DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 69 – 77

Generalized energy of Hopf vector fields on Berger's 3-spheres.

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Abstract. The aim of this paper is to study the behavior of Hopf vector fields with respect to the generalized energy functionals, on the 3-sphere when we consider the metrics obtained by performing the canonical variation of the Hopf fibration. These metrics are known as Berger's metrics.

1. Introduction

A smooth vector field V on a Riemannian manifold (M, g) can be seen as a map into its tangent bundle endowed with the Sasaki metric g^S, defined by g. If $\tilde{g}$ is another metric on M, we define the generalized energy $E_{\tilde{g}}$ as the energy of the map $V : (M, \tilde{g}) \to (TM, g^S)$. These energies were introduced in [**4**] to study the relationship between the volume and the energy of vector fields. In particular, if we take either $\tilde{g} = g$ or $\tilde{g} = V^*g^S$, the generalized energy turns out to be, up to constant factors, the energy and the volume of the vector field respectively.

On a compact M, the critical vector fields of all these functionals should be parallel, so it is usual to restrict the functionals to the submanifold of unit vector fields. Obviously, if M admits unit parallel vector fields, they are the absolute minimizers.

The geometrically simplest manifolds admitting unit vector fields but not parallel ones are odd-dimensional round spheres. It is well known that Hopf fibration $\pi : S^{2m+1} \longrightarrow \mathbb{C}P^m$ determines a foliation of S^{2m+1} by great circles and that a unit vector field can be chosen as a generator of this distribution. It is given by $V = JN$ where N represents the unit normal to the sphere and J the usual complex structure on $\mathbb{R}^{2m+2}$. V is the standard Hopf vector field, but it is usual to call also Hopf vector field any vector field obtained as the image of N by any complex structure.

Many authors have studied the value of the infimum and the regularity of minimizers on the spheres, endowed with the usual metric (see for example [**1**], [**10**], [**6**] and [**2**]). In particular, for dimension 3, Gluck and Ziller showed in [**7**] that Hopf vector fields on the round sphere are the absolute minimizers of the volume and the analogous result for the energy was shown by Brito in [**3**]. For the study of these questions in another 3-dimensional manifolds, see for example [**8**].

2000 *Mathematics Subject Classification.* 53C20, 58E15, 53C25, 58E20.
Key words and phrases. Generalized energy functionals; unit vector fields on spheres; critical points; Hopf fibration; Berger's metrics.
This paper is in final form and no version of it will be published elswhere.

In all these works the metric considered on the sphere is the canonical one, but in [**5**], Gil-Medrano and the author studied the behavior of the Hopf vector field with respect to the volume and the energy when we consider on the sphere the canonical variation of the Riemannian submersion given by the Hopf fibration. The metrics so constructed are known as Berger's metrics, they consist in a 1-parameter variation g_μ for $\mu > 0$. Moreover, they also studied the subset of $\mathbb{R}^+ \times \mathbb{R}^+$ of pairs (μ, λ) such that V^μ is stable as a critical point of the generalized energy E_{g_λ} on the spheres of dimension greater than three.

In this paper, following the methods developed in [**2**] and used in [**5**], we are going to determine the behavior of the unit Hopf vector field V^μ with respect the generalized energy E_{g_λ} on Berger's 3-spheres. This paper is organized as follows:

We devote section 2 to recall the definitions and to state the results we will need in the sequel.

In section 3, we study the values of λ and μ for which the Hopf vector field V^μ is stable as a critical point of E_{g_λ}. We show, for example, that if $\mu \leqslant 8/3$, then V^μ is stable if and only if $\lambda \leqslant \frac{(\mu-2)^2}{\mu}$, and that if $8/3 < \mu \leqslant 4$ and $\lambda \leqslant \frac{(\mu-3)^2}{\mu-2}$, V^μ is a stable critical point. Nevertheless, there exist values of the parameters λ and μ, for which the question is still open.

To finish the paper, thanks to the special structure of the 3-sphere, we prove that if $\mu < 1$, then the Hopf vector field V^μ is , up to sign, the only minimizer of the generalized energy E_{g_λ} for $\lambda \leqslant \mu$.

2. Definitions and known results

Given a Riemannian manifold (M, g), the Sasaki metric g^S on the tangent bundle TM is defined, using g and its Levi-Civita connection ∇, as follows:

$$g^S(\zeta_1, \zeta_2) = g(\pi_* \circ \zeta_1, \pi_* \circ \zeta_2) + g(\kappa \circ \zeta_1, \kappa \circ \zeta_2),$$

where $\pi : TM \to M$ is the projection and κ is the connection map of ∇. We will consider also its restriction to the tangent sphere bundle, obtaining the Riemannian manifold (T^1M, g^S).

As in [**4**], for each metric $\tilde{g}$ on M we can define the generalized energy of the vector field V, denoted $E_{\tilde{g}}(V)$, as the energy of the map $V : (M, \tilde{g}) \to (TM, g^S)$ that is given by

$$E_{\tilde{g}}(V) = \frac{1}{2} \int_M \operatorname{tr} L_{(\tilde{g},V)} \, dv_{\tilde{g}},$$

where $L_{(\tilde{g},V)}$ is the endomorphism determined by $V^*g^S(X, Y) = \tilde{g}(L_{(\tilde{g},V)}(X), Y)$. This energy can also be written as

$$E_{\tilde{g}}(V) = \frac{1}{2} \int_M \sqrt{\det P_{\tilde{g}}} \, \operatorname{tr}(P_{\tilde{g}}^{-1} \circ L_V) \, dv_g \tag{1}$$

where $P_{\tilde{g}}$ and L_V are defined by $\tilde{g}(X, Y) = g(P_{\tilde{g}}(X), Y)$, $V^*g^S(X, Y) = g(L_V(X), Y)$, respectively. By the definition of the Sasaki metric, $L_V = \operatorname{Id} + (\nabla V)^t \circ \nabla V$.

In particular, for $\tilde{g} = g$, $E_g(V)$ is the energy of the vector field V and for $\tilde{g} = V^*g^S$, the generalized energy is proportional to the volume functional. In fact,

$$F(V) = \frac{2}{n} E_{V^*g^S}(V).$$

The first variation of the generalized energy has been computed in [**4**]. It has been also shown that on a compact M, a critical vector field of any of these generalized energies should be parallel. This is one of the reasons why it is usual to restrict the functionals to the submanifold of unit vector fields and so, critical points are those V which are stationary for variations consisting on unit vector fields, or equivalently with variational field orthogonal to V.

For now on, we are going to consider the restriction of these functionals to the submanifold of unit vector fields.

Proposition 1. ([**4**]) *Let (M,g) be a Riemannian manifold, a unit vector field V is a critical point of $E_{\tilde{g}}$ if and only if*

$$\omega_{(V,\tilde{g})}(V^{\perp}) = \{0\},$$

where $V^{\perp}$ denotes the orthogonal with respect to the metric g of the 1-dimensional distribution generated by V, $\omega_{(V,\tilde{g})} = C_1^1 \nabla K_{(V,\tilde{g})}$ and $K_{(V,\tilde{g})} = \sqrt{\det P_{\tilde{g}}}\; P_{\tilde{g}}^{-1} \circ (\nabla V)^t$.

Remark. For a $(1,1)$-tensor field K, if $\{E_i\}$ is a g-orthonormal local frame,

$$C_1^1 \nabla K(X) = \sum_i g((\nabla_{E_i} K)X, E_i).$$

Theorem 1. ([**6**]) *Let V be a critical point of $E_{\tilde{g}}$, the Hessian of $E_{\tilde{g}}$ at V acting on $A \in V^{\perp}$ is given by*

$$(HessE_{\tilde{g}})_V(A) = \int_M \|A\|^2 \omega_{(V,\tilde{g})}(V)\, dv_g + \int_M \sqrt{\det P_{\tilde{g}}}\; \mathrm{tr}\left(P_{\tilde{g}}^{-1} \circ (\nabla A)^t \circ \nabla A\right) dv_g.$$

Hopf vector fields on the sphere are tangent to the fibres of the Hopf fibration $\pi : (S^3, g) \to (S^2, \overline{g})$, where g is the usual metric of curvature 1 and $\overline{g}$ is the the usual metric of curvature 4. This map is a Riemannian submersion with totally geodesic fibres whose tangent space is generated by the unit vector field $V = JN$, where N is the unit normal to the sphere and J is the usual complex structure of $\mathbb{R}^4$; in other words, $V(p) = ip$.

The canonical variation of the submersion is the one-parameter family of metrics (S^3, g_μ), $\mu > 0$, defined by

$$g_\mu|_{V^\perp} = g|_{V^\perp}, \qquad g_\mu(V,V) = \mu g(V,V), \qquad g_\mu(V, V^\perp) = 0. \tag{2}$$

For all $\mu > 0$, the map $\pi : (S^3, g_\mu) \to (S^2, \overline{g})$ is a Riemannian submersion with totally geodesic fibres. (S^3, g_μ) is known as a Berger's sphere and we will call $V^\mu = \frac{1}{\sqrt{\mu}} V$ the Hopf vector field. It is a unit Killing vector field with geodesic flow.

We denote by $\bar{\nabla}$ the Levi-Civita connection on $\mathbb{R}^4$. The Levi-Civita connection ∇ on (S^3, g) is $\nabla_X Y = \bar{\nabla}_X Y - < \bar{\nabla}_X Y, N > N$ and $\bar{\nabla}_X V = J\bar{\nabla}_X N = JX$. Therefore $\nabla_V V = 0$ and if $X \in V^\perp$ then $\nabla_X V = JX$.

Using Koszul formula, one obtains the relation of ∇^μ, the Levi-Civita connection of the metric g_μ, with ∇

$$\nabla^\mu_V X = \nabla_V X + (\mu - 1)\nabla_X V, \qquad \nabla^\mu_X V = \mu \nabla_X V, \qquad \nabla^\mu_X Y = \nabla_X Y, \tag{3}$$

for all X, Y in $V^\perp$.

By straightforward computations it can be seen that the sectional curvature K_μ of (S^3, g_μ) takes the value

$$K_\mu(\sigma) = 1 + (1-\mu)g(X, JY)^2,$$

if $\sigma \subset V^\perp$ and $\{X, Y\}$ is an orthonormal basis and it takes the value $K_\mu(\sigma) = \mu$, if the plane σ contains de vector V^μ. Consequently, the Ricci tensor has the form

$$Ric_\mu(V^\mu, V^\mu) = 2\mu, \qquad Ric_\mu(X, V^\mu) = 0,$$
$$Ric_\mu(X, Y) \;=\; 2(2-\mu)g(X, Y), \tag{4}$$

for all X, Y in $V^\perp$, and the scalar curvature is given by

$$S_\mu = 2(4-\mu).$$

PROPOSITION 2 ([**5**]). *For all $\mu, \lambda > 0$, the map*

$$V^\mu : (S^{2m+1}, g_\lambda) \to (T^1(S^{2m+1}), g_\mu^S)$$

is harmonic.

As a consequence, the Hopf vector field V^μ is a critical point of the generalized energy E_{g_λ}, for all $\lambda > 0$.

3. Stability of Hopf vector fields on S^3

To study the stability of Hopf vector fields it is useful to find several expressions of the Hessian of E_{g_λ}. These expressions have been obtained in [**5**] relating the integral of $\|\nabla^\mu A\|^2$ with the integral of $\|\pi \circ D^{\mathbb{C}} A\|^2_{V^\perp}$ and that of $\|\bar{D}^{\mathbb{C}} A\|^2_{V^\perp}$, where $D^{\mathbb{C}}$ and $\bar{D}^{\mathbb{C}}$ are the differential operators independent of μ defined as follows:

$$D^{\mathbb{C}}_X W = \bar{\nabla}_{JX} W - J\bar{\nabla}_X W \quad \text{ and } \quad \bar{D}^{\mathbb{C}}_X W = \bar{\nabla}_{JX} W + J\bar{\nabla}_X W.$$

Then, if $V^\perp$ is the distribution $Span(x, Jx)^\perp$ on $\mathbb{C}^2 \setminus \{0\}$ and $\pi : T(\mathbb{C}^2 \setminus \{0\}) \to V^\perp$ is the natural projections $\{x\} \times \mathbb{C}^2 \to V^\perp_x$, we denote by $\|\pi \circ D^{\mathbb{C}} W\|_{V^\perp}$ the norm of $\pi \circ D^{\mathbb{C}} W_{|V^\perp} : V^\perp \to V^\perp$ and analogously we denote by $\|\bar{D}^{\mathbb{C}} W\|^2_{V^\perp}$ the norm of $\pi \circ \bar{D}^{\mathbb{C}} W_{|V^\perp} = \bar{D}^{\mathbb{C}} W_{|V^\perp} : V^\perp \to V^\perp$.

As a particular case of the expressions developed in [**5**] we have

PROPOSITION 3. *Let V^μ be the unit vector field on (S^3, g_μ) and A a vector field orthogonal to V^μ, then*

$$a)(HessE_{g_\lambda})_{V^\mu}(A) = \sqrt{\lambda/\mu} \int_{S^3} \Big(-2\mu\|A\|^2 + \|\nabla^\mu A\|^2 + (\mu/\lambda - 1)\|\nabla^\mu_{V^\mu} A\|^2 \Big) dv_\mu.$$

$$b)\ (HessE_{g_\lambda})_{V^\mu}(A) = \sqrt{\lambda/\mu} \int_{S^3} \Big((4 - 3\mu - \lambda)\|A\|^2 + \frac{1}{2}\|\pi \circ D^C A\|^2_{V^\perp} + \mu/\lambda \|\nabla^\mu_{V^\mu} A + \frac{\lambda}{\sqrt{\mu}} JA\|^2 \Big) dv_\mu.$$

$$c)(HessE_{g_\lambda})_{V^\mu}(A) = \sqrt{\lambda/\mu} \int_{S^3} \Big((-4 + \mu - \lambda)\|A\|^2 + \frac{1}{2}\|\overline{D}^C A\|^2_{V^\perp} + \mu/\lambda \|\nabla^\mu_{V^\mu} A - \frac{\lambda}{\sqrt{\mu}} JA\|^2 \Big) dv_\mu.$$

The instability results for the spheres of higher dimensions with respect the functionals E_{g_λ}, have been obtained by showing that the Hessian is negative when acting on the vector fields $A_a = a - \langle a, V\rangle V - \langle a, N\rangle N = a - \bar{f}_a V - f_a N$ for all $a \in \mathbb{R}^{2m+2}$, $a \neq 0$ (see [**5**]). As a particular case we can state,

LEMMA 1. *Let V^μ be the Hopf unit vector field on (S^3, g_μ), for each $a \in \mathbb{R}^4$, $a \neq 0$ we have:*

$$(HessE_{g_\lambda})_{V^\mu}(A_a) \;=\; \frac{\sqrt{\lambda}}{2}|a|^2\Big(-\mu + 2 + \frac{(\mu-1)^2}{\lambda}\Big)vol(S^3).$$

From here we obtain,

PROPOSITION 4. *On (S^3, g_μ), if $2 - \mu + \frac{(\mu-1)^2}{\lambda} < 0$, or equivalently, if $\lambda > (\mu-1)^2/(\mu-2)$ and $\mu > 2$, then V^μ is an unstable critical point of the generalized energy E_{g_λ}.* But, in this case, thanks to the special structure of the 3-sphere, we can do better. In fact, if i, j, k represent the imaginary unit quaternions and we take $V = iN$, $E_1 = jN$ and $E_2 = kN$, then $\{V^\mu, E_1, E_2\}$ is an adapted g_μ-orthonormal frame where each vector is a Killing vector field on the round sphere. If we use Proposition 3 to compute the Hessian on the directions E_i with $i = 1,\ 2$ we obtain,

LEMMA 2.

$$(HessE_{g_\lambda})_{V^\mu}(E_i) = \sqrt{\lambda}\Big(-\mu + \frac{(\mu-2)^2}{\lambda}\Big)\mathrm{vol}(S^3),$$

where $i = 1, 2$.

PROOF. We will use expression a) of Proposition 3. In [**9**] it has been shown that

$$\nabla_{E_1}E_1 = \nabla_{E_2}E_2 = \nabla_V V = 0,$$
$$\nabla_{E_1}V = -\nabla_V E_1 = E_2,\ \nabla_{E_2}V = -\nabla_V E_2 = -E_1,\ \nabla_{E_1}E_2 = -\nabla_{E_2}E_1 = -V.$$

Then, it is easy to see that

$$\|\nabla^\mu E_i\|^2 = \frac{(\mu-2)^2}{\mu} + \mu \quad \text{and} \quad \|\nabla^\mu_{V^\mu} E_i\|^2 = \frac{(\mu-2)^2}{\mu},$$

which completes the proof. □

Consequently,

PROPOSITION 5. *On (S^3, g^μ), if $\lambda > (\mu-2)^2/\mu$ then the Hopf vector field V^μ is an unstable critical point of E_{g_λ}.*

Remark. Since $(\mu-1)^2\mu > (\mu-2)^3$ for $\mu > 2$, the above Proposition improves the instability result of Proposition 4.

To solve the problem we would have to determine the behavior of Hopf vector fields when λ and μ verify the inequality $(\mu-2)^2/\mu \geqslant \lambda$. In order to this, we are going to follow the techniques developed in [**2**] and used in [**5**] for discussing these kind of questions in higher dimensional spheres.

If we consider a vector field A on S^3, orthogonal to the Hopf vector field, as a map $A : S^3 \longrightarrow V^\perp \subset \mathbb{C}^2$ where $V^\perp$ here represents the distribution $V_x^\perp = Span\{x, Jx\}^\perp$, we can write

$$A_l(p) = \frac{1}{2\pi}\int_0^{2\pi} A(e^{i\theta}p)e^{-il\theta}d\theta \in V_p^\perp$$

so that
$$A(p) = \sum_{l \in Z} A_l(p)$$
is the Fourier series of A. Since $A_l(e^{i\theta}p) = e^{il\theta}A_l(p)$ then
$$\nabla_V A = \bar{\nabla}_V A = \sum_{l \in Z} ilA_l = \sum_{l \in Z} lJA_l$$
and
$$\|\nabla^{\mu}_{V^{\mu}} A_l + \alpha JA_l\|^2 = \frac{1}{\mu}(l - 1 + \mu + \alpha\sqrt{\mu})^2 \; \|A_l\|^2.$$
If $\mathcal{C}(p)$ denotes the fibre of the Hopf fibration $\pi : S^3 \longrightarrow S^2$ passing through p, and for $l \neq q$,
$$\int_{\mathcal{C}(p)} < A_l, A_q >= 0.$$
By the construction of Berger's metrics, this fact is independent of μ and so, the following Lemma, shown in [**2**] for the volume functional in the case $\mu = 1$, remains valid

Lemma 3.
$$(HessE_{g_\lambda})_{V^\mu}(A) \;\; = \;\; \sum_{l \in Z} (HessE_{g_\lambda})_{V^\mu}(A_l).$$
Now, we can prove the following

Theorem 2. *On (S^3, g_μ) the Hopf unit vector field V^μ is stable as a critical point of the functionals E_{g_λ} in the following cases:*

a) *If $\mu \leqslant 8/3$, for $\lambda \leqslant \frac{(\mu-2)^2}{\mu}$,*
b) *If $8/3 < \mu \leqslant 4$, for $\lambda \leqslant \frac{(\mu-3)^2}{\mu-2}$,*
c) *If $\mu > 4$, for $\lambda \leqslant \mu - 4$.*

Proof. The condition c) is a direct consequence of expression c) of Proposition 3. Moreover, notice that,
$$\frac{(\mu-3)^2}{\mu-2} < \frac{(\mu-2)^2}{\mu} \quad \text{for} \;\; \mu > 8/3 \tag{5}$$
and
$$\frac{(\mu-2)^2}{\mu} \leqslant \frac{(\mu-3)^2}{\mu-2} \quad \text{for} \;\; 2 < \mu \leqslant 8/3. \tag{6}$$
For a) and b), thanks to Lemma 3, we only have to show that, under the hypothesis, $(HessE_{g_\lambda})_{V^\mu}(A_l) \geqslant 0$ for all $l \in \mathbb{Z}$.

Using expression b) of Proposition 3 we have that
$$(HessE_{g_\lambda})_{V^\mu}(A_l) \geqslant \sqrt{\frac{\lambda}{\mu}}\; e_1(\lambda, \mu, l) \int_{S^3} \|A_l\|^2 dv_\mu,$$
where $e_1(\lambda, \mu, l) = -3\mu + 4 - \lambda + \frac{1}{\lambda}(l - 1 + \mu + \lambda)^2$.

If we assume a) or b), then
$$e_1(\lambda, \mu, l) = -\mu + 2 + 2l + \frac{1}{\lambda}(l - 1 + \mu)^2 \geqslant 0$$

for all $l \in \mathbb{Z}^+$ when $\mu \leqslant 2$. Now, if $\mu > 2$ we have that $e_1(\lambda, \mu, l) \geqslant e_1(\lambda, \mu, 0)$ and that

$$e_1(\lambda, \mu, 0) = -\mu + 2 + \frac{(\mu-1)^2}{\lambda} \geqslant -\mu + \frac{(\mu-2)^2}{\lambda} \geqslant 0.$$

Consequently, $(HessE_{g_\lambda})_{V^\mu}(A_l) \geqslant 0$ for all $l \in \mathbb{Z}^+$.

For the negative values of l, using expression c) of Proposition 3 we obtain that

$$(HessE_{g_\lambda})_{V^\mu}(A_l) \geqslant \sqrt{\frac{\lambda}{\mu}} e_2(\lambda, \mu, l) \int_{S^3} \|A_l\|^2 \mathrm{dv}_\mu,$$

where

$$e_2(\lambda, \mu, l) = \mu - 4 - \lambda + \frac{1}{\lambda}(l - 1 + \mu - \lambda)^2. \tag{7}$$

Since $e_2(\lambda, \mu, l) \geqslant -\mu - 2(l+1)$, then $e_2(\lambda, \mu, l) \geqslant 0$ for all $l \leqslant -3$ if $\mu \leqslant 4$. For $l = -1, -2$, if we assume a) or b) then,

$$e_2(\lambda, \mu, -1) = -\mu + (\mu - 2)^2/\lambda \geqslant 0,$$

and

$$e_2(\lambda, \mu, -2) = -\mu + 2 + (\mu - 3)^2/\lambda \geqslant 0,$$

by (5) and (6) respectively. $\square$

The above Theorem, jointly with the corresponding instability result, solves completely the problem when $\mu \leqslant 8/3$. For other values of μ there exist values of λ for which the behavior of Hopf vector fields is still an open question (see Figure 1).

It is known that on the 3-dimensional round sphere Hopf vector fields are the absolute minimizers of the energy (see [**3**]), but for Berger's 3-sphere the situation is quite different, as can be seen in [**5**]. It has been shown there, that Hopf vector fields are the only minimizers of the energy if and only if $\mu \leqslant 1$.

THEOREM 3. *On (S^3, g_μ) with $\mu \leqslant 1$, the unit Hopf vector field is, up to sign, the only minimizer of the generalized energy E_{g_λ} for $\lambda < \mu$.*

PROOF. Let X be a unit vector field on (S^3, g_μ), then

$$E_{g_\lambda}(X) = \frac{\sqrt{\lambda}}{2}(\mu/\lambda+2)vol(S^3) + \frac{\sqrt{\lambda/\mu}}{2}\int_{S^3}(\mu/\lambda\|\nabla^\mu_{V^\mu}X\|^2 + \|\nabla^\mu_{E_1}X\|^2 + \|\nabla^\mu_{E_2}X\|^2)\mathrm{dv}_\mu.$$

Since $\mu > \lambda$,

$$\frac{2}{\sqrt{\lambda}}E_{g_\lambda}(X) \geqslant (\frac{\mu}{\lambda} + 2)vol(S^3) + \tag{8}$$

$$+\frac{1}{\sqrt{\mu}}\int_{S^3}(\|\nabla^\mu_{V^\mu}X\|^2 + \|\nabla^\mu_{E_1}X\|^2 + \|\nabla^\mu_{E_2}X\|^2)\mathrm{dv}_\mu$$

$$\geqslant (\frac{\mu}{\lambda} + 2)vol(S^3) + \frac{1}{\sqrt{\mu}}\int_{S^3} Ric_\mu(X, X)\mathrm{dv}_\mu, \tag{9}$$

as can be seen in [**3**].

Using 4 we have that if $\mu < 1$, then

$$Ric_\mu(X, X) \geqslant Ric_\mu(V^\mu, V^\mu) = 2\mu$$

for all unit X, with equality if and only if $X = \pm V^\mu$.

For $\mu = 1$, g_1 is the usual metric on the sphere, so $Ric(X,X) = 2$ for all unit X and it is known (see [**3**]) that the equality in (9) holds if and only if X is a unit Killing vector field. Moreover, the equality in (8) holds only if $\|\nabla_V X\|^2 = 0$. In addition, it is easy to see that these two conditions (i.e, X is a unit Killing vector field and $\|\nabla_V X\|^2 = 0$) are satisfied if and only if $X = \pm V$.

Therefore, under the hypothesis of the Theorem

$$E_{g_\lambda}(X) \geqslant \frac{\sqrt{\lambda}}{2}(\mu/\lambda + 2)vol(S^3) + \mu\sqrt{\lambda}vol(S^3),$$

and the equality holds only for the unit Hopf vector field. □

Remark. It is worthy to recall that if $\mu \leqslant 1$ the sphere with the metrics g_μ is isometrically immersed as a geodesic sphere in the complex projective space.

The results obtained in Theorems 2 and 3 can be represented graphically on $\mathbb{R}^+ \times \mathbb{R}^+$ as can be seen in Figure 1.

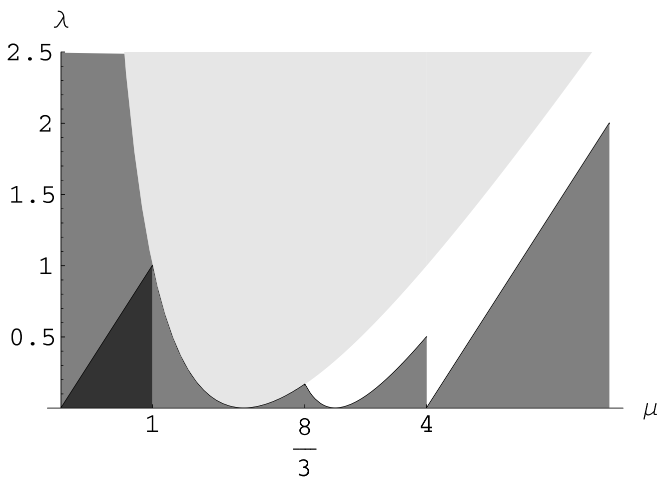

FIGURE 1. The light gray region is the subset of $\mathbb{R}^+ \times \mathbb{R}^+$ of pairs (μ, λ) such that V^μ is unstable as a critical point of E_{g_λ}. The stability domain is painted in dark gray and the darkest region corresponds to the values of (μ, λ) for which V^μ is absolute minimizer of E_{g_λ}. The question is still open for the white region.

Acknowledgments

The author wants to thank O. Gil-Medrano for her collaboration in the preparation of this paper. Partially supported by DGI (Spain) and FEDER Project MTM 2004-06015-C02-01, a grant AVCiTGRUPOS03/169 and by a Research Grant from *Ministerio de Educación y Cultura"*

References

[1] V. Borrelli, F. Brito and O. Gil-Medrano, *The infimum of the Energy of unit vector fields on odd-dimensional spheres*, Ann. Global Anal. Geom. 23 (2003) 129-140.

[2] V. Borrelli and O. Gil-Medrano, *A critical radius for unit Hopf vector fields on spheres*, Preprint.

[3] F. Brito, *Total Bending of flows with mean curvature correction*, Diff. Geom. and its Appl. 12 (2000), 157-163.

[4] O. Gil-Medrano, *Relationship between volume and energy of vector fields*, Diff. Geom. and its Appl. 15 (2001), 137-152.

[5] O. Gil-Medrano and A. Hurtado, *Volume, energy and generalized energy of unit vector fields on Berger's spheres. Stability of Hopf vector fields*, Proc. Roy. Soc. Edinburgh Sect. A. to appear.

[6] O. Gil-Medrano and E. Llinares-Fuster, *Second variation of Volume and Energy of vector fields. Stability of Hopf vector fields*, Math. Ann. 320 (2001), 531-545.

[7] H. Gluck and W. Ziller, *On the volume of a unit vector field on the three-sphere*, Comment. Math. Helv. 61 (1986), 177-192.

[8] J. C. González-Dávila and L. Vanhecke, *Energy and volume of unit vector fields on three-dimensional Riemannian manifolds*, Differential Geom. Appl. 16 (2002), 225-244.

[9] G. Wiegmink, *Total bending of vector fields on the Sphere S^3*, Diff. Geom. and its Appl. 6 (1996), 219-236.

[10] C.M.Wood, *On the energy of a unit vector field*, Geometriae Dedicata 64 (1997), 319-330.

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