{-1}(P) = \{\mathbf{v} \in V \setminus \{0\} \mid \mathbf{v} \in P\}$.

Consider the pullback $\varphi^* J : V \backslash \{0\} \to V \otimes V^*$. Hence with respect to a designated orthonormal basis of V, noting that J and $\varphi^* J$ are skew–symmetric matrix–valued functions, we may represent it in the form

$$\varphi^* J = \Sigma_{\mu.\nu} J^{\mu\nu} \frac{\partial}{\partial x_\mu} \wedge dx_\nu \ .$$

Note moreover that the isomorphism $V^* \cong V$ via the Euclidean inner product allows us to define a 2–form $\omega_\Sigma := \Sigma_{\mu.\nu} J^{\mu\nu} dx_\mu \wedge dx_\nu$. If $i_{\mathbf{v}}$ denotes contraction of a form by the position vector $\mathbf{v} = \frac{1}{2}\Sigma_\eta v_\eta \frac{\partial}{\partial \eta}$, then it is easily shown in [2] that

$$d\omega_\Sigma = 0 \Rightarrow \omega_\Sigma(\mathbf{v}) = d(i_{\mathbf{v}} \omega_\Sigma) \ .$$

Now define the 1–form $\lambda_\Sigma := i_{\mathbf{v}} \omega_\Sigma$ for ω_Σ closed, noting that if $[J]$ denotes the matrix of $\varphi^* J$, then we may express λ_Σ in terms of the inner product as

$$\lambda_\Sigma(\mathbf{w}) = -(\mathbf{w}, [J] \cdot \mathbf{v})$$

for all $\mathbf{w} \in V$. Moreover, let X_λ be the vector field defined by $X_\lambda(\mathbf{v}) = -[J] \cdot \mathbf{v}$. It is easily checked that λ_Σ, together with J restricted to $ker(\lambda) \cap T\mathbb{S}^3$ defines a pseudohermitian structure on $\mathbb{S}^3$, with periodic orbits comprising a circle–fibration over Σ. But it is a fundamental fact that all such fibrations are smoothly equivalent, as bundles over $\mathbb{S}^2$, to the standard one. Distinctions among these structures must therefore involve the metric, as is confirmed by the following

THEOREM 1. *(cf. [2]) Let J be the osculating complex structure of an elliptic line congruence $\Sigma \subset \mathbb{G}r_2(V)$, such that the skew–symmetric 2–form ω_Σ on $V \setminus \{0\}$ is closed. Then the 1–form λ, such that $\lambda_{\mathbf{v}} := i_{\mathbf{v}} \omega_\Sigma$, defines a fillable, hence tight, contact structure on $\mathbb{S}^3$. Moreover, two such structures are equivalent via a diffeomorphism δ of $\mathbb{S}^3$ if and only if $\delta \in \mathbb{O}(4)$.*

2. J–holomorphic curves and linking with periodic orbits

Let (M, λ, J) be an arbitrary pseudohermitian structure on a manifold M. Define a bundle projection map $\pi : TM \to \xi$ such that

$$\pi(\mathbf{v}) = \mathbf{v} - \lambda_p(\mathbf{v}) X(p)$$

for $\mathbf{v} \in T_p M$. Consider $(\mathbf{v}, \alpha) \in T(M \times \mathbb{R})$ and define a natural extension of J to an endomorphism $\tilde{J}$ of $T(M \times \mathbb{R})$ such that

$$\tilde{J}(\mathbf{v}, \alpha) = (J \cdot \pi(\mathbf{v}) + \alpha X, \ -\lambda(\mathbf{v})) \ .$$

On $M \times \mathbb{R}$ one can now introduce the notion of "J-holomorphic curves" as in symplectic geometry (cf.,e.g., [3]). Given a Riemann surface Σ, a map $\psi : \Sigma \to M$, and a function $a : \Sigma \to \mathbb{R}$, the map $\tilde{\psi} = (\psi, a) : \Sigma \to M \times \mathbb{R}$ is J–holomorphic if $\tilde{\psi}_* \circ \mathbf{i} = \tilde{J} \circ \tilde{\psi}$. If we define a local complex coordinate $z = \eta + \mathbf{i}\zeta$ on a neighbourhood of Σ, then the local form of the above equation is

$$\frac{\partial \tilde{\psi}}{\partial \eta} + \tilde{J} \frac{\partial \tilde{\psi}}{\partial \zeta} = 0 \ .$$

This may be unpacked further to yield an elliptic system of equations

$$\pi(\frac{\partial \psi}{\partial \eta}) + J\pi(\frac{\partial \psi}{\partial \zeta}) = 0$$

$$\lambda(\psi_\eta) = a_\zeta \ ; \quad \lambda(\psi_\zeta) = -a_\eta \ .$$

As a consequence of these equations it is well–known that

$$\psi^* d\lambda = \triangle a \geqslant 0$$

hence the integral

$$\int_\Sigma \psi^* d\lambda \geqslant 0 \quad (*) \ ,$$

and vanishes if and only if ψ is a constant map. On the other hand Stokes' theorem implies that the integral indeed vanishes if the boundary of Σ is empty, hence attention is focused on Riemann surfaces punctured at a finite number of points. The "energy" of a J–holomorphic curve is defined in terms of $\tilde{\psi}$ rather than ψ, hence an appropriate extension of $d\lambda$ to $M \times \mathbb{R}$ must be constructed before one can write the formula for energy

$$\mathcal{E}(\tilde{\psi}) = \int_\Sigma \tilde{\psi}^* d\lambda$$

(cf., e.g., [4]). Restricting Σ to a punctured disc $D \setminus \{0\} \subset \mathbb{C}$ with coordinate $z = e^{-(\rho + \mathbf{i}\varphi)}$, and identifying this domain with a cylinder $(\varphi, \rho) \in \mathbb{S}^1 \times [a_0, \infty)$, the "charge" of the puncture is defined as

$$T = \lim_{\rho \to \infty} \int_{\mathbb{S}^1} \psi^* \lambda \ .$$

Fundamental to our discussion is a theorem of Hofer, Wysocki and Zehnder [5], which states that if $\tilde{\psi}$ is a finite–energy cylinder, then T is a finite quantity, and moreover

$$\lim_{\rho \to \infty} \psi(\varphi, \rho) = \gamma(T\varphi) \ ,$$

i.e., the image of the cylinder is smoothly asymptotic to a periodic orbit of the Reeb flow (if we confine our attention to contact manifolds M which are foliated by periodic orbits, such as the examples of the previous section, then there is no need to mention the "non–degeneracy" condition that is required of the orbit in order to obtain smooth convergence in a more general setting). Moreover, T is an integer multiple of the minimal period τ of the orbit – a fact that invites comparison with multiplicity of singular plane curves $\Gamma \subset \mathbb{C}^2$. Our aim is to understand the extent to which J–holomorphic cylinders in symplectisations actually *link* with the periodic orbits which they approach asymptotically. In this sense the case of algebroid knots K_Γ linking with great circles in the 3–sphere suggests a classical model, since $\mathbb{S}^3 \times \mathbb{R}$ is clearly diffeomorphic to $\mathbb{C}^2 \setminus \{0\}$ (we will return to this matter below). By analogy with the work of McDuff [7], Sikorav [9] and others we show that a suitably constructed diffeomorphism applied to a tubular neighbourhood of $\mathcal{P}$ allows $\tilde{\psi}$ to be represented locally by a holomorphic parametrization. The following theorem offers a convenient specialisation of a more general statement in [2]. Let $\mathcal{U}$ be a tubular neighbourhood of some periodic orbit $\mathcal{P}$ corresponding to the asymptotic limit of a finite–energy J–holomorphic curve $\tilde{\psi}$. The aim is to find a diffeomorphism h between $\mathcal{U} \times \mathbb{R}$ and $\Delta \times \mathbb{S}^1 \times \mathbb{R}$, where Δ denotes a disc in $\mathbb{R}^2$, such that the projection of $h \circ \tilde{\psi}$ onto Δ corresponds to a holomorphic function, while the projection onto $\mathbb{S}^1 \times \mathbb{R}$ is essentially the logarithm of a holomorphic function. In this way the asymptotic behaviour of $\tilde{\psi}$ is seen to recover the classical theory of iterated torus knots, or "links", of isolated plane singularities.

THEOREM 2. *Let $\tilde{\psi} : D\setminus\{0\} \to M \times \mathbb{R}$ be a J–holomorphic curve of finite energy and charge n at $z = 0$, asymptotic to a periodic orbit $\mathcal{P}$ within a manifold fibred by periodic orbits. Consider a tubular neighbourhood of $\mathcal{P}$ in M, diffeomorphic to $\Delta\times\mathbb{S}^1$ such that $\{0\}\times\mathbb{S}^1 \approx \mathcal{P}$. There exists a smooth change of coordinates h in $\Delta\times\mathbb{S}^1\times\mathbb{R}$ such that the map $h\circ\tilde{\psi}$ can be expressed in the form*

$$(F(z), \frac{1}{2\pi\mathbf{i}}\log(\rho) - G(z,\bar{z}))\ ,$$

where F and ρ are holomorphic on $D\setminus\{0\}$, $F(0) = 0$ and $ord_0(\rho) = n$. Moreover, the function G belongs to $C^2(D)$ and is uniformly bounded by $K|z|$.

Full details of this local comparison theorem are given in [2]. In particular, the function G admits a decomposition of the form $G = g\circ F + H$ with respect to a smooth function g and a holomorphic function H. A representation of (ψ, a) corresponding to the classical local parametrization of plane curves is then obtained by further defining a coordinate

$$v := \rho(z)e^{-G} = \rho(z)e^{-g\circ F - H}$$

and letting $\rho(z)e^{-H} = f(z)^n$ for some holomorphic function f on $D' \subseteq D$, with $f'(0) \neq 0$. Now $z' := f(z)$ implies $w = F(z) = F\circ f^{-1}(z')$, while $v = (z')^n e^{-g\circ F\circ f^{-1}(z')}$. >From the final coordinate diffeomorphism $w' = w$; $v' = ve^{g(w,\bar{w})}$, it now follows that

$$w' = F\circ f^{-1}(z')\ ; \quad v' = (z')^n\ .$$

The above result does not apply directly to the rational ellipsoid, since in this instance periodic orbits do not all have the same period. Specifically, if $\mathcal{P}$ is one of the exceptional orbits then all nearby orbits have minimal period corresponding to an integer multiple m of the period $\tau(\mathcal{P})$. Given a tubular neighbourhood $\mathcal{U}$ of $\mathcal{P}$, we construct a solid torus $\Delta'\times\mathbb{S}^1$ with contact form $\tilde{\lambda}$, and a smooth m–fold covering map $\beta : \mathcal{U} \to \Delta'\times\mathbb{S}^1$ such that $\lambda\mid_{\mathcal{U}} = \beta^*\tilde{\lambda}$ and $\beta_*\circ j_0 = j_0'\circ\beta_*$ (where $j_0 = J_0\mid_{ker(\lambda)}$ and $j_0' = J_0\mid_{ker(\tilde{\lambda})}$). Pseudoholomorphic maps asymptotic to $\mathcal{P}$ in $\mathcal{U}$ then project pseudoholomorphically via β into $\Delta'\times\mathbb{S}^1$, which is fibred by periodic orbits of the (well–defined) Reeb vector field $\beta_* X$ (cf. [2]).

Some of the key ideas behind the above theorem are illustrated by the classical model itself, treating $\mathbb{C}^2\setminus\{0\}$ as diffeomorphic to $\mathbb{S}^3\times\mathbb{R}$ equipped with the standard contact structure. Singular plane curves transform neatly to J–holomorphic curves in this way (as has also been observed, from a slightly different point of view, in [4]). Hence consider $M = \mathbb{S}^3 \subseteq \mathbb{C}^2$, with

$$(\lambda_0)_{\mathbf{v}} = (*, J_0\cdot\mathbf{v})\mid_{T\mathbb{S}^3}$$

the standard contact form defined with respect to the complex structure J_0 of $\mathbb{C}^2$ and the Euclidean inner product of $\mathbb{R}^4$. M is then defined by the equation $|w_1|^2+|w_2|^2 = 1$ with respect to complex coordinates in $\mathbb{C}^2$, while

$$\lambda_0 = \mathrm{Re}(-\mathbf{i}(\bar{w}_1 dw_1 + \bar{w}_2 dw_2))\ .$$

Consider $\Phi(z) : D \to \mathbb{C}^2$ a complex–analytic curve defined on a neighbourhood of the origin in $\mathbb{C}$ such that $\Phi(z) = (z^n, \rho(z))$ and $ord_0(\rho) \geqslant n+1$, hence Φ has a singularity of multiplicity n at $(0,0)$. The corresponding map $(\psi, a) : D \to \mathbb{S}^3\times\mathbb{R}$ is given by

$$\psi(z) = [|\rho|^2 + |z|^{2n}]^{-\frac{1}{2}}(z^n, \rho(z))\ ,$$

$$a(z) = -\frac{1}{2}\ln(|z|^{2n} + |\rho|^2) = -n\ln|z| - \frac{1}{2}\ln(1 + |z|^{-2n}|\rho|^2)\ .$$

The periodic orbit $\mathcal{P}$ corresponds simply to the circle defined by $\{w_2 = 0\} \cap \mathbb{S}^3$ and is associated with the degenerate tangent cone of Φ at $(0,0)$. It is, moreover, a simple calculation to verify that the charge of ψ at the origin is n. Consider the proper holomorphic map $\sigma : \mathcal{O}_{\mathbb{P}_1}(-1) \to \mathbb{C}^2$, where $\mathcal{O}_{\mathbb{P}_1}(-1)$ denotes the complex line bundle of Chern class equal to –1 on the Riemann Sphere $\mathbb{P}_1(\mathbb{C}) = \sigma^{-1}(\mathbf{0})$, with $\sigma : \mathcal{O}_{\mathbb{P}_1}(-1) \setminus \sigma^{-1}(\mathbf{0}) \cong \mathbb{C}^2 \setminus \{0\}$, given in local coordinates by the quadratic transformation $w_1 = \mu$, $w_2 = \mu\nu$. Noting that $|\mu|^2(1 + |\nu|^2) = 1$ on the chart of $\sigma^{-1}(\mathbb{S}^3)$ corresponding to $\nu \neq \infty$, we have

$$\sigma^*\lambda_0 = \mathrm{Re}(-\mathbf{i}(\mu^{-1}d\mu + \bar{\nu}|\mu|^2 d\nu))\ ;$$

moreover, for each $\nu \in \mathbb{P}_1(\mathbb{C}) \setminus \{\infty\}$, the Hopf fibration corresponding to

$$\varpi : \sigma^{-1}(\mathbb{S}^3) \to \mathbb{P}_1(\mathbb{C}) \approx \mathbb{S}^2$$

has fibres $\varpi^{-1}(\nu) = \{\mu \in \mathbb{C} \mid |\mu| = (1 + |\nu|^2)^{-\frac{1}{2}}\}$. Let $\nu = x + \mathbf{i}y$, then $\mu = |\mu|e^{\mathbf{i}\vartheta} = (1 + x^2 + y^2)^{-\frac{1}{2}}e^{\mathbf{i}\vartheta}$, $h := \sigma\ |_{\varpi^{-1}(\mathbb{P}_1\setminus\{\infty\})}$ implies

$$h^*\lambda_0 = d\vartheta + (1 + x^2 + y^2)^{-1}(xdy - ydx)\ ,$$

while $h^*X_{\lambda_0} = \frac{\partial}{\partial\vartheta}$. Now

$$\Psi(z) := h^{-1}\psi(z) = (\arg(z^n)\ ,\ F(z)) \ \text{ where } F(z) := z^{-n}\rho(z)\ .$$

Clearly $\nu = F(z)$ is holomorphic, so it remains to show that

$$h^*\lambda_0(\Psi_\eta) = a_\zeta\ ;\ h^*\lambda_0(\Psi_\zeta) = -a_\eta\ .$$

Note that $a(z) = -\frac{1}{2}\ln((\eta^2 + \zeta^2)^n(1 + x^2 + y^2))$ implies

$$a_\zeta = \frac{-n\zeta}{\eta^2 + \zeta^2} - \frac{xx_\zeta + yy_\zeta}{1 + x^2 + y^2}\ ,$$

while $\Psi_\eta = (x_\eta, y_\eta, \vartheta_\eta) = (x_\eta, y_\eta, \frac{-n\zeta}{\eta^2+\zeta^2})$ implies

$$h^*\lambda_0(\Psi_\eta) = \frac{-n\zeta}{\eta^2 + \zeta^2} + \frac{xy_\eta - yx_\eta}{1 + x^2 + y^2}\ .$$

But $\nu = x + \mathbf{i}y = F(z)$ is holomorphic, hence $x_\eta = y_\zeta$, $x_\zeta = -y_\eta$ yields $a_\zeta = h^*\lambda_0(\Psi_\eta)$, and similarly for a_η. Moreover (cf. [2]), we can compute the function $G(z,\bar{z}) = -\ln(1 + |F|^2)^{\frac{1}{2}}$.

On the other hand, modulo rescaling with respect to $t = \frac{\vartheta}{2\pi}$ so that

$$\hat{a}(z) = \frac{-1}{2\pi}\ln(|z|^n(1 + |F|^2)^{\frac{1}{2}})\ ,$$

we have

$$\begin{aligned} t + \mathbf{i}\hat{a} &= \frac{1}{2\pi}(arg(z^n) + \mathbf{i}\{-\ln|z|^n - \ln(1 + |F|^2)^{\frac{1}{2}}\}) \\ &= \frac{1}{2\pi\mathbf{i}}(\log(z^n) - G)\ , \end{aligned}$$

in accordance with the statement of Theorem 2.

Let $\Gamma \subset \mathbb{C}^2$ be the locus of the plane curve parametrized by Φ. We remark in conclusion that although the multiplicity of the singular point of the strict transform $\overline{\sigma^{-1}(\Gamma) \setminus \sigma^{-1}(0)}$ is less than n, the charge at $z = 0$ of Ψ is easily seen to be conserved by the diffeomorphism h. The singular plane curve corresponding to the strict transform (assuming the singularity has not been resolved by a single quadratic transformation) is in fact asymptotic (viewed locally as a J–holomorphic curve Ψ') to a distinct periodic orbit within a new 3–sphere bounding a neighbourhood of the transformed singularity. However, the linking of the transform of K_Γ with the original periodic orbit of $\sigma^{-1}(\mathbb{S}^3)$ is topologically unaffected.

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