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Liouville theorem for geometric structures arising from minimal rational curves

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1. Introduction-Liouville theorem in conformal geometry

The classical theorem of Liouville in conformal geometry ([DFN, 15.2]), states that any local conformal automorphism of the sphere of dimension $\geqslant 3$ comes from a global conformal automorphism of the sphere. Our aim is to discuss some generalizations of Liouville's theorem, in particular, for certain geometric structures that arise naturally from algebraic geometry. We will survey a number of results related to this topic, omitting technical details and proofs.

Before we proceed, let us mention that a lot of results in the direction of generalizing Liouville theorem have been obtained by various people in many different settings which we have no expertise to discuss here. We will restrict ourselves to a few results which are relevant for our purpose. Above all, we will strictly narrow our discussion to holomorphic situation.

In holomorphic category, the analogue of the sphere of dimension $\geqslant 3$ is the hyperquadric $\mathbf{Q}_n \subset \mathbf{P}_{n+1}$ with $n \geqslant 3$, namely, the complex hypersurface of the complex projective space $\mathbf{P}_{n+1} = \{[z_0; z_1; \cdots ; z_{n+1}]\}$ given by a non-degenerate quadratic equation

$$\mathbf{Q}_n := (z_0^2 + \cdots + z_{n+1}^2 = 0).$$

This is a homogeneous complex manifold and at each point $x \in \mathbf{Q}_n$ the isotropy action on the projectivized holomorphic tangent space $\mathbf{P}T_x(\mathbf{Q}_n)$ has a unique closed orbit $\mathcal{W}_x \subset \mathbf{P}T_x(\mathbf{Q}_n)$. As a projective subvariety, $\mathcal{W}_x \subset \mathbf{P}T_x(\mathbf{Q}_n)$ is isomorphic to a lower dimensional hyperquadric $\mathbf{Q}_{n-2} \subset \mathbf{P}_{n-1}$. Thus the union of $\mathcal{W}_x$ as x varies over $\mathbf{Q}_n$ defines a holomorphic conformal structure on $\mathbf{Q}_n$. Then Liouville's theorem implies that a local automorphism of the conformal structure on $\mathbf{Q}_n$ can be extended to a global automorphism of the complex manifold $\mathbf{Q}_n$. More precisely, we may restate the theorem as follows.

Theorem 1 (Liouville) *Let U_1 and U_2 be two connected open subsets of $\mathbf{Q}_n$. Suppose there exists a biholomorphic map $\varphi : U_1 \to U_2$ such that for each $x \in U_1$, the derivative $d\varphi_x : \mathbf{P}T_x(U_1) \to \mathbf{P}T_{\varphi(x)}(U_2)$ satisfies*

$$d\varphi_x(\mathcal{W}_x) = \mathcal{W}_{\varphi(x)}.$$

Then there exists a biholomorphic automorphism $\Phi : \mathbf{Q}_n \to \mathbf{Q}_n$ such that $\varphi = \Phi|_{U_1}$.

There are two natural ways to generalize this. Regarding $\mathbf{Q}_n$ as the flat model of holomorphic conformal geometry, we may generalize this to the flat models of other

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holomorphic geometric structures. This will be discussed in Section 2. Another possibility is to generalize it to certain non-flat conformal structures, which will be discussed in Section 3.

2. Liouville theorem for G/P

The hyperquadric is an example of irreducible Hermitian symmetric spaces of compact type. Recall that a compact complex homogeneous space G/P where G is a complex simple Lie group and P is a parabolic subgroup is an irreducible Hermitian symmetric space of compact type if the isotropy representation of P on the tangent space $T_o(G/P)$ at a base point $o = [P] \in G/P$ is irreducible. Let $\mathcal{W}_o \in \mathbf{P}T_o(G/P)$ be the highest weight orbit of the isotropy representation. Then the translates of $\mathcal{W}_o$ by the group action G defines a holomorphic fiber subbundle $\mathcal{W}$ of $\mathbf{P}T(G/P)$. This geometric structure is exactly the conformal structure discussed above when $G/P = \mathbf{Q}_n$. It is natural to ask whether the analogue of Liouville theorem holds for this geometric structure.

There is an obvious exception, the complex projective space $\mathbf{P}_n$. For $\mathbf{P}_n = G/P$, the isotropy action on $\mathbf{P}T_o(\mathbf{P}_n)$ is transitive and $\mathcal{W}_o = \mathbf{P}T_o(\mathbf{P}_n)$. Thus the geometric structure defined by $\mathcal{W} = \mathbf{P}T(\mathbf{P}_n)$ is just the full complex structure. The analogue of Theorem 1 is definitely false in this case. A biholomorphic map $\varphi : U_1 \to U_2$ between any two open subsets of $\mathbf{P}_n$ automatically satisfies the void condition $d\varphi_x(\mathcal{W}_x) = \mathcal{W}_{\varphi(x)}$ for each $x \in U_1$ and there is no reason why φ can be extended to a biholomorphic automorphism of $\mathbf{P}_n$.

However if we exclude $\mathbf{P}_n$, Liouville theorem does hold for every irreducible Hermitian symmetric space G/P.

Theorem 2 (Kobayashi-Nagano, Tanaka) *Let U_1 and U_2 be two connected open subsets of an irreducible Hermitian symmetric space G/P different from the projective space. Suppose there exists a biholomorphic map $\varphi : U_1 \to U_2$ such that for each $x \in U_1$, the derivative $d\varphi_x : \mathbf{P}T_x(U_1) \to \mathbf{P}T_{\varphi(x)}(U_2)$ satisfies*

$$d\varphi_x(\mathcal{W}_x) = \mathcal{W}_{\varphi(x)}.$$

Then there exists a biholomorphic automorphism $\Phi : G/P \to G/P$ such that $\varphi = \Phi|_{U_1}$.

This follows from the fact proved in [KN] and [Ta1] that the Lie algebra of infinitesimal automorphism of the geometric structure $\mathcal{W} \subset \mathbf{P}T(G/P)$ is exactly the Lie algebra of G when G/P is not a projective space. This implies that φ induces an automorphism of the Lie algebra from which one can deduce Theorem 2.

A further generalization of this geometric structure was studied by Tanaka [Ta2]. To simplify the matter, let us concentrate on the case G/P where G is a complex simple Lie group and P is a maximal parabolic subgroup. Irreducible Hermitian symmetric spaces are special examples of this case. The isotropy representation of P on $T_o(G/P)$ is not necessarily irreducible, but has a unique irreducible subspace. Let $\mathcal{W}_o \subset \mathbf{P}T_o(G/P)$ be the highest weight orbit of the unique irreducible subspace. Then its translates $\mathcal{W} \subset \mathbf{P}T(G/P)$ defines a geometric structure. What is different from the Hermitian symmetric space case is that this geometric structure is not necessarily flat. In fact, it is flat only in the Hermitian symmetric case. For this geometric structure, Liouville theorem holds:

Theorem 3 (Yamaguchi [Ya]) *Let G be a complex simple Lie group and P be a maximal parabolic subgroup. Assume that G/P is not the projective space. Let U_1 and U_2 be two connected open subsets of G/P. Suppose there exists a biholomorphic map $\varphi : U_1 \to U_2$ such that for each $x \in U_1$, the derivative $d\varphi_x : \mathbf{P}T_x(U_1) \to \mathbf{P}T_{\varphi(x)}(U_2)$ satisfies*

$$d\varphi_x(\mathcal{W}_x) = \mathcal{W}_{\varphi(x)}.$$

Then there exists a biholomorphic automorphism $\Phi : G/P \to G/P$ such that $\varphi = \Phi|_{U_1}$.

In fact, [Ya] computed the Lie algebra of infinitesimal automorphisms of the geometric structure, from which Theorem 3 follows.

3. Liouville theorem for hypersurfaces

The hyperquadric $\mathbf{Q}_n \subset \mathbf{P}_{n+1}$ is a special case of complex hypersurfaces in $\mathbf{P}_{n+1}$. Let $X \subset \mathbf{P}_{n+1}, n \geqslant 3$, be a compact complex smooth hypersurface. By Chow's theorem, X is the zero set of a homogeneous polynomial $F(z_0, \ldots, z_{n+1})$ of some degree d in the homogeneous coordinates $[z_0; z_1; \cdots ; z_{n+1}]$ of $\mathbf{P}_{n+1}$. Given a point $x \in X$, the second fundamental form of X at x is a quadratic form

$$II_x : S^2T_x(X) \to N_x$$

where N_x is the normal space of X at x, is given by the Hessian of F at x. When $d \geqslant 2$ and x is a general point of X, the second fundamental form is non-degenerate and its zero set $\mathcal{W}_x \subset \mathbf{P}T_x(X)$ defines a smooth hyperquadric. As $\mathcal{W}_x$ varies over general points $x \in X$, we get a holomorphic conformal structure. In other words, on a Zariski-open subset of the hypersurface X, there is a natural conformal structure defined by the second fundamental form of X in $\mathbf{P}_{n+1}$. When $d = 2$, i.e., when $X = \mathbf{Q}_n$, the zero set of the second fundamental form $\mathcal{W}_x$ is exactly the highest weight orbit of the isotropy representation we encountered in Section 1. Thus as a generalization of Theorem 1, it is natural to ask whether Liouville theorem holds for this conformal structure for $d \geqslant 3$ and the answer is Yes.

Theorem 4 (Jensen-Musso) *Let U_1 and U_2 be two connected open subsets of a smooth hypersurface $X \subset \mathbf{P}_{n+1}$ of degree $\geqslant 2$. Suppose there exists a biholomorphic map $\varphi : U_1 \to U_2$ such that for a general point $x \in U_1$, the derivative $d\varphi_x : \mathbf{P}T_x(U_1) \to \mathbf{P}T_{\varphi(x)}(U_2)$ satisfies*

$$d\varphi_x(\mathcal{W}_x) = \mathcal{W}_{\varphi(x)}.$$

Then there exists a biholomorphic automorphism $\Phi : X \to X$ such that $\varphi = \Phi|_{U_1}$.

This follows from a more general result proved in [JM]. In fact, one needs not assume that the open subsets U_1 and U_2 belong to the same hypersurface X and the following stronger statement holds.

Theorem 5 (Jensen-Musso) *Let X_1 and X_2 be two smooth hypersurfaces in $\mathbf{P}_{n+1}$. Let $U_1 \subset X_1$ and $U_2 \subset X_2$ be connected open subsets. Suppose there exists a biholomorphic map $\varphi : U_1 \to U_2$ such that for a general point $x \in U_1$, the derivative $d\varphi_x : \mathbf{P}T_x(U_1) \to \mathbf{P}T_{\varphi(x)}(U_2)$ satisfies*

$$d\varphi_x(\mathcal{W}_x) = \mathcal{W}_{\varphi(x)}.$$

Then there exists a biholomorphic map $\Phi : X_1 \to X_2$ such that $\varphi = \Phi|_{U_1}$.

The proof of this theorem uses moving frame method and the calculation is rather involved. It has to be mentioned that the main result of [JM] contains much more than Theorem 5. They showed that to verify the existence of φ satisfying the above condition, it suffices to check certain third-order invariants of the hypersurfaces. Just to get Theorem 5, there proof can be simplified following the arguments in [JM, Section 4].

4. Describing the geometric structures via lines

Theorem 3 and Theorem 4 provide two different generalization of Theorem 1. Both are concerned with geometric structures defined by holomorphic fiber subbundles of the projectivized holomorphic tangent bundles. However the definitions of two geometric structures are of different nature. The structure on G/P is defined using the isotropy representation of P while the structure on hypersrufaces is defined by the second fundamental form. It is very interesting to note that when we restrict to a certain class of G/P and hypersurfaces, there is a way to describe the two geometric structures in a common setting, by using lines.

First, consider a smooth hypersurface X of degree d in $\mathbf{P}_{n+1}$. Pick a point $x \in X \subset \mathbf{P}_{n+1}$ and consider the set Lines_x of all lines on $\mathbf{P}_{n+1}$ passing through x. Since a line in $\mathbf{P}_{n+1}$ is determined uniquely by a point on it and the tangent direction at that point, we have a natural identification

$$\mathrm{Lines}_x \cong \mathbf{P}T_x(\mathbf{P}_{n+1}).$$

Let $\mathcal{K}_x \subset \mathrm{Lines}_x$ be the subset consisting of lines through x lying on X. Since the degree of X is d, $\mathcal{K}_x$ is the set of lines through x which have order of contact $\geqslant d$ with x. Under the identification of Lines_x and $\mathbf{P}T_x(\mathbf{P}_{n+1})$, the condition 'order of contact $\geqslant d$' is given by d homogenous equations of respective degree $1, 2, \ldots, d$ in a homogeneous coordinate of $\mathbf{P}T_x(\mathbf{P}_{n+1})$. Thus when $d \leqslant n$, the set $\mathcal{K}_x$ is non-empty. Moreover, one can show that these d equations define a complex submanifold $\mathcal{C}_x$ of dimension $n-d$ in $\mathbf{P}T_x(\mathbf{P}_{n+1})$ when x is a general point of X. Among these equations, the equation of degree 1 defines the hyperplane in $\mathbf{P}T_x(\mathbf{P}_{n+1})$ which is exactly the projectivized tangent space of X at x. Thus $\mathcal{C}_x$ lies in $\mathbf{P}T_x(X)$ and

$$\mathcal{C}_x \cong \{\alpha \in \mathbf{P}T_x(X), \alpha \text{ is tangent to a line on } X \text{ through } x\}.$$

As x varies over general points of X, the complex submanifolds $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ defines a geometric structure on X. This geometric structure has not been studied classically. But note that there exists a unique hyperquadric in $\mathbf{P}T_x(X)$ containing the submanifold $\mathcal{C}_x$ for general $x \in X$. This hyperquadric is the equation of degree 2 defining $\mathcal{C}_x$ and is nothing but the zero set $\mathcal{W}_x$ of the second fundamental form of X at x. In other words, the conformal structure $\mathcal{W}_x$ discussed in the previous section is determined by the geometric structure $\mathcal{C}_x$ coming from the lines passing through x, when $d \leqslant n$.

Next, consider G/P with P a maximal parabolic subgroup. The parabolic group P is associated with a simple root of the Lie algebra $\mathbf{g}$ of G in the following manner.

Choose a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$ and the root system $\Phi \subset \mathbf{h}^*$ of $\mathbf{g}$ with respect to $\mathbf{h}$. Fix a system of simple roots $\{\alpha_1, \ldots, \alpha_l\}$ and a distinguished choice of a simple root α_i. Given an integer $k, -m \leqslant k \leqslant m$, we define Φ_k as the set of all roots $\sum_{q=1}^{l} m_q\alpha_q$ with $m_i = k$. Here m is the largest integer such that $\Phi_m \neq 0$. For $\alpha \in \Phi$, let $\mathbf{g}_\alpha$ be the corresponding root space. Define

$$\begin{aligned} \mathbf{g}_0 &= \mathbf{h} \oplus \bigoplus_{\alpha\in\Phi_0} \mathbf{g}_\alpha \\ \mathbf{g}_k &= \bigoplus_{\alpha\in\Phi_k} \mathbf{g}_\alpha, \quad k \neq 0. \end{aligned}$$

The decomposition $\mathbf{g} = \bigoplus_{k=-m}^{m} \mathbf{g}_k$ gives a graded Lie algebra structure on $\mathbf{g}$. Define

$$\mathbf{p} \quad = \quad \mathbf{g}_0 \oplus \mathbf{g}_{-1} \oplus \cdots \oplus \mathbf{g}_{-m}.$$

We say that $\mathbf{p}$ is the maximal parabolic subalgebra associated to the simple root α_i. The closed subgroup of G corresponding to the subalgebra $\mathbf{p}$ is a maximal parabolic subgroup. Conversely, any maximal parabolic subgroup P arises this way, associated to a simple root.

The quotient map $G \to G/P$ defines a P-principal bundle on G/P. The left action of P on the reductive group $L = P/U$ where U is the unipotent radical of P, induces an L-principal bundle $\mathbf{L}$ on G/P. There exists a homogeneous line bundle $\mathcal{L}$ on G/P whose first Chern class generates $H^2(G/P, \mathbf{Z})$. This line bundle $\mathcal{L}$ is associated to $\mathbf{L}$ by a 1-dimensional representation of L. This representation can be described as follows. Let α_i be the simple root defining P. Let $H_{\alpha_i} \in \mathbf{h}$ be its coroot. The center of the reductive group $L = P/U$ has Lie algebra $\mathbf{C}H_{\alpha_i}$. Hence a $\mathbf{Z}$-functional on $\mathbf{Z}H_{\alpha_i}$ induces a character of L, giving rise to a homogeneous line bundle on G/P. The line bundle $\mathcal{L}$ is the one associated to the functional having value 1 on H_{α_i}.

For example, when $G/P = \mathbf{P}_1$, $\mathbf{g} = \mathbf{sl}_2$ has a unique simple root and corresponding coroot. A functional having value $k \in \mathbf{Z}$ on the coroot gives rise to the line bundle $\mathcal{O}(k)$ on $\mathbf{P}_1$.

It is well-known that $\mathcal{L}$ is very ample, namely, the holomorphic sections of $\mathcal{L}$ over G/P defines an equivariant projective embedding

$$G/P \subset \mathbf{P}\Gamma(G/P, \mathcal{L})^*.$$

On our rational homogeneous space G/P, there are lines under the embedding defined by $\mathcal{L}$. Let α_i be the simple root defining P and $H_{\alpha_i} \in \mathbf{h}$ be its coroot. Let $\mathbf{s}_{\alpha_i} \subset \mathbf{g}$ be the subalgebra isomorphic to $\mathbf{sl}_2$ such that $\mathbf{s}_{\alpha_i} \cap \mathbf{h} = \mathbf{C}H_{\alpha_i}$ and H_{α_i} is the coroot for $\mathbf{s}_{\alpha_i}$. The orbit of $o \in G/P$ under the subgroup $S_{\alpha_i} \subset G$ with Lie algebra $\mathbf{s}_{\alpha_i}$ is a complex curve to be denoted by C_{α_i}. Note that the character of L defining $\mathcal{L}$ has value 1 on H_{α_i}. Thus C_{α_i} is a line under the embedding of G/P defined by $\mathcal{L}$. Under the natural identification of $\mathbf{g}_1$ as a subspace of the tangent space $T_o(G/P)$, H_{α_i} is a tangent vector of the line C_{α_i} at the point $o \in G/P$.

Now assume that α_i is a long simple root. Then we can show ([HM3, Proposition 1]) that any line on G/P under the above projective embedding is conjugate to C_{α_i}, namely, the set of all lines on G/P is homogeneous under the natural action of G. Denote by $\mathcal{C}_o \subset \mathbf{P}T_o(G/P)$ the tangent vectors to lines passing through the base point o. Then $\mathcal{C}_o$ is the unique closed orbit of the isotropy action of P on $\mathbf{P}T_o(G/P)$. Thus when P is associated to a long simple root, the geometric structure $\mathcal{W}_x \subset \mathbf{P}T_x(G/P)$ is exactly $\mathcal{C}_x \subset \mathbf{P}T_x(G/P)$, the tangents to lines on G/P passing through x.

When α_i is not a long root, $\mathcal{C}_x$ may be different from $\mathcal{W}_x$. But it is possible to recover $\mathcal{W}_x$ from $\mathcal{C}_x$ as a certain distinguished submanifold.

In summary, we have seen that for G/P associated with a long simple root and hypersurfaces in $\mathbf{P}_{n+1}$ of degree $\leqslant n$, the geometric structure $\mathcal{W}_x$ we discussed in Section 2 and Section 3 is equivalent to the one arising from $\mathcal{C}_x$, which is the subset of the projectivized tangent space consisting of tangent vectors to lines through x under a suitable embedding of the manifold in a projective space.

5. Geometric structure defined by the variety of minimal rational tangents

In the previous section, we have seen that we can describe the natural geometric structure on some G/P and hypersurfaces in terms of lines under a suitable projective embedding. The notion of lines seems to depend on the choice of the projective embedding. However, we can define these lines in a more intrinsic manner. This approach makes it possible to define such geometric structures on a large class of complex manifolds. Let us start from the general set-up.

Let X be a compact Kähler manifold with the second Betti number $b_2 = 1$. A rational curve on X means the image of a non-constant holomorphic map from $\mathbf{P}_1$ to X. We will assume that X is covered by rational curves, in other words, for each point $x \in X$, there exists a rational curve passing through x. This assumption on the existence of rational curves looks strong. But by the celebrated result of Mori [Mo], if X is a Fano manifold, i.e., a compact Kähler manifold of positive Ricci curvature, then X is covered by rational curves. This provides many examples of X satisfying our assumptions. Conversely, it is easy to see that a compact Kähler manifold with $b_2 = 1$ which is covered by rational curves is a Fano manifold. Thus our assumption is equivalent to saying that X is a Fano manifold with $b_2 = 1$. Let us recall that a Fano manifold is always simply connected.

Now let $x \in X$ be a general point. A rational curve C passing through x is a minimal rational curve if its homology class $[C] \in H_2(X, \mathbf{Z})/\text{torsion} \cong \mathbf{Z}$ is minimal among all rational curves through x. Suppose X can be embedded in some projective space $\mathbf{P}_N$ such that X is covered by lines under this embedding. Then any line lying on X passing through a general point will be a minimal rational curve. But there are many examples of X satisfying our assumptions whose minimal rational curves cannot be realized as lines under any embedding of X in a projective space. For example, let $X \subset \mathbf{P}_{n+1}$ be a smooth hypersurface of degree $n+1$ for $n \geqslant 3$. Then X is a Fano manifold with $b_2 = 1$ and X is not covered by lines, but conics. For a general point $x \in X$, conics through x lying on X are the minimal rational curves. It is easy to see that these conics cannot become lines under any projective embedding of X.

Thus the concept of minimal rational curves are more general than the notion of lines. But there are some properties of lines that hold for minimal rational curves. The most important one is the following.

Bend-and-Break Lemma *Let $x \in X$ be a general point. Suppose there exists a positive-dimensional family of minimal rational curves through x. Then the intersection of all the curves in this family is x.*

In other words, given another point $y \neq x$, there exist only finitely many minimal rational curves passing through both x and y. For a comprehensive discussion of this lemma and other properties of minimal rational curves, we recommend [Ko].

Now, let

$$\mathcal{C}_x \subset \mathbf{P}T_x(X)$$

be the closure of the union of vectors tangent to some minimal rational curve through x which is immersed at x. Then $\mathcal{C}_x$ is a complex analytic subvariety of the projective space $\mathbf{P}T_x(X)$. We call it the *variety of minimal rational tangents* at x. In all the known examples, $\mathcal{C}_x$ is smooth for general x, but it is not known whether this always holds for every Fano manifold with $b_2 = 1$.

As x varies over general point of X, the varieties of minimal rational tangents define a geometric structure on an open subset of X. This geometric structure has some unusual features. Among others, the subvarieties $\mathcal{C}_{x_1} \subset \mathbf{P}T_{x_1}(X)$ and $\mathcal{C}_{x_2} \subset \mathbf{P}T_{x_2}(X)$ may not be isomorphic when x_1 and x_2 are two distinct points of X. One may say that this is a 'non-transitive' geometric structure. Of course, for G/P or any almost-homogeneous manifold, $\mathcal{C}_{x_1} \cong \mathcal{C}_{x_2}$ for two general points x_1 and x_2. However, for many examples of X, we expect that the non-transitivity holds. One problem here is that it is not easy to check the non-transitivity. For example, when X is a hypersurface of degree d with $3 \leqslant d \leqslant n+1$ in $\mathbf{P}_{n+1}$, we do not know whether the non-transitivity holds.

As an example where the non-transitivity is easy to see, let us consider the moduli space M of stable bundles of rank 2 with a fixed determinant of degree 1 over a compact Riemann surface R of genus $\geqslant 4$. It is well-known that M is a Fano manifold with $b_2 = 1$. By definition, a point of M corresponds to a holomorphic vector bundle V of rank 2 on R and two distinct points correspond to two distinct vector bundles. As was noticed in [Hw], $\mathcal{C}_x$ is isomorphic to the projectivization of the vector bundle V_x corresponding to the point $x \in M$. Thus when $x \neq y$ for y in an analytic neighborhood of x, $\mathcal{C}_x$ and $\mathcal{C}_y$ are not isomorphic.

Such non-transitive geometric structures have seldom been studied by differential geometers because such structures do not arise naturally in differential geometry. But here we have very natural non-transitive geometric structures arising from algebraic geometric examples, namely, Fano manifolds with $b_2 = 1$. It is very interesting to study the properties of these geometric structures. Among many properties, here we want to discuss the following question, which is the analogue of Liouville theorem.

Question *Let X be a Fano manifold with $b_2 = 1$. Let U_1 and U_2 be two connected open subsets of X. Suppose there exists a biholomorphic map $\varphi : U_1 \to U_2$ such that for a general point $x \in U_1$, the derivative $d\varphi_x : \mathbf{P}T_x(U_1) \to \mathbf{P}T_{\varphi(x)}(U_2)$ satisfies*

$$d\varphi_x(\mathcal{C}_x) = \mathcal{C}_{\varphi(x)}.$$

Does there exist a biholomorphic automorphism $\Phi : X \to X$ such that $\varphi = \Phi|_{U_1}$?

This is false for $X \cong \mathbf{P}_n$. As noticed before, for $X = \mathbf{P}_n$, $\mathcal{C}_x = \mathbf{P}T_x(X)$ at every point x and the condition

$$d\varphi_x(\mathcal{C}_x) = \mathcal{C}_{\varphi(x)}$$

is void. Conversely, the result of Cho-Miyaoka-Shepherd-Barron [CMS] implies that if $\mathcal{C}_x = \mathbf{P}T_x(X)$ for a general point on a Fano manifold X, then X must be a projective space. This suggests that it is very likely that $\mathbf{P}_n$ is the only exception for an affirmative answer to the Question.

How would one prove an affirmative answer to the Question? We know two cases where it holds, Theorem 3 for G/P and Theorem 4 for the hypersurfaces of degree

$\leqslant n$ in $\mathbf{P}_{n+1}$. Strictly speaking Theorem 3 applies only when P is associated to a long simple root. But when P is associated to a short root, we can recover $\mathcal{W}_x$ from $\mathcal{C}_x$ and get an affirmative answer to the Question by Theorem 3 (see [HM1]). The proofs for these two cases are quite different. The proof for G/P is essentially a Lie algebra computation while the proof for the hypersurfaces is a moving-frame computation. From the very definition of our geometric structures, we have to replace these computations by some geometric arguments involving the minimal rational curves.

6. Liouville theorem when the variety of minimal rational tangents has non-degenerate Gauss map

An affirmative answer to the Question in the previous section was given in [HM2] when $\mathcal{C}_x$ satisfies certain non-degeneracy condition. This non-degeneracy condition is in terms of the Gauss map of $\mathcal{C}_x$. Recall that given a complex analytic subvariety $V \subset \mathbf{P}_{n-1}$ of pure dimension ℓ, the Gauss map of V is a holomorphic map $\gamma : Sm(V) \to Gr(\ell+1; \mathbf{C}^n)$ where $Sm(V)$ is the smooth locus of V and $Gr(\ell+1;\mathbf{C}^n)$ is the Grassmannian of subspaces of dimension $\ell+1$ in $\mathbf{C}^n$, defined by

$$\gamma(v) := \text{ the affine tangent space of } V \text{ at } v \in Sm(V).$$

We say that V has non-degenerate Gauss map if the image of γ has dimension ℓ. By the result of Zak [Za], V has non-degenerate Gauss map if V is a smooth complex submanifold which is not the union of finitely many linear subspaces in $\mathbf{P}_{n-1}$.

We have an affirmative answer to the Question when $\mathcal{C}_x$ is positive dimensional and has non-degenerate Gauss map. In fact, under these assumptions, we have the following stronger result which generalizes Theorem 5.

Theorem 6 (Hwang-Mok [HM2]) *Let X_1 and X_2 be two Fano manifolds with second Betti number 1. Assume that $\mathcal{C}_x$ for a general $x \in X_1$ is positive-dimensional and has non-degenerate Gauss map as a projective subvariety of $\mathbf{P}T_x(X_1)$. Let $U_1 \subset X_1$ and $U_2 \subset X_2$ be connected open subsets. Suppose there exists a biholomorphic map $\varphi : U_1 \to U_2$ such that for a general point $x \in U_1$, the derivative $d\varphi_x : \mathbf{P}T_x(U_1) \to \mathbf{P}T_{\varphi(x)}(U_2)$ satisfies*

$$d\varphi_x(\mathcal{C}_x) = \mathcal{C}_{\varphi(x)}.$$

Then there exists a biholomorphic map $\Phi : X_1 \to X_2$ such that $\varphi = \Phi|_{U_1}$.

There are many examples of Fano manifolds with second Betti number 1 such that $\mathcal{C}_x$ is positive-dimensional and has non-degenerate Gauss map. The condition that $\mathcal{C}_x$ is positive-dimensional is somewhat restrictive because there are many examples where $\mathcal{C}_x$ is zero-dimensional, like hypersurfaces of degree n or $n+1$ in $\mathbf{P}_{n+1}$. On the other hand, the non-degeneracy of the Gauss map seems to hold for all examples excepting $\mathbf{P}_n$. In particular, it holds for G/P, hypersurfaces of degree $\leqslant n+1$ and the moduli of vector bundles discussed in the previous section.

Let us briefly discuss the main ideas of the proof of Theorem 6. With some over-simplification, the proof consists of two components, the 'local' part and the 'extension' part. The local part is the following lemma. curves.

Lemma *In the setting of Theorem 6, let $\mathcal{K}_1$ (resp. $\mathcal{K}_2$) be the family of holomorphic curves covering U_1 (resp. U_2) given by the minimal rational curves. Then φ sends a general member of $\mathcal{K}_1$ to a general member of $\mathcal{K}_2$.*

This is proved in [HM1]. The proof follows from the study of certain differential systems on the variety $\mathcal{C}^1$, which is the union of the tangent vectors to members of $\mathcal{K}_1$. Roughly speaking, $\mathcal{C}^1$ has a local holomorphic foliation by the tangential lifts of members of $\mathcal{K}_1$ and the proof follows from the fact that this foliation is determined by the geometry of $\mathcal{C}^1$ inside $\mathbf{P}T(U_1)$ via natural differential systems on it.

Using this Lemma, we can extend φ to a global biholomorphism Φ via analytic continuation along the rational curves. This is the 'extension' part of the proof. Let us explain why such extension is possible. Suppose we want to extend the map φ to φ' defined on a larger open set than U_1. Choose a general minimal rational curve C which intersects U_1. Pick a general point $y \in C$. We want to define $\varphi'(y)$. Of course, we have $\varphi'(y) = \varphi(y)$ if $y \in C \cap U_1$. Now for general $y \in C$, by the assumption that $\mathcal{C}_y$ is of positive dimension, we can find a family of rational curves

$$\{C^1_t, t \in \mathbf{C}, |t| < 1\}$$

such that $C^1_0 = C$, $y \in C^1_t$ and $C^1_t \cap U_1 \neq \emptyset$ for all t. By Lemma,

$$\varphi(C^1_t \cap U_1) = C^2_t \cap U_2$$

for some minimal rational curve C^2_t on X_2. We claim that

$$\bigcap_{|t|<1} C^2_t \neq \emptyset.$$

This is by continuity argument. In fact, this is true when y lies on $C \cap U_1$ because the intersection contains $\varphi(y)$ in that case. Thus by continuity, this should hold for general $y \in C$. But then by Bend-and-Break Lemma in Section 5, the intersection must be a single point. Thus define

$$\varphi'(y) := \bigcap_{|t|<1} C^2_t.$$

This extends φ along general points of C. Since a Zariski open subset of X_1 is covered by minimal rational curves, this analytic continuation defines an extension of φ as a possibly multi-valued rational map. The multi-valuedness can be removed by a topological argument using the simple-connectedness of X_1 and then a standard argument in algebraic geometry shows that the extended map is biholomorphic.

7. Open question-Liouville theorem when the variety of minimal rational tangents is finite

As we mentioned in Section 6, the non-degeneracy condition in Theorem 6 holds for practically all examples except the projective space. Thus Theorem 6 gives a fairly satisfactory answer to the Question as long as the variety of minimal rational tangents is positive-dimensional. However when the variety of minimal rational tangents is zero-dimensional, i.e., a finite set of points, Theorem 6 cannot be applied.

As a matter of fact, the extension theorem as stated in Theorem 6 is false in this case as the following example shows. This example appeared in [Sc]. Let $X_2 \subset \mathbf{P}_4$ be a smooth hypersurface of degree 3, for example,

$$X_2 = (x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0) \subset \mathbf{P}_4 = \{[x_0 : x_1 : x_2 : x_3 : x_4]\}.$$

Then the second Betti number of X_2 is 1 and as we saw in Section 3, X_2 is covered by lines. Consider the singular hypersurface $\hat{X}_2$ of degree 3 in $\mathbf{P}_5 = \{[x_0 : x_1 : x_2 : x_3 : x_4 : x_5]\}$ defined by

$$\hat{X}_2 := (x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0).$$

Let X_1 be the intersection of $\hat{X}_2$ and a hyperquadric in $\mathbf{P}_5 = \{[x_0 : x_1 : x_2 : x_3 : x_4 : x_5]\}$ defined by

$$X_1 = (x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0) \cap (a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 + a_5 x_5^2 = 0).$$

Then X_1 is a 3-dimensional complex manifold for a general choice of the complex coefficients $a_0, \ldots, a_5$. Moreover, X_1 is a Fano manifold with $b_2 = 1$. The projection of $\mathbf{P}_5$ to $\mathbf{P}_4$ given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \longrightarrow [x_0 : x_1 : x_2 : x_3 : x_4]$$

defines a holomorphic map $\varphi : X_1 \to X_2$ which is generically two-to-one. One can check that the inverse images of lines on X_2 under φ are minimal rational curves in X_1. Thus if we restrict φ to a small open subset in X_1, then it preserves the variety of minimal rational tangents in the sense of Theorem 6, however φ is not a biholomorphic map from X_1 to X_2.

It is still possible that the analogue of Liouville theorem holds, i.e., the answer to the Question is affirmative, even when the variety of minimal rational tangents is finite. This is the case for hypersurfaces of degree n in $\mathbf{P}_{n+1}$ by Theorem 4. This is also true for the Mukai-Umemura threefolds [UM]. These are $SL_2(\mathbf{C})$-almost homogeneous Fano manifolds with $b_2 = 1$. The minimal rational curves at a point on the open orbit of $SL_2(\mathbf{C})$ arise from an orbit of a Cartan subgroup of $SL_2(\mathbf{C})$. Using this, one can verify the Liouville property. Excepting these examples, we do not know any other cases with finite $\mathcal{C}_x$ where the Liouville property holds.

Let us just mention that when $\mathcal{C}_x$ is finite for general $x \in X$, we have finitely many minimal rational curves through a general point. Thus in an analytic open neighborhood of a general point, minimal rational curves define a web-structure. The Question is then the problem of extending the automorphisms of the web structure. The geometry of this web is governed by global properties of minimal rational curves. Some results on this web was obtained in [HM4]. However the analogue of Liouville theorem seems to be out of reach by the method developed there.

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