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Natural connections given by general linear and classical connections

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Abstract. We assume a vector bundle $p : \boldsymbol{E} \to \boldsymbol{M}$ with a general linear connection K and a classical linear connection Λ on $\boldsymbol{M}$. We prove that all classical linear connections on the total space $\boldsymbol{E}$ naturally given by (Λ, K) form a 15-parameter family. Further we prove that all connections on $J^1\boldsymbol{E}$ naturally given by (Λ, K) form a 14-parameter family. Both families of connections are described geometrically.

Introduction

The reduction theorems for classical (linear) connections on manifolds are very powerful tools to classify natural (invariant) operators (of any finite order) from the bundle of classical connections and some tensor bundle to another tensor bundle. By reduction theorems all such operators reduce to operators defined on tensor bundles only (operators of curvature tensor fields of classical connections in question, given tensor fields and covariant differentials of the curvature and given tensor fields). For the general theory of natural bundles and operators see [**10, 11, 12, 14**] and for the proof of reduction theorems for classical connections and their applications see [**10, 13**].

The reduction idea for general linear connections on vector bundles is also possible. It is a gauge version of reduction theorems for the gauge group $G = GL(n, \mathbb{R})$. For operators of order one the first reduction theorem for general linear connection is in fact the special case of the Utiyama's theorem, see [**1, 10**]. In [**6**] the reduction theorems for general linear connections were proved for operators of any finite orders with values in a gauge-natural bundle of order (1,0). In this gauge situation auxiliary classical connections on base manifolds have to be used.

Sometimes we need to study natural operators which have values in a natural or a gauge-natural bundle of higher order. In this case we can use higher order valued reduction theorems, for the case of classical connections see [**7**] and for the case of general linear connections see [**8**]. Typical operators of this type are natural tensor fields on the tangent or the cotangent bundle of a manifold with a classical connection or natural tensor fields on the total space of a vector bundle with a general linear connection and a classical connection on the base manifold.

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In this paper we use the higher order valued reduction theorems for general linear connections, [**8**], to clasify all classical linear connections on the total space of a vector bundle $p : \boldsymbol{E} \to \boldsymbol{M}$ and connections on the first jet prolongation $J^1\boldsymbol{E}$ naturally induced by a general linear connection K on $\boldsymbol{E}$ and a classical connection Λ on $\boldsymbol{M}$. We shall prove that there is a 15-parameter family of classical linear connections $\widetilde{D}(\Lambda, K)$ on the total space $\boldsymbol{E}$ and a 14-parameter family of connections $\widetilde{\Gamma}(\Lambda, K)$ on $J^1\boldsymbol{E}$. We present also the geometrical description of both families and we show that $\widetilde{\Gamma}(\Lambda, K)$ can be obtained from $\widetilde{D}(\Lambda, K)$ by using the operator χ described in [**9**]. The result obtained for classical linear connections on $\boldsymbol{E}$ coincides with the classification of [**3**] (see also [**10**]).

All manifolds and maps are assumed to be smooth. The sheaf of (local) sections of a fibered manifold $p : \boldsymbol{Y} \to \boldsymbol{X}$ is denoted by $C^\infty(\boldsymbol{Y})$, $C^\infty(\boldsymbol{Y}, \mathbb{R})$ denotes the sheaf of (local) functions.

1. Higher order valued reduction theorems for general linear connections

In what follows let $G = GL(n, \mathbb{R})$ be the group of linear automorphisms of $\mathbb{R}^n$ with coordinates (a^i_j), $i, j = 1, \dots, n$. Let $\mathcal{M}_m$ be the category of m-dimensional C^∞-manifolds and smooth embeddings. Let $\mathcal{VB}_{m,n}$ be the category of vector bundles with m-dimensional bases, n-dimensional fibers and local fiber linear diffeomorphisms and let $\mathcal{PB}_m(G)$ be the category of smooth principal G-bundles with m-dimensional bases and smooth G-bundle maps (φ, f), where the map $f \in \mathrm{Mor}\mathcal{M}_m$. Then any vector bundle $(p : \boldsymbol{E} \to \boldsymbol{M}) \in \mathrm{Ob}\,\mathcal{VB}_{m,n}$ can be considered as a zero order G-gauge-natural bundle given by the G-gauge-natural bundle functor of associated bundles $\mathcal{PB}_m(G) \to \mathcal{VB}_{m,n}$.

Local linear fiber coordinates on $\boldsymbol{E}$ will be denoted by $(x^A) = (x^\lambda, y^i)$, $A = 1, \dots, m+n$, $\lambda = 1, \dots, m$, $i = 1, \dots, n$. The induced fiber coordinates on $T\boldsymbol{E}$ or $T^*\boldsymbol{E}$ will be denoted by $(x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i)$ or $(x^\lambda, y^i, \dot{x}_\lambda, \dot{y}_i)$ and the induced local bases of sections of $T\boldsymbol{E}$ or $T^*\boldsymbol{E}$ will be denoted by $(\partial_\lambda, \partial_i)$ or (d^λ, d^i), respectively.

We define a *general linear connection* on $\boldsymbol{E}$ to be a linear section $K : \boldsymbol{E} \to J^1\boldsymbol{E}$. Considering the contact morphism $J^1\boldsymbol{E} \to T^*\boldsymbol{M} \otimes T\boldsymbol{E}$ over the identity of $T\boldsymbol{M}$, a linear connection can be regarded as a $T\boldsymbol{E}$-valued 1-form $K : \boldsymbol{E} \to T^*\boldsymbol{M} \otimes T\boldsymbol{E}$ projecting onto the identity of $T\boldsymbol{M}$.

The coordinate expression of a linear connection K is of the type

$$K = d^\lambda \otimes \big(\partial_\lambda + K_j{}^i{}_\lambda\, y^j\, \partial_i\big)\,, \qquad \text{with} \qquad K_j{}^i{}_\lambda \in C^\infty(\boldsymbol{M}, \mathbb{R})\,. \tag{1}$$

Linear connections can be regarded as sections of a (1,1)-order G-gauge-natural bundle $\mathrm{Lin}\,\boldsymbol{E} \to \boldsymbol{M}$, [**1, 6, 10**]. The standard fiber of the functor Lin will be denoted by $R = \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^{m*}$, elements of R will be said to be *formal linear connections*, the induced coordinates on R will be said to be *formal symbols* of formal linear connections and will be denoted by $(K_j{}^i{}_\lambda)$.

The curvature tensor field $R[K] : \boldsymbol{M} \to \boldsymbol{E}^* \otimes \boldsymbol{E} \otimes \bigwedge^2 T^*\boldsymbol{M}$ given by K is a natural operator $R[K] : C^\infty(\mathrm{Lin}\,\boldsymbol{E}) \to C^\infty(\boldsymbol{E}^* \otimes \boldsymbol{E} \otimes \Lambda^2 T^*\boldsymbol{M})$ which is of order one. Its coordinate expression is

$$\begin{aligned} R[K] &= R[K]_j{}^i{}_{\lambda\mu}\, d^j \otimes \partial_i \otimes d^\lambda \wedge d^\mu \\ &= -2(\partial_\lambda K_j{}^i{}_\mu + K_j{}^p{}_\lambda K_p{}^i{}_\mu)\, d^j \otimes \partial_i \otimes d^\lambda \wedge d^\mu\,. \end{aligned}$$

We define a *classical connection* on $\boldsymbol{M}$ to be a linear connection on the tangent vector bundle $p_{\boldsymbol{M}} : T\boldsymbol{M} \to \boldsymbol{M}$ with the coordinate expression

$$\Lambda = d^\lambda \otimes \big(\partial_\lambda + \Lambda_\nu{}^\mu{}_\lambda \, \dot{x}^\nu \, \dot\partial_\mu\big) \,, \quad \Lambda_\mu{}^\lambda{}_\nu \in C^\infty(\boldsymbol{M}, \mathbb{R}) \,. \tag{2}$$

Classical connections can be regarded as sections of a 2nd order natural bundle $\mathrm{Cla}\,\boldsymbol{M} \to \boldsymbol{M}$, [**10**]. The standard fiber of the functor Cla will be denoted by $Q = \mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes \mathbb{R}^{m*}$, elements of Q will be said to be *formal classical connections*, the induced coordinates on Q will be said to be *formal Christoffel symbols* of formal classical connections and will be denoted by $(\Lambda_\mu{}^\lambda{}_\nu)$. The natural subbundle of symmetric classical connections will be denoted by $\mathrm{Cla}_\tau \boldsymbol{M} \to \boldsymbol{M}$ with local fiber coordinates $(x^\lambda, \Lambda_\mu{}^\lambda{}_\nu)$, $\Lambda_\mu{}^\lambda{}_\nu = \Lambda_\nu{}^\lambda{}_\mu$.

The curvature tensor field of a classical connection is a natural operator

$$R[\Lambda] : C^\infty(\mathrm{Cla}\,\boldsymbol{M}) \to C^\infty(T^*\boldsymbol{M} \otimes T\boldsymbol{M} \otimes \bigwedge^2 T^*\boldsymbol{M})$$

which is of order one.

Let us denote by $\boldsymbol{E}^{p,r}_{q,s} := \otimes^p \boldsymbol{E} \otimes \otimes^q \boldsymbol{E}^* \otimes \otimes^r T\boldsymbol{M} \otimes \otimes^s T^*\boldsymbol{M}$ the tensor product over $\boldsymbol{M}$ and recall that $\boldsymbol{E}^{p,r}_{q,s}$ is a vector bundle which is a G-gauge-natural bundle of order $(1,0)$.

A classical connection Λ on $\boldsymbol{M}$ and a linear connection K on $\boldsymbol{E}$ induce the linear tensor product connection $K^p_q \otimes \Lambda^r_s := \otimes^p K \otimes \otimes^q K^* \otimes \otimes^r \Lambda \otimes \otimes^s \Lambda^*$ on $\boldsymbol{E}^{p,r}_{q,s}$

$$K^p_q \otimes \Lambda^r_s : \boldsymbol{E}^{p,r}_{q,s} \to T^*\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} T\boldsymbol{E}^{p,r}_{q,s}$$

which can be considered as a linear section

$$K^p_q \otimes \Lambda^r_s : \boldsymbol{E}^{p,r}_{q,s} \to J^1 \boldsymbol{E}^{p,r}_{q,s} \,.$$

Then we define, [**5**], the *covariant differential of a section* $\Phi : \boldsymbol{M} \to \boldsymbol{E}^{p,r}_{q,s}$ *with respect to the pair of connections* (Λ, K) as a section of $\boldsymbol{E}^{p,r}_{q,s} \otimes T^*\boldsymbol{M}$ given by

$$\nabla^{(\Lambda,K)} \Phi = j^1 \Phi - (K^p_q \otimes \Lambda^r_s) \circ \Phi \,.$$

In what follows we set $\nabla = \nabla^{(\Lambda,K)}$ and $\varphi^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s;\nu} = \nabla_\nu \varphi^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s}$. We shall denote by ∇^i the i-th iterated covariant differential and we shall put

$$\nabla^{(k,r)} := (\nabla^k, \dots, \nabla^r), \quad r \geqslant k, \quad \nabla^{(r)} := \nabla^{(0,r)}.$$

Let us denote by $C_{C,i}\boldsymbol{M}$ the i-th *curvature bundle of symmetric classical connections* given as the image of the natural operator

$$\nabla^i R[\Lambda] : J^{i+1}\,\mathrm{Cla}_\tau \boldsymbol{M} \to T^*\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} T\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} \bigwedge^2 T^*\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} \otimes^i T^*\boldsymbol{M} \,.$$

We shall put $C^{(r)}_C \boldsymbol{M} := C_{C,0}\boldsymbol{M} \underset{\boldsymbol{M}}{\times} \dots \underset{\boldsymbol{M}}{\times} C_{C,r}\boldsymbol{M}$ and $\mathrm{pr}^r_k : C^{(r)}_C \boldsymbol{M} \to C^{(k)}_C \boldsymbol{M}$, $r \geqslant k$, the canonical projection.

Similarly let us denote by $C_{L,i}\boldsymbol{E}$ the i-th *curvature bundle of general linear connections* given as the image of the natural operator

$$\nabla^i R[K] : J^{i-1}\,\mathrm{Cla}_\tau \boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{i+1}\,\mathrm{Lin}\,\boldsymbol{E} \to \boldsymbol{E}^* \underset{\boldsymbol{M}}{\otimes} \boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \bigwedge^2 T^*\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} \otimes^i T^*\boldsymbol{M} \,.$$

We shall put $C^{(r)}_L\boldsymbol{E} := C_{L,0}\boldsymbol{E} \underset{\boldsymbol{M}}{\times} \dots \underset{\boldsymbol{M}}{\times} C_{L,r}\boldsymbol{E}$ and $\mathrm{pr}^r_k : C^{(r)}_L\boldsymbol{M} \to C^{(k)}_L\boldsymbol{M}$, $r \geqslant k$, the canonical projection.

We denote by $J^{k-2}\operatorname{Cla}_\tau \boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{k-1}\operatorname{Lin}\boldsymbol{E} \underset{\boldsymbol{M}}{\times} C^{(k-2,s-1)}_C\boldsymbol{M} \underset{\boldsymbol{M}}{\times} C^{(k-1,r-1)}_L\boldsymbol{E}$, $s \geqslant r-2$, the pullback of

$$\mathrm{pr}^{s-1}_{k-3} \times \mathrm{pr}^{r-1}_{k-2} : C^{(s-1)}_C\boldsymbol{M} \underset{\boldsymbol{M}}{\times} C^{(r-1)}_L\boldsymbol{E} \to C^{(k-3)}_C\boldsymbol{M} \underset{\boldsymbol{M}}{\times} C^{(k-2)}_L\boldsymbol{E}$$

with respect to the surjective submersion, [**6**],

$$(\nabla^{(k-3)}R[\Lambda], \nabla^{(k-2)}R[K]) : J^{k-2}\operatorname{Cla}_\tau \boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{k-1}\operatorname{Lin}\boldsymbol{E} \to C^{(k-3)}_C \underset{\boldsymbol{M}}{\times} C^{(k-2)}_L\boldsymbol{E}\,.$$

Then the first k-th order valued reduction theorem can be formulated as follows, for the poof see [**8**],

Theorem 1. *Let $s \geqslant r-2$, $r+1, s+2 \geqslant k \geqslant 1$. Let F be a G-gauge-natural bundle of order k. All natural differential operators*

$$f : C^\infty(\operatorname{Cla}_\tau \boldsymbol{M} \underset{\boldsymbol{M}}{\times} \operatorname{Lin}\boldsymbol{E}) \to C^\infty(F\boldsymbol{E})$$

which are of order s with respect to symmetric classical connections and of order r with respect to general linear connections are of the form

$$f(j^s\Lambda, j^rK) = g(j^{k-2}\Lambda, j^{k-1}K, \nabla^{(k-2,s-1)}R[\Lambda], \nabla^{(k-1,r-1)}R[K])$$

where g is a unique natural operator

$$g : J^{k-2}\operatorname{Cla}_\tau \boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{k-1}\operatorname{Lin}\boldsymbol{E} \underset{\boldsymbol{M}}{\times} C^{(k-2,s-1)}_C\boldsymbol{M} \underset{\boldsymbol{M}}{\times} C^{(k-1,r-1)}_L\boldsymbol{E} \to F\boldsymbol{E}\,.$$

Further let us denote by $Z_i\boldsymbol{E}$ the image of the operator

$$(\nabla^{(i-2)}R[\Lambda], \nabla^{(i-2)}R[K], \nabla^{(i)}\Phi) :$$

$$J^{i-1}\operatorname{Cla}_\tau \boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{i-1}\operatorname{Lin}\boldsymbol{E} \underset{\boldsymbol{M}}{\times} J^i\boldsymbol{E}^{p_1,p_2}_{q_1,q_2} \to C^{(i-2)}_C\boldsymbol{M} \underset{\boldsymbol{M}}{\times} C^{(i-2)}_L\boldsymbol{E} \underset{\boldsymbol{M}}{\times} \boldsymbol{E}^{p_1,p_2}_{q_1,q_2} \underset{\boldsymbol{M}}{\otimes} \otimes^i T^*\boldsymbol{M}$$

We shall put $Z^{(r)}\boldsymbol{E} := Z_0\boldsymbol{E} \underset{\boldsymbol{M}}{\times} \dots \underset{\boldsymbol{M}}{\times} Z_r\boldsymbol{E}$ and $\mathrm{pr}^r_k : Z^{(r)}\boldsymbol{E} \to Z^{(k)}\boldsymbol{E}$, $r \geqslant k$, the canonical projection.

We denote by $J^{k-2}\operatorname{Cla}_\tau \boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{k-2}\operatorname{Lin}\boldsymbol{E} \underset{\boldsymbol{M}}{\times} J^{k-1}\boldsymbol{E}^{p_1,p_2}_{q_1,q_2} \underset{\boldsymbol{M}}{\times} Z^{(k,r)}\boldsymbol{E}$ the pullback of

$$\mathrm{pr}^r_{k-1} : Z^{(r)}\boldsymbol{E} \to Z^{(k-1)}\boldsymbol{E}$$

with respect to the surjective submersion, [**6**],

$$(\nabla^{(k-3)}R[\Lambda], \nabla^{(k-3)}R[K], \nabla^{(k-1)}) :$$

$$J^{k-2}\operatorname{Cla}_\tau \boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{k-2}\operatorname{Lin}\boldsymbol{E} \underset{\boldsymbol{M}}{\times} J^{k-1}\boldsymbol{E}^{p_1,p_2}_{q_1,q_2} \to Z^{(k-1)}\boldsymbol{E}\,.$$

Then the second k-th order valued reduction theorem can be formulated as follows, [**8**],

Theorem 2. *Let F be a G-gauge-natural bundle of order $k \geqslant 1$ and let $r+1 \geqslant k$. All natural differential operators*

$$f : C^\infty(\operatorname{Cla}_\tau \boldsymbol{M} \underset{\boldsymbol{M}}{\times} \operatorname{Lin}\boldsymbol{E} \underset{\boldsymbol{M}}{\times} \boldsymbol{E}^{p_1,p_2}_{q_1,q_2}) \to C^\infty(F\boldsymbol{E})$$

of order r with respect sections of $\boldsymbol{E}^{p_1,p_2}_{q_1,q_2}$ are of the form

$$
\begin{aligned}
&f(j^{r-1}\Lambda, j^{r-1}K, j^r\Phi) = \\
&= g(j^{k-2}\Lambda, j^{k-2}K, j^{k-1}\Phi, \nabla^{(k-2,r-2)}R[\Lambda], \nabla^{(k-2,r-2)}R[K], \nabla^{(k,r)}\Phi)
\end{aligned}
$$

where g is a unique natural operator

$$
g : J^{k-2}\,\mathrm{Cla}_\tau\,\boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{k-2}\,\mathrm{Lin}\,\boldsymbol{E} \underset{\boldsymbol{M}}{\times} J^{k-1}\boldsymbol{E}^{p_1,p_2}_{q_1,q_2} \underset{\boldsymbol{M}}{\times} Z^{(k,r)}\boldsymbol{M} \to F\boldsymbol{E}\,.
$$

Remark 1. The order $(r-1)$ of the above operators with respect to linear and symmetric classical connections is the minimal order we have to use. The second reduction theorem can be easily generalized for any operators of orders s_1 or s_2 with respect to connections Λ or K, respectively, where $s_1, s_2 \geqslant r-1$, $s_1 \geqslant s_2 - 1$. Then

$$
\begin{aligned}
&f(j^{s_1}\Lambda, j^{s_2}K, j^r\Phi) = \\
&= g(j^{k-2}\Lambda, j^{k-2}K, j^{k-1}\Phi, \nabla^{(k-2,s_1-1)}R[\Lambda], \nabla^{(k-2,s_2-1)}R[K], \nabla^{(k,r)}\Phi)\,.
\end{aligned}
$$

Remark 2. The above higher order valued valued reduction theorems deal with symmetric classical connections on the base manifolds. If Λ is a non-symmetric classical connection, then there is its unique splitting $\Lambda = \widetilde{\Lambda} + T$, where $\widetilde{\Lambda}$ is the symmetric classical connection obtained from Λ by symmetrization, i.e., $\widetilde{\Lambda}_\mu{}^\lambda{}_\nu = \frac{1}{2}\,(\Lambda_\mu{}^\lambda{}_\nu + \Lambda_\nu{}^\lambda{}_\mu)$, and T is the torsion $(1,2)$-tensor, i.e., $T_\mu{}^\lambda{}_\nu = \frac{1}{2}\,(\Lambda_\mu{}^\lambda{}_\nu - \Lambda_\nu{}^\lambda{}_\mu)$. Then any finite order natural operator for Λ and K is of the form, $s \geqslant r-2$,

$$
\begin{aligned}
f(j^s\Lambda, j^rK) &= f(j^s\widetilde{\Lambda}, j^rK, j^sT) = \\
&= g(j^{k-2}\widetilde{\Lambda}, j^{k-2}K, j^{k-1}T, \widetilde{\nabla}^{(k-2,s-1)}R[\widetilde{\Lambda}], \widetilde{\nabla}^{(k-2,r-1)}R[K], \widetilde{\nabla}^{(k,s)}T)\,,
\end{aligned}
$$

where $\widetilde{\ }$ refers to $\widetilde{\Lambda}$.

2. Natural classical connections on the total space of a vector bundle

A classical connection D on $\boldsymbol{E}$ is given by $D : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ over the identity of $T\boldsymbol{E}$. In coordinates

$$
D = d^C \otimes (\partial_C + D_B{}^A{}_C\,\dot{x}^B\,\dot{\partial}_A)\,, \quad D_B{}^A{}_C \in C^\infty(\boldsymbol{E}, \mathbb{R})\,. \tag{3}
$$

Given a general linear connection K on $\boldsymbol{E}$ and a classical connection Λ on the base manifol $\boldsymbol{M}$ we have an induced natural classical connection $D(\Lambda, K)$ on $\boldsymbol{E}$ given by, [**2, 10**],

Proposition 1. *There exists a unique classical connection $D = D(\Lambda, K)$ on the total space $\boldsymbol{E}$ with the following properties*

$$
\nabla^D_{h^K(X)} h^K(Y) = h^K(\nabla^\Lambda_X Y), \quad \nabla^D_{h^K(X)} s^V = (\nabla^K_X s)^V,
$$
$$
\nabla^D_{s^V} h^K(X) = 0, \quad \nabla^D_{s^V} \sigma^V = 0\,,
$$

for all vector fields X, Y on $\boldsymbol{M}$ and all sections s, σ of $\boldsymbol{E}$, where h^K is the horisontal lift with respect to K, $\nabla^K, \nabla^\Lambda, \nabla^D$ are covariant differentials with respect to K, Λ, D, respectively, and s^V, σ^V denote the vertical lifts of the sections s, σ, respectively.

In coordinates

$$\begin{aligned}
&D_\mu{}^\lambda{}_\nu = \Lambda_\mu{}^\lambda{}_\nu\,, \quad D_\mu{}^\lambda{}_k = 0\,, \quad D_j{}^\lambda{}_\nu = 0\,, \quad D_j{}^\lambda{}_k = 0\,,\\
&D_\mu{}^i{}_\nu = \big(\partial_\nu K_p{}^i{}_\mu - K_r{}^i{}_\nu K_p{}^r{}_\mu + K_p{}^i{}_\rho \Lambda_\mu{}^\rho{}_\nu\big)\, y^p\,,\\
&D_\mu{}^i{}_k = K_k{}^i{}_\mu\,, \quad D_j{}^i{}_\nu = K_j{}^i{}_\nu\,, \quad D_j{}^i{}_k = 0\,.
\end{aligned}$$

Remark 3. $\mathrm{Cla}\,\boldsymbol{E}$ is a G-gauge-natural bundle of order (2,2) and $D(\Lambda, K)$ defines the natural operator $D : C^\infty(\mathrm{Cla}\,\boldsymbol{M} \underset{\boldsymbol{M}}{\times} \mathrm{Lin}\,\boldsymbol{E}) \to C^\infty(\mathrm{Cla}\,\boldsymbol{E})$ which is of order zero with respect to Λ and of order one with respect to K.

The difference of any two classical connections on $\boldsymbol{E}$ is a tensor field on $\boldsymbol{E}$ of the type $(1,2)$. So, having the connection $D(\Lambda, K)$, all classical connections on $\boldsymbol{E}$ naturally given by K and Λ are of the type $D(\Lambda, K) + \Phi(\Lambda, K)$, where $\Phi(\Lambda, K)$ is a natural (1,2)-tensor field on $\boldsymbol{E}$. Hence, the problem of classiffication of natural classical connections on $\boldsymbol{E}$ is reduced to the problem of classifffication of natural tensor fields on $\boldsymbol{E}$.

Any tensor field on $\boldsymbol{E}$ is a section of a G-gauge-natural bundle of order $(1,1)$. Then, by Theorem 2 and Remark 2, we get

Corollary 1. *Let Φ be a tensor field on $\boldsymbol{E}$ naturally given by a classical connection Λ on $\boldsymbol{M}$ (in order s) and by a general linear connection K on $\boldsymbol{E}$ (in order r, $s \geqslant r-2$). Then*

$$\Phi(u, j^s\Lambda, j^r K) = \Psi(u, \widetilde{\nabla}^{(s)}T, \widetilde{\nabla}^{(s-1)}R[\widetilde{\Lambda}], K, \widetilde{\nabla}^{(r-1)}R[K])\,,$$

where $u \in \boldsymbol{E}$ and $\widetilde{}$ refers to the classical symmetrized connection $\widetilde{\Lambda}$.

Now we can use the above Corollary 1 to classify (1,2)-tensor fields on $\boldsymbol{E}$. We have

Lemma 1. *All (1,2)-tensor fields on $\boldsymbol{E}$ naturally given by Λ (in order s) and by K (in order r, $s \geqslant r-2$) are of the maximal order 1 (with respect to both connections) and form a 15-parameter family of operators with the coordinate expression given by*

$$\begin{aligned}
\Phi(\Lambda, K) &= \big(a_1 T_\mu{}^\lambda{}_\nu + a_2\, \delta^\lambda_\mu T_\rho{}^\rho{}_\nu + a_3\, \delta^\lambda_\nu T_\mu{}^\rho{}_\rho\big)\, d^\mu \otimes \partial_\lambda \otimes d^\nu\\
&+ \Big(y^i\,(b_1 T_\rho{}^\rho{}_\mu T_\sigma{}^\sigma{}_\nu + b_2 T_\sigma{}^\rho{}_\mu T_\rho{}^\sigma{}_\nu + b_3 T_\rho{}^\rho{}_\sigma T_\mu{}^\sigma{}_\nu + c_1 T_\rho{}^\rho{}_{\mu;\nu} + c_2 T_\rho{}^\rho{}_{\nu;\mu}\\
&+ c_3 T_\mu{}^\rho{}_{\nu;\rho} + d_1 \widetilde{R}_\rho{}^\rho{}_{\mu\nu} + d_2 \widetilde{R}_\mu{}^\rho{}_{\rho\nu} + e_2 R_p{}^p{}_{\mu\nu}) + e_1 R_j{}^i{}_{\mu\nu}\, y^j\\
&+ (a_3 - h_2)\, T_\mu{}^\rho{}_\rho K_j{}^i{}_\nu\, y^j + (a_2 - h_1)\, T_\rho{}^\rho{}_\nu K_j{}^i{}_\mu\, y^j + a_1 T_\mu{}^\rho{}_\nu K_j{}^i{}_\rho\, y^j\Big)\, d^\mu \otimes \partial_i \otimes d^\nu\\
&+ h_1\, \delta^i_j\, T_\rho{}^\rho{}_\nu\, d^j \otimes \partial_i \otimes d^\nu + h_2\, \delta^i_k\, T_\rho{}^\rho{}_\mu\, d^\mu \otimes \partial_i \otimes d^k\,,
\end{aligned}$$

where a_i, b_i, c_i, $i = 1,2,3$, d_j, e_j, h_j, $j = 1,2$, are real coefficients.

Proof. By the general theory of natural operators, [**10**], we have to classify all equivariant mappings between standard fibers of G-gauge-natural bundles in question. $T^*\boldsymbol{E} \otimes T\boldsymbol{E} \otimes T^*\boldsymbol{E}$ is a G-gauge-natural bundle of order $(1,1)$ where the action of the group $W^{(1,1)}_{m,n}G$ on the standard fiber $S_F = \mathbb{R}^{(m+n)*} \otimes \mathbb{R}^{(m+n)} \otimes \mathbb{R}^{(m+n)*}$ is given by

$$\bar{\Phi}_j{}^i{}_k = a^i_r\,\Phi_s{}^r{}_t\,\tilde{a}^s_j\,\tilde{a}^t_k + a^i_{p\rho}\,y^p\,\Phi_s{}^\rho{}_t\,\tilde{a}^s_j\,\tilde{a}^t_k\,, \tag{4}$$

$$\begin{aligned}\bar{\Phi}_j{}^i{}_\nu &= a^i_r\,\Phi_s{}^r{}_\tau\,\tilde{a}^s_j\,\tilde{a}^\tau_\nu + a^i_r\,\Phi_s{}^r{}_t\,\tilde{a}^s_j\,\tilde{a}^t_{p\nu}\,a^p_k\,y^k\\ &\quad + a^i_{p\rho}\,y^p\,\Phi_s{}^\rho{}_t\,\tilde{a}^s_j\,\tilde{a}^t_{k\nu}\,a^k_q\,y^q + a^i_{p\rho}\,y^p\,\Phi_s{}^\rho{}_\tau\,\tilde{a}^s_j\,\tilde{a}^\tau_\nu\,,\end{aligned} \tag{5}$$

$$\begin{aligned}\bar{\Phi}_\mu{}^i{}_k &= a^i_r\,\Phi_\sigma{}^r{}_t\,\tilde{a}^\sigma_\mu\,\tilde{a}^t_k + a^i_r\,\Phi_s{}^r{}_t\,\tilde{a}^s_{p\mu}\,a^p_q\,y^q\tilde{a}^t_k\\ &\quad + a^i_{p\rho}\,y^p\,\Phi_s{}^\rho{}_t\,\tilde{a}^s_{q\mu}\,a^q_l\,y^l\,\tilde{a}^t_k + a^i_{p\rho}\,y^p\,\Phi_\sigma{}^\rho{}_t\,\tilde{a}^\sigma_\mu\,\tilde{a}^t_k\,,\end{aligned} \tag{6}$$

$$\begin{aligned}\bar{\Phi}_\mu{}^i{}_\nu &= a^i_r\,\Phi_\sigma{}^r{}_\tau\,\tilde{a}^\sigma_\mu\,\tilde{a}^\tau_\nu + a^i_r\,\Phi_s{}^r{}_\tau\,\tilde{a}^s_{p\mu}\,a^p_q\,y^q\tilde{a}^\tau_\nu\\ &\quad + a^i_r\,\Phi_\sigma{}^r{}_t\,\tilde{a}^\sigma_\mu\,\tilde{a}^t_{p\nu}\,a^p_l\,y^l + a^i_r\,\Phi_s{}^r{}_t\,\tilde{a}^s_{p\mu}\,a^p_l\,y^l\tilde{a}^t_{q\nu}\,a^p_m\,y^m\\ &\quad + a^i_{p\rho}\,y^p\,\Phi_s{}^\rho{}_t\,\tilde{a}^s_{q\mu}\,a^q_l\,y^l\,\tilde{a}^t_{n\nu}\,a^n_m\,y^m + a^i_{p\rho}\,y^p\,\Phi_\sigma{}^\rho{}_t\,\tilde{a}^\sigma_\mu\,\tilde{a}^t_{q\nu}\,a^q_l\,y^l\\ &\quad + a^i_{p\rho}\,y^p\,\Phi_s{}^\rho{}_\tau\,\tilde{a}^s_{q\mu}\,a^q_l\,y^l\tilde{a}^\tau_\nu + a^i_{p\rho}\,y^p\,\Phi_\sigma{}^\rho{}_\tau\,\tilde{a}^\sigma_\mu\,\tilde{a}^\tau_\nu\,,\end{aligned} \tag{7}$$

$$\bar{\Phi}_j{}^\lambda{}_k = a^\lambda_\rho\,\Phi_s{}^\rho{}_t\,\tilde{a}^s_j\,\tilde{a}^t_k\,, \tag{8}$$

$$\bar{\Phi}_j{}^\lambda{}_\nu = a^\lambda_\rho\,\Phi_s{}^\rho{}_\tau\,\tilde{a}^s_j\,\tilde{a}^\tau_\nu + a^\lambda_\rho\,\Phi_s{}^\rho{}_t\,\tilde{a}^s_j\,\tilde{a}^t_{p\nu}\,a^p_k\,y^k\,, \tag{9}$$

$$\bar{\Phi}_\mu{}^\lambda{}_k = a^\lambda_\rho\,\Phi_\sigma{}^\rho{}_t\,\tilde{a}^\sigma_\mu\,\tilde{a}^t_k + a^\lambda_\rho\,\Phi_s{}^\rho{}_t\,\tilde{a}^s_{p\mu}\,a^p_q\,y^q\tilde{a}^t_k\,, \tag{10}$$

$$\begin{aligned}\bar{\Phi}_\mu{}^\lambda{}_\nu &= a^\lambda_\rho\,\Phi_\sigma{}^\rho{}_\tau\,\tilde{a}^\sigma_\mu\,\tilde{a}^\tau_\nu + a^\lambda_\rho\,\Phi_s{}^\rho{}_\tau\,\tilde{a}^s_{p\mu}\,a^p_q\,y^q\tilde{a}^\tau_\nu\\ &\quad + a^\lambda_\rho\,\Phi_\sigma{}^\rho{}_t\,\tilde{a}^\sigma_\mu\,\tilde{a}^t_{p\nu}\,a^p_l\,y^l + a^\lambda_\rho\,\Phi_s{}^\rho{}_t\,\tilde{a}^s_{p\mu}\,a^p_l\,y^l\tilde{a}^t_{q\nu}\,a^q_m\,y^m\,.\end{aligned} \tag{11}$$

By Corollary 1 equivariant mappings $\Phi : \mathbb{R}^n \times T^s_m Q \times T^r_m R \to S_F$ between standard fibers are in the form

$$\Phi_B{}^A{}_C(y^i, \Lambda_\mu{}^\lambda{}_{\nu,\boldsymbol{\alpha}}, K_j{}^i{}_{\mu,\boldsymbol{\beta}}) = \Psi_B{}^A{}_C(y^i, T_\mu{}^\lambda{}_{\nu;\boldsymbol{\gamma}}, \widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\boldsymbol{\varepsilon}})\,,$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\varepsilon}$ are multiindices of the ranks $0 \leqslant \|\boldsymbol{\alpha}\|, \|\boldsymbol{\gamma}\| \leqslant s$, $0 \leqslant \|\boldsymbol{\beta}\| \leqslant r$, $0 \leqslant \|\boldsymbol{\delta}\| \leqslant s-1$, $0 \leqslant \|\boldsymbol{\varepsilon}\| \leqslant r-1$, and $(\Lambda_\mu{}^\lambda{}_{\nu,\boldsymbol{\alpha}})$ or $(K_j{}^i{}_{\mu,\boldsymbol{\beta}})$ are the induced coordinates on $T^s_m Q$ or $T^r_m R$, respectively.

First, let us consider the equivariance of $\Psi_B{}^A{}_C$ with respect to the fiber homotheties, i.e., with respect to $(c\,\delta^i_j)$. Then, from (8), we get

$$c^{-2}\,\Psi_j{}^\lambda{}_k = \Psi_j{}^\lambda{}_k(c\,y^i, T_\mu{}^\lambda{}_{\nu;\boldsymbol{\gamma}}, \widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\boldsymbol{\varepsilon}})\,.$$

Multiplying it by c^2 and letting $c \to 0$, we obtain

$$\Psi_j{}^\lambda{}_k = \Phi_j{}^\lambda{}_k = 0\,. \tag{12}$$

Similarly, from (9), (10) and (4), we have

$$\begin{aligned}c^{-1}\,\Psi_j{}^\lambda{}_\nu &= \Psi_j{}^\lambda{}_\nu(c\,y^i, T_\mu{}^\lambda{}_{\nu;\boldsymbol{\gamma}}, \widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\boldsymbol{\varepsilon}})\,,\\ c^{-1}\,\Psi_\mu{}^\lambda{}_k &= \Psi_\mu{}^\lambda{}_k(c\,y^i, T_\mu{}^\lambda{}_{\nu;\boldsymbol{\gamma}}, \widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\boldsymbol{\varepsilon}})\,,\\ c^{-1}\,\Psi_j{}^i{}_k &= \Psi_j{}^i{}_k(c\,y^i, T_\mu{}^\lambda{}_{\nu;\boldsymbol{\gamma}}, \widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\boldsymbol{\varepsilon}})\,,\end{aligned}$$

which, by multiplying by c and letting $c \to 0$, imply

$$\Psi_j{}^\lambda{}_\nu = \Phi_j{}^\lambda{}_\nu = 0\,, \quad \Psi_\mu{}^\lambda{}_k = \Phi_\mu{}^\lambda{}_k = 0\,, \quad \Psi_j{}^i{}_k = \Phi_j{}^i{}_k = 0\,. \tag{13}$$

Further, from (5), (6) and (11), we have

$$\Psi_j{}^i{}_\nu = \Psi_j{}^i{}_\nu(c\,y^i, T_\mu{}^\lambda{}_{\nu;\gamma}, \widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\varepsilon})\,,$$
$$\Psi_\mu{}^i{}_k = \Psi_\mu{}^i{}_k(c\,y^i, T_\mu{}^\lambda{}_{\nu;\gamma}, \widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\varepsilon})\,,$$
$$\Psi_\mu{}^\lambda{}_\nu = \Psi_\mu{}^\lambda{}_\nu(c\,y^i, T_\mu{}^\lambda{}_{\nu;\gamma}, \widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\varepsilon})\,,$$

which implies, by letting $c \to 0$, that $\Psi_j{}^i{}_\nu, \Psi_\mu{}^i{}_k$ and $\Psi_\mu{}^\lambda{}_\nu$ are independent of y^i.

Finally, from (7), we have

$$c\,\Psi_\mu{}^i{}_\nu = \Psi_\mu{}^i{}_\nu(c\,y^i, T_\mu{}^\lambda{}_{\nu;\gamma}, \widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\varepsilon})\,.$$

By the homogeneous function theorem, [**10**], $\Psi_\mu{}^i{}_\nu$ is linear in y^p, i.e.,

$$\Psi_\mu{}^i{}_\nu = F_\mu{}^i{}_{\nu p}(T_\mu{}^\lambda{}_{\nu;\gamma}, \widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\varepsilon})\,y^p\,.$$

So we have equivariant functions $\Psi_j{}^i{}_\nu$, $\Psi_\mu{}^i{}_k$, $\Psi_\mu{}^\lambda{}_\nu$ and $F_\mu{}^i{}_{\nu p}$ of variables $T_\mu{}^\lambda{}_{\nu;\gamma}$, $\widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}$, $K_j{}^i{}_\mu$, $R_j{}^i{}_{\lambda\mu;\varepsilon}$.

Now we consider the base homotheties, i.e., the equivariance with respect to $(c\,\delta^\lambda_\mu)$. Then

$$c^{-1}\,\Psi_\mu{}^\lambda{}_\nu = \Psi_\mu{}^\lambda{}_\nu(c^{-(\|\gamma\|+1)}\,T_\mu{}^\lambda{}_{\nu;\gamma}, c^{-(\|\boldsymbol{\delta}\|+2)}\,\widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, c^{-1}\,K_j{}^i{}_\mu, c^{-(\|\varepsilon\|+2)}\,R_j{}^i{}_{\lambda\mu;\varepsilon})\,.$$

The homogeneous function theorem implies that $\Psi_\mu{}^\lambda{}_\nu$ is a polynomial function with exponents a_i in $T_\mu{}^\lambda{}_{\nu;\gamma}$, $i = \|\boldsymbol{\gamma}\|$, b_j in $R_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}$, $j = \|\boldsymbol{\delta}\|$, c in $K_j{}^i{}_\mu$ and d_k in $R_j{}^i{}_{\lambda\mu;\varepsilon}$, $k = \|\boldsymbol{\varepsilon}\|$, such that

$$-1 = -\sum_{i=0}^{s}(i+1)\,a_i - \sum_{j=0}^{s-1}(j+2)\,b_j - c - \sum_{k=0}^{r-1}(k+2)\,d_k\,. \tag{14}$$

The above equation (14) has solutions in integers only for $a_0 = 1$ and the other coefficients vahish or $c = 1$ and the other coefficients vanish. This implies that $\Psi_\mu{}^\lambda{}_\nu$ is of the type

$$\Psi_\mu{}^\lambda{}_\nu = A^{\lambda\sigma\tau}_{\mu\nu\rho}\,T_\sigma{}^\rho{}_\tau + L^{\lambda j\rho}_{\mu\nu i}\,K_j{}^i{}_\rho\,,$$

where $A^{\lambda\sigma\tau}_{\mu\nu\rho}$ and $L^{\lambda j\rho}_{\mu\nu i}$ are absolute invariant tensors, i.e., products of the Kronecker symbols. Then

$$\Psi_\mu{}^\lambda{}_\nu = a_1\,T_\mu{}^\lambda{}_\nu + a_2\,\delta^\lambda_\mu\,T_\rho{}^\rho{}_\nu + a_3\,\delta^\lambda_\nu\,T_\rho{}^\rho{}_\mu + l_1\,\delta^\lambda_\mu K_i{}^i{}_\nu + l_2\,\delta^\lambda_\nu K_i{}^i{}_\mu\,.$$

Finally, the equivariance of $\Psi_\mu{}^\lambda{}_\nu$ with respect to elements of the type $(\delta^i_j, \delta^\lambda_\mu, a^i_{j\lambda})$ implies $l_1 = l_2 = 0$ and we have

$$\Psi_\mu{}^\lambda{}_\nu = \Phi_\mu{}^\lambda{}_\nu = a_1\,T_\mu{}^\lambda{}_\nu + a_2\,\delta^\lambda_\mu\,T_\rho{}^\rho{}_\nu + a_3\,\delta^\lambda_\nu\,T_\rho{}^\rho{}_\mu\,. \tag{15}$$

Similarly, for $\Psi_j{}^i{}_\nu$ and $\Psi_\mu{}^i{}_k$, we get polynomial functions with exponents satisfying the same equation (14). So we have

$$\Psi_j{}^i{}_\nu = H^{i\sigma\tau}_{j\nu\rho}\,T_\sigma{}^\rho{}_\tau + M^{im\rho}_{j\nu k}\,K_m{}^k{}_\rho\,,\quad \Psi_\mu{}^i{}_k = H^{i\sigma\tau}_{\mu k\rho}\,T_\sigma{}^\rho{}_\tau + M^{im\rho}_{\mu kp}\,K_m{}^p{}_\rho\,,$$

where coefficients are absolute invariant tensors, i.e.,

$$\Psi_j{}^i{}_\nu = h_1\,\delta^i_j\,T_\rho{}^\rho{}_\nu + m_1\,K_j{}^i{}_\nu + m_2\,\delta^i_j\,K_p{}^p{}_\nu\,,$$
$$\Psi_\mu{}^i{}_k = h_2\,\delta^i_k\,T_\rho{}^\rho{}_\mu + m_3\,K_k{}^i{}_\mu + m_4\,\delta^i_k\,K_p{}^p{}_\mu\,.$$

The equivariance with respect to elements of the type $(\delta^i_j, \delta^\lambda_\mu, a^i_{j\lambda})$ implies $m_i = 0$, $i = 1, \dots, 4$, and we have

$$\Psi_j{}^i{}_\nu = \Phi_j{}^i{}_\nu = h_1\, \delta^i_j\, T_\rho{}^\rho{}_\nu\,, \quad \Psi_\mu{}^i{}_k = \Phi_\mu{}^i{}_k = h_2\, \delta^i_k\, T_\rho{}^\rho{}_\mu\,. \tag{16}$$

Finally, let us discuss $F_\mu{}^i{}_{\nu j}$. Considering base homotheties and the homogeneous function theorem we get that $F_\mu{}^i{}_{\nu j}$ is polynomial with exponents satisfying (14) with -2 on the left hand side. Then we have the following 6 possible solutions: $a_0 = 2$ and the other exponents vanish; $a_1 = 1$ and the other exponents vanish; $a_0 = 1$, $c = 1$ and the other exponents vanish; $b_0 = 1$ and the other exponents vanish; $c = 2$ and the other exponents vanish; $d_0 = 1$ and the other exponents vanish. Then

$$\begin{aligned} F_\mu{}^i{}_{\nu j} = {} & B^{i\sigma_1\tau_1\sigma_2\tau_2}_{\mu\nu j\rho_1\rho_2}\, T_{\sigma_1}{}^{\rho_1}{}_{\tau_1}\, T_{\sigma_2}{}^{\rho_2}{}_{\tau_2} + C^{i\sigma\tau_1\tau_2}_{\mu\nu j\rho}\, T_\sigma{}^\rho{}_{\tau_1;\tau_2} + N^{i\sigma\tau_1 q\tau_2}_{\mu\nu j\rho p}\, T_\sigma{}^\rho{}_{\tau_1}\, K_q{}^p{}_{\tau_2} \\ & + D^{i\sigma\tau_1\tau_2}_{\mu\nu j\rho}\, \tilde R_\sigma{}^\rho{}_{\tau_1\tau_2} + P^{iq_1\tau_1 q_2\tau_2}_{\mu\nu j p_1 p_1}\, K_{q_1}{}^{p_1}{}_{\tau_1}\, K_{q_2}{}^{p_2}{}_{\tau_2} + E^{iq\tau_1\tau_2}_{\mu\nu jp}\, R_q{}^p{}_{\tau_1\tau_2}\,, \end{aligned}$$

where all coefficients are absolute invariant tensors, i.e.,

$$\begin{aligned} F_\mu{}^i{}_{\nu j} = {} & \delta^i_j\, (b_1\, T_\rho{}^\rho{}_\mu\, T_\sigma{}^\sigma{}_\nu + b_2\, T_\sigma{}^\rho{}_\mu\, T_\rho{}^\sigma{}_\nu + b_3\, T_\rho{}^\rho{}_\sigma\, T_\mu{}^\sigma{}_\nu) \\ & + \delta^i_j\, (c_1\, T_\rho{}^\rho{}_{\mu;\nu} + c_2\, T_\rho{}^\rho{}_{\nu;\mu} + c_3\, T_\mu{}^\rho{}_{\nu;\rho}) \\ & + \delta^i_j\, (n_1\, T_\rho{}^\rho{}_\mu\, K_p{}^p{}_\nu + n_2\, T_\rho{}^\rho{}_\nu\, K_p{}^p{}_\mu + n_3\, T_\mu{}^\rho{}_\nu\, K_p{}^p{}_\rho) \\ & + n_4\, T_\mu{}^\rho{}_\rho\, K_j{}^i{}_\nu + n_5\, T_\rho{}^\rho{}_\nu\, K_j{}^i{}_\mu + n_6\, T_\mu{}^\rho{}_\nu\, K_j{}^i{}_\rho \\ & + d_1\, \delta^i_j\, \widetilde R_\rho{}^\rho{}_{\mu\nu} + d_2\, \delta^i_j\, \widetilde R_\mu{}^\rho{}_{\rho\nu} + e_1\, \delta^i_j\, R_p{}^p{}_{\mu\nu} + e_2\, R_j{}^i{}_{\mu\nu} \\ & + \delta^i_j\, (p_1\, K_p{}^p{}_\mu\, K_q{}^q{}_\nu + p_2\, K_q{}^p{}_\mu\, K_p{}^q{}_\nu) \\ & + p_3\, K_j{}^i{}_\mu\, K_p{}^p{}_\nu + p_4\, K_j{}^i{}_\nu\, K_p{}^p{}_\mu + p_5\, K_p{}^i{}_\mu\, K_j{}^p{}_\nu + p_6\, K_p{}^i{}_\nu\, K_j{}^p{}_\mu\,. \end{aligned}$$

The equivariance with respect to elements of the type $(\delta^i_j, \delta^\mu_\nu, a^i_{j\mu})$ implies $n_1 = n_2 = n_3 = 0$, $n_4 = a_3 - h_2$, $n_5 = a_2 - h_1$, $n_6 = a_1, p_i = 0$, $i = 1 \dots, 6$, and the other coefficients are arbitrary. Then

$$\begin{aligned} \Psi_\mu{}^i{}_\nu = \Phi_\mu{}^i{}_\nu = F_\mu{}^i{}_{\nu j}\, y^j = {} & y^i\, (b_1\, T_\rho{}^\rho{}_\mu\, T_\sigma{}^\sigma{}_\nu + b_2\, T_\sigma{}^\rho{}_\mu\, T_\rho{}^\sigma{}_\nu + b_3\, T_\rho{}^\rho{}_\sigma\, T_\mu{}^\sigma{}_\nu) \\ & + y^i\, (c_1\, T_\rho{}^\rho{}_{\mu;\nu} + c_2\, T_\rho{}^\rho{}_{\nu;\mu} + c_3\, T_\mu{}^\rho{}_{\nu;\rho}) \\ & + y^j\, \big((a_3 - h_2)\, T_\mu{}^\rho{}_\rho\, K_j{}^i{}_\nu + (a_2 - h_1)\, T_\rho{}^\rho{}_\nu\, K_j{}^i{}_\mu + a_1\, T_\mu{}^\rho{}_\nu\, K_j{}^i{}_\rho\big) \\ & + d_1\, y^i\, \widetilde R_\rho{}^\rho{}_{\mu\nu} + d_2\, y^i\, \widetilde R_\mu{}^\rho{}_{\rho\nu} + e_1\, y^i\, R_p{}^p{}_{\mu\nu} + e_2\, R_j{}^i{}_{\mu\nu}\, y^j\,. \end{aligned} \tag{17}$$

Summerizing all results (12), (13), (15), (16) and (17) we get Lemma 1. □

As a direct consequence of Lemma 1 we have

COROLLARY 2. *All natural operators transforming* Λ *and* K *into classical connections on* $\boldsymbol{E}$ *are of the maximal order one and form 15-parameter family*

$$\widetilde D(\Lambda, K) = D(\Lambda, K) + \Phi(\Lambda, K)\,,$$

where $D(\Lambda, K)$ *is the connection given by Proposition 1 and* $\Phi(\Lambda, K)$ *is the 15-parameter family of natural tensor fields given by Lemma 1.*

REMARK 4. In [**3**] (see also [**10**], Proposition 54.3) the same result was obtained by direct calculations without using the reduction theorems. Our result coincides with the result of [**3, 10**] but our base of the 15-parameter family of operators of Lemma 1 differ from the base used in [**3, 10**].

Now we shall describe the geometrical construction of the operators from Lemma 1. First let us recall that we have the canonical immersions $\iota_{T^*\boldsymbol{M}} : T^*\boldsymbol{M} \to T^*\boldsymbol{E}$ and $\iota_{V\boldsymbol{E}} : V\boldsymbol{E} \to T\boldsymbol{E}$. The connection K defines the horisontal lift $h^K : T\boldsymbol{M} \to T\boldsymbol{E}$ and the vertical projection $\nu_K : T\boldsymbol{E} \to V\boldsymbol{E}$.

Then we have the following subfamilies of natural operators given by Λ and K.

A) Λ gives 3-parameter family of (1,2)-tensor fields on $\boldsymbol{M}$, [**10**], given by

$$S(\Lambda) = a_1 T + a_2 I_{T\boldsymbol{M}} \otimes \hat{T} + a_3 \hat{T} \otimes I_{T\boldsymbol{M}} , \tag{18}$$

where T is the torsion tensor of Λ, $\hat{T}$ is its contraction and $I_{T\boldsymbol{M}} : M \to T\boldsymbol{M} \otimes T^*\boldsymbol{M}$ is the identity tensor. Then the first 3-parameter subfamily of operators of Lemma 1 is given by $h^K(S(\Lambda)) \equiv \iota_{T^*\boldsymbol{M}} \otimes h^K \otimes \iota_{T^*\boldsymbol{M}}(S(\Lambda))$.

B) Λ and K define naturally the following 9-parameter family of (0,2) tensor fields on $\boldsymbol{M}$, [**10**], given by

$$\begin{aligned} G(\Lambda, K) = {} & b_1\, C^{12}_{13}(T \otimes T) + b_2\, C^{12}_{31}(T \otimes T) + b_3\, C^{12}_{12}(T \otimes T) \\ & + c_1\, C^1_1 \widetilde{\nabla} T + c_2\, \overline{C^1_1 \widetilde{\nabla} T} + c_3\, C^1_3 \widetilde{\nabla} T + d_1\, C^1_1 R[\widetilde{\Lambda}] + d_2\, C^1_2 R[\widetilde{\Lambda}] + e_1\, C^1_1 R[K] , \end{aligned}$$

where C^{ij}_{kl} is the contraction with respect to indicated indices and $\overline{C^1_1 \widetilde{\nabla} T}$ denotes the conjugated tensor obtained by the exchange of subindices. The second 9-parameter subfamily of operators from Lemma 1 is then given by $L \otimes G(\Lambda, K) \equiv \iota_{T^*\boldsymbol{M}} \otimes \iota_{V\boldsymbol{E}} \otimes \iota_{T^*\boldsymbol{M}}(L \otimes G(\Lambda, K))$, where $L = y^i\, \partial_i$ is the Liouville vertical vector field on $\boldsymbol{E}$.

C) The value of the curvature tensor $R[K]$ applied on the Liouville vector field is in $T^*\boldsymbol{M} \otimes V\boldsymbol{E} \otimes T^*\boldsymbol{M}$. Then $R[K](L) \equiv \iota_{T^*\boldsymbol{M}} \otimes \iota_{V\boldsymbol{E}} \otimes \iota_{T^*\boldsymbol{M}}(R[K](L))$ is the operator standing by e_2 in Lemma 1.

D) Finally, if we consider ν_K as the vertical valued 1-form $\nu_K : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}$ with coordinate expression

$$\nu_K = (d^i - K_j{}^i{}_\lambda\, y^j\, d^\lambda) \otimes \partial_i ,$$

the last 2-parameter subfamily of operators from Lemma 1 is obtained by applying the morphism $\iota_{T^*\boldsymbol{M}} \otimes \iota_{V\boldsymbol{E}} \otimes \mathrm{id}_{T^*\boldsymbol{E}}$ on

$$H(\Lambda, K) = h_1\, \nu_K \otimes \hat{T} + h_2\, \hat{T} \otimes \nu_K .$$

Summarizing the above constructions we get

THEOREM 3. *All classical connections on $\boldsymbol{E}$ naturally given by Λ (in order s) and by K (in order r, $s \geqslant r - 2$) are of the maximal order one and are of the form*

$$\widetilde{D}(\Lambda, K) = D(\Lambda, K) + h^K\big(S(\Lambda)\big) + L \otimes G(\Lambda, K) + e_2\, R[K](L) + H(\Lambda, K) .$$

COROLLARY 3. *All natural operators transforming a general linear connection K on $\boldsymbol{E}$ and a symmetric classical connection Λ on $\boldsymbol{M}$ into classical connections on $\boldsymbol{E}$ are of the maximal order one and form the following 4-parameter family*

$$\widetilde{D}(\Lambda, K) = D(\Lambda, K) + L \otimes \big(d_1\, C^1_1 R[\widetilde{\Lambda}] + d_2\, C^1_2 R[\widetilde{\Lambda}] + e_1\, C^1_1 R[K]\big) + e_2\, R[K](L) .$$

3. Natural connections on the 1st jet prolongation of vector bundles

Assume the 1-jet prolongation $\pi^1_0 : J^1\boldsymbol{E} \to \boldsymbol{E}$ with the induced fiber coordinate chart $(x^\lambda, y^i; y^i_\lambda)$. Then $J^1\boldsymbol{E}$ is a (1,1)-order G-gauge-natural bundle with the standard fiber $\mathbb{R}^n \times \mathbb{R}^n \otimes \mathbb{R}^{m*}$ and the induced action of $W^{(1,1)}_m G$ given in coordinates by

$$\bar{y}^i = a^i_p\, y^p\,, \quad \bar{y}^i_\lambda = a^i_p\, y^p_\rho\, \tilde{a}^\rho_\lambda + a^i_{p\rho}\, y^p\, \tilde{a}^\rho_\lambda\,.$$

A connection Γ on $J^1\boldsymbol{E}$ is given by $\Gamma : J^1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ^1\boldsymbol{E}$ over the identity of $T\boldsymbol{E}$ with coordinate expression

$$\Gamma = d^A \otimes \left(\partial_A + \Gamma_A{}^i_\lambda\, \partial_i^\lambda\right), \quad \Gamma_A{}^i_\lambda \in C^\infty(J^1\boldsymbol{E}, \mathbb{R})\,. \tag{19}$$

A connection Γ is affine if and only if $\Gamma_A{}^i_\lambda = \Gamma_A{}^i_\lambda{}^\mu_p\, y^p_\mu + \Gamma_A{}^i_\lambda{}^\cdot_\cdot$.

Remark 5. If Γ_1 and Γ_2 are two connections on $J^1\boldsymbol{E}$, then the difference $\Gamma_1 - \Gamma_2 : J^1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes VJ^1\boldsymbol{E}$ and, by using the identification $VJ^1\boldsymbol{E} = J^1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{M} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}$, we get

$$\Gamma_1 - \Gamma_2 = \varphi : J^1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{M} \otimes V\boldsymbol{E}\,.$$

In **[9]** we have described a natural operator χ transforming a classical connection on the total space of a fibered manifold and a classical connection on the base manifold into a connection on the 1st jet prolongation of the fibered manifold. Applying this operator on a classical connection D on the total space of a vector bundle $\boldsymbol{E} \to \boldsymbol{M}$ we get

Proposition 2. *Let D be a classical connection on $\boldsymbol{E}$ and Λ be a classical connection on $\boldsymbol{M}$. Then we have a connection $\Gamma(\Lambda, D) = \chi(D)$ on $\pi^1_0 : J^1\boldsymbol{E} \to \boldsymbol{E}$ such that*

$$\Gamma_A{}^i_\lambda = D_A{}^i{}_j\, y^j_\lambda + D_A{}^i{}_\lambda - y^i_\mu\, (D_A{}^\mu{}_j\, y^j_\lambda + D_A{}^\mu{}_\lambda)\,.$$

Remark 6. The connection $\Gamma(\Lambda, D)$ is independent of Λ, but the geometric construction of the operator χ depends on Λ essentially, see **[9]**.

Now, applying the operator χ on the connection $D(\Lambda, K)$ from Proposition 1, we get

Theorem 4. *A general linear connection K on $\boldsymbol{E}$ and a classical connection Λ on $\boldsymbol{M}$ give naturally the connection $\Gamma(\Lambda, K) = \chi(D(\Lambda, K))$ on $J^1\boldsymbol{E}$ with the coordinate expression*

$$\begin{aligned}
\Gamma_\mu{}^i_\lambda &= K_j{}^i{}_\mu\, y^j_\lambda + (\partial_\lambda K_j{}^i{}_\mu - K_p{}^i{}_\lambda\, K_j{}^p{}_\mu + K_j{}^i{}_\rho\, \Lambda_\mu{}^\rho{}_\lambda)\, y^j\,,\\
\Gamma_j{}^i_\lambda &= K_j{}^i{}_\lambda\,.
\end{aligned}$$

Corollary 4. *All natural operators transforming a (general) linear connection K on $\boldsymbol{E}$ (in order r) and a classical connection Λ on $\boldsymbol{M}$ (in order s, $s \geqslant r-2$) into connections on $J^1\boldsymbol{E}$ are of the form*

$$\widetilde{\Gamma}(\Lambda, K) = \Gamma(\Lambda, K) + \varphi(\Lambda, K)\,,$$

where $\Gamma(K, \Lambda)$ is the connection from Theorem 4 and $\varphi(\Lambda, K)$ is a natural operator

$$\varphi : J^1\boldsymbol{E} \underset{\boldsymbol{M}}{\times} J^s\,\mathrm{Cla}\,\boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^r\,\mathrm{Lin}\,\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{M} \otimes V\boldsymbol{E}\,.$$

So to classify natural connections on $J^1\boldsymbol{E}$ it is sufficient to classify natural operators φ.

Lemma 2. *All natural tensor fields $\varphi(\Lambda, K) : J^1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{M} \otimes V\boldsymbol{E}$ naturally given by a classical connection Λ on $\boldsymbol{M}$ (in order s) and by a general linear connection K on $\boldsymbol{E}$ (in order r, $s \geqslant r-2$) are of the maximal order one (with respect to both* *connections) and form a 14-parameter family of operators with coordinate expression given by*

$$\begin{aligned}\varphi(\Lambda, K) = \Big(& \big(a_1\, T_\lambda{}^\rho{}_\mu + a_2\, T_\sigma{}^\sigma{}_\mu\, \delta^\rho_\lambda + a_3\, T_\lambda{}^\sigma{}_\sigma\, \delta^\rho_\mu\big)\, y^i_\rho + y^i\, \big(b_1\, T_\rho{}^\rho{}_\lambda\, T_\sigma{}^\sigma{}_\mu + b_2\, T_\sigma{}^\rho{}_\lambda\, T_\rho{}^\sigma{}_\mu \\ & + b_3\, T_\rho{}^\rho{}_\sigma\, T_\lambda{}^\sigma{}_\mu + c_1\, T_\rho{}^\rho{}_{\lambda;\mu} + c_2\, T_\rho{}^\rho{}_{\mu;\lambda} + c_3\, T_\lambda{}^\rho{}_{\mu;\rho}\big) \\ & - \big(a_3\, T_\rho{}^\rho{}_\lambda\, K_j{}^i{}_\mu + (a_2 + h_1)\, T_\rho{}^\rho{}_\mu\, K_j{}^i{}_\lambda - a_1\, T_\lambda{}^\rho{}_\mu\, K_j{}^i{}_\rho\big)\, y^j \\ & + y^i\, \big(d_1\, R_\rho{}^\rho{}_{\lambda\mu} + d_2\, R_\lambda{}^\rho{}_{\rho\mu}\big) + e_1\, y^i\, R_p{}^p{}_{\lambda\mu} + e_2\, R_j{}^i{}_{\lambda\mu}\, y^j \Big)\, d^\lambda \otimes d^\mu \otimes \partial_i \\ & + h_1\, \delta^i_j\, T_\rho{}^\rho{}_\mu\, d^j \otimes d^\mu \otimes \partial_i\,,\end{aligned}$$

where all coefficients are real numbers.

Proof. We have the action of the group $W^{(1,1)}_{m,n}G$ on the standard fiber $S_F = \mathbb{R}^{(m+n)*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^n$ of $T^*\boldsymbol{E} \otimes T^*\boldsymbol{M} \otimes V\boldsymbol{E}$ given by

$$\bar\varphi_\lambda{}^i_\mu = a^i_p\, \varphi_\rho{}^p_\sigma\, \tilde a^\rho_\lambda\, \tilde a^\sigma_\mu + a^i_p\, \varphi_q{}^p_\sigma\, \tilde a^q_{r\lambda}\, a^r_s\, y^s\, \tilde a^\sigma_\mu\,, \quad \bar\varphi_j{}^i_\mu = a^i_p\, \varphi_q{}^p_\sigma\, \tilde a^q_j\, \tilde a^\sigma_\mu\,. \tag{20}$$

The corresponding equivariant mapping $\varphi : \mathbb{R}^n \times \mathbb{R}^n \otimes \mathbb{R}^{m*} \times T^s_m Q \times T^r_m R \to S_F$ is given, by Corollary 1, by

$$\varphi_A{}^i_\lambda(y^i, y^i_\lambda, \Lambda_\mu{}^\lambda{}_{\nu,\boldsymbol{\alpha}}, K_j{}^i{}_{\mu,\boldsymbol{\beta}}) = \psi_A{}^i_\lambda(y^i, y^i_\lambda, T_\mu{}^\lambda{}_{\nu;\gamma}, \widetilde R_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\boldsymbol{\varepsilon}})\,,$$

where the multiindices $\boldsymbol{\alpha}, \dots, \boldsymbol{\varepsilon}$ are as in the proof of Lemma 1.

The equivariance with respect to fiber homotheties $(c\,\delta^i_j)$ implies

$$c\,\psi_\lambda{}^i_\mu = \psi_\lambda{}^i_\mu(c\,y^i, c\,y^i_\lambda, T_\mu{}^\lambda{}_{\nu;\gamma}, \widetilde R_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\varepsilon})\,.$$

which gives, by the homogeneous function theorem, that $\psi_\lambda{}^i_\mu$ is linear in y^i and y^i_λ, i.e.

$$\psi_\lambda{}^i_\mu = f^i_{\lambda\mu j}\, y^j + g^{i\nu}_{\lambda\mu j}\, y^j_\nu\,.$$

First, we shall discuss $g^{i\nu}_{\lambda\mu j}$. We get, from the equivariance with respect to base homotheties $(c\,\delta^\lambda_\mu)$,

$$c^{-1}\, g^{i\nu}_{\lambda\mu j} = g^{i\nu}_{\lambda\mu j}(c^{-(\|\gamma\|+1)}\, T_\mu{}^\lambda{}_{\nu;\gamma}, c^{-(\|\boldsymbol{\delta}\|+2)}\, \widetilde R_\mu{}^\lambda{}_{\nu_1\nu_2;\boldsymbol{\delta}}, c^{-1}\, K_j{}^i{}_\lambda, c^{-(\|\varepsilon\|+2)}\, R_j{}^i{}_{\lambda\mu;\varepsilon})$$

which implies that $g^{i\nu}_{\lambda\mu j}$ is polynomial with exponents satisfying (14). We have the following 2 possible solutions: $a_0 = 1$ and the other exponents vanish; $c = 1$ and the other exponents vanish. Then

$$\begin{aligned} g^{i\nu}_{\lambda\mu j} = A^{i\nu\sigma\tau}_{\lambda\mu j\rho}\, T_\sigma{}^\rho{}_\tau + L^{i\nu q\sigma}_{\lambda\mu jp}\, K_q{}^p{}_\sigma &= a_1\, \delta^i_j\, T_\lambda{}^\nu{}_\mu + a_2\, \delta^i_j\, \delta^\nu_\lambda\, T_\rho{}^\rho{}_\mu + a_3\, \delta^i_j\, \delta^\nu_\mu\, T_\lambda{}^\rho{}_\rho \\ & + l_1\, \delta^\nu_\lambda\, K_j{}^i{}_\mu + l_2\, \delta^\nu_\mu\, K_j{}^i{}_\lambda + l_3\, \delta^i_j\, \delta^\nu_\lambda\, K_p{}^p{}_\mu + l_4\, \delta^i_j\, \delta^\nu_\mu\, K_p{}^p{}_\lambda \end{aligned}$$

Similarly, for $f^i_{\lambda\mu j}$ we get, from the equivariance with respect to base homotheties $(c\,\delta^\lambda_\mu)$,

$$c^{-2}\, f^i_{\lambda\mu j} = f^i_{\lambda\mu j}(c^{-(\|\gamma\|+1)}\, T_\mu{}^\lambda{}_{\nu;\gamma}, c^{-(\|\boldsymbol{\delta}\|+2)}\, \widetilde R_\mu{}^\lambda{}_{\nu_1\nu_2;\boldsymbol{\delta}}, c^{-1}\, K_j{}^i{}_\lambda, c^{-(\|\varepsilon\|+2)}\, R_j{}^i{}_{\lambda\mu;\varepsilon})$$

which implies that $f^i_{\lambda\mu j}$ is polynomial with exponents satisfying (14) with -2 on the left hand side. We have the following 6 possible solutions: $a_0 = 2$ and the other exponents vanish; $a_1 = 1$ and the other exponents vanish; $a_0 = 1$, $c = 1$ and the other exponents vanish; $b_0 = 1$ and the other exponents vanish; $c = 2$ and the other exponents vanish; $d_0 = 1$ and the other exponents vanish. Then

$$\begin{aligned}
f^i_{\lambda\mu j} &= B^{i\sigma_1\tau_1\sigma_2\tau_2}_{\lambda\mu j\rho_1\rho_2}\, T_{\sigma_1}{}^{\rho_1}{}_{\tau_1}\, T_{\sigma_2}{}^{\rho_2}{}_{\tau_2} + C^{i\sigma\tau_1\tau_2}_{\lambda\mu j\rho}\, T_\sigma{}^\rho{}_{\tau_1;\tau_2} + D^{i\sigma\tau_1\tau_2}_{\lambda\mu j\rho}\, R_\sigma{}^\rho{}_{\tau_1\tau_2} \\
&\quad + P^{iq_1\tau_1q_2\tau_2}_{\lambda\mu jp_1p_2}\, K_{q_1}{}^{p_1}{}_{\tau_1}\, K_{q_2}{}^{p_2}{}_{\tau_2} + E^{iq\tau_1\tau_2}_{\lambda\mu jp}\, R_q{}^p{}_{\tau_1\tau_2} + N^{i\sigma\tau_1q\tau_2}_{\lambda\mu j\rho p}\, T_\sigma{}^\rho{}_{\tau_1}\, K_q{}^p{}_{\tau_2} \\
&= b_1\, \delta^i_j\, T_\rho{}^\rho{}_\lambda\, T_\sigma{}^\sigma{}_\mu + b_2\, \delta^i_j\, T_\sigma{}^\rho{}_\lambda\, , T_\rho{}^\sigma{}_\mu + b_3\, \delta^i_j\, T_\rho{}^\rho{}_\sigma\, T_\lambda{}^\sigma{}_\mu + c_1\, \delta^i_j\, T_\rho{}^\rho{}_{\lambda;\mu} \\
&\quad + c_2\, \delta^i_j\, T_\rho{}^\rho{}_{\mu;\lambda} + a_9\, \delta^i_j\, T_\lambda{}^\rho{}_{\mu;\rho} + d_1\, \delta^i_j\, R_\rho{}^\rho{}_{\lambda\mu} + d_2\, \delta^i_j\, R_\lambda{}^\rho{}_{\rho\mu} \\
&\quad + p_1\, \delta^i_j\, K_p{}^p{}_\lambda\, K_q{}^q{}_\mu + p_2\, \delta^i_j\, K_q{}^p{}_\lambda\, K_p{}^q{}_\mu + p_3\, K_j{}^i{}_\lambda\, K_p{}^p{}_\mu \\
&\quad + p_4\, K_j{}^i{}_\mu\, K_p{}^p{}_\lambda + p_5\, K_p{}^i{}_\lambda\, K_j{}^p{}_\mu + p_6\, K_p{}^i{}_\mu\, K_j{}^p{}_\lambda + e_1\, R_j{}^i{}_{\lambda\mu} + e_2\, \delta^i_j\, R_p{}^p{}_{\lambda\mu} \\
&\quad + n_1\, T_\lambda{}^\rho{}_\rho\, K_j{}^i{}_\mu + n_2\, T_\rho{}^\rho{}_\mu\, K_j{}^i{}_\lambda + n_3\, T_\lambda{}^\rho{}_\mu\, K_j{}^i{}_\rho + n_4\, \delta^i_j\, T_\rho{}^\rho{}_\lambda\, K_p{}^p{}_\mu \\
&\quad + n_5\, \delta^i_j\, T_\rho{}^\rho{}_\mu\, K_p{}^p{}_\lambda + n_6\, \delta^i_j\, T_\lambda{}^\rho{}_\lambda\, K_p{}^p{}_\rho\, .
\end{aligned}$$

Further, from the equivariancy with respect to fiber homotheties, we have

$$\psi_j{}^i_\mu = \psi_j{}^i_\mu(c\, y^i, c\, y^i_\lambda, T_\mu{}^\lambda{}_{\nu;\gamma}, \widetilde{R}_\mu{}^\lambda{}_{\nu\kappa;\boldsymbol{\delta}}, K_j{}^i{}_\mu, R_j{}^i{}_{\lambda\mu;\varepsilon})\, .$$

which implies, by the homogeneous function theorem, that $\psi_j{}^i_\mu$ is independent of y^i and y^i_λ.

If we suppose the equivariance with respect to base homotheties $(c\, \delta^\lambda_\mu)$. We have

$$c^{-1}\, \psi_j{}^i_\mu = \psi_j{}^i_\mu(c^{-(\|\gamma\|+1)}\, T_\mu{}^\lambda{}_{\nu;\gamma}, c^{-(\|\boldsymbol{\delta}\|+2)}\, \widetilde{R}_\mu{}^\lambda{}_{\nu_1\nu_2;\boldsymbol{\delta}}, c^{-1}\, K_j{}^i{}_\lambda, c^{-(\|\varepsilon\|+2)}\, R_j{}^i{}_{\lambda\mu;\varepsilon})$$

which implies that $\psi_j{}^i_\mu$ is polynomial with exponents satisfying the equation (14), i.e.

$$\psi_j{}^i_\mu = H^{i\sigma\tau}_{j\mu\rho}\, T_\sigma{}^\rho{}_\tau + M^{iq\rho}_{j\mu p}\, K_q{}^p{}_\rho = h_1\, \delta^i_j\, T_\rho{}^\rho{}_\mu + m_1\, K_j{}^i{}_\mu + m_2\, \delta^i_j\, K_p{}^p{}_\mu\, .$$

Finally, the equivariance of $\psi_j{}^i_\mu$ with respect to elements of the type $(\delta^i_j, \delta^\lambda_\mu, a^i_{j\lambda})$ implies $m_1 = m_2 = 0$ and we have

$$\psi_j{}^i_\mu = h_1\, \delta^i_j\, T_\rho{}^\rho{}_\mu \tag{21}$$

and the equivariance of $\psi_\lambda{}^i_\mu$ with respect to elements of the type $(\delta^i_j, \delta^\lambda_\mu, a^i_{j\lambda})$ implies $p_i = 0$, $i = 1, \dots, 10$, $n_4 = n_5 = n_6 = 0$, $n_1 = -a_3$, $n_2 = -a_2 - h_1$, $n_3 = -a_1$ and the other coefficients are arbitrary. Then

$$\begin{aligned}
\psi_\lambda{}^i_\mu = \varphi_\lambda{}^i_\mu &= g^{i\nu}_{\lambda\mu j}\, y^j_\nu + f^i_{\lambda\mu j}\, y^j \\
&= a_1\, T_\lambda{}^\nu{}_\mu\, y^i_\nu + a_2\, T_\rho{}^\rho{}_\mu\, y^i_\lambda + a_3\, T_\lambda{}^\rho{}_\rho\, y^i_\mu + b_1\, y^i\, T_\rho{}^\rho{}_\lambda\, T_\sigma{}^\sigma{}_\mu + b_2\, y^i\, T_\sigma{}^\rho{}_\lambda\, T_\rho{}^\sigma{}_\mu \\
&\quad + b_3\, y^i\, T_\rho{}^\rho{}_\sigma\, T_\lambda{}^\sigma{}_\mu + c_1\, y^i\, T_\rho{}^\rho{}_{\lambda;\mu} + c_2\, y^i\, T_\rho{}^\rho{}_{\mu;\lambda} + c_3\, y^i\, T_\lambda{}^\rho{}_{\mu;\rho} \\
&\quad - a_3\, T_\lambda{}^\rho{}_\rho\, K_j{}^i{}_\mu\, y^j - (a_2 + a_{10})\, T_\rho{}^\rho{}_\mu\, K_j{}^i{}_\lambda\, y^j - a_1\, T_\lambda{}^\rho{}_\mu\, K_j{}^i{}_\rho\, y^j \\
&\quad + d_1\, y^i\, R_\rho{}^\rho{}_{\lambda\mu} + d_2\, y^i\, R_\lambda{}^\rho{}_{\rho\mu} + e_1\, y^i\, R_p{}^p{}_{\lambda\mu} + e_2\, R_j{}^i{}_{\lambda\mu}\, y^j\, .
\end{aligned} \tag{22}$$

Summerizing (21) and (22) we get Lemma 2. □

THEOREM 5. *All natural operators transforming a linear connection K on $\boldsymbol{E}$ and a classical linear connection Λ on $\boldsymbol{M}$ into connections on $\pi^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ are of the maximal order one and form the 14-parameter family*

$$\widetilde{\Gamma}(\Lambda, K) = \chi(D(\Lambda, K)) + \varphi(\Lambda, K)\, ,$$

where $\chi(D(\Lambda, K))$ is the connection given by Theorem 4 and $\varphi(\Lambda, K)$ is the 14-parameter family of natural tensor fields given by Lemma 2.

Let us recall that we have the natural complementary contact maps

$$\delta : J^1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{M} \to T\boldsymbol{E}\,, \quad \theta : J^1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to V\boldsymbol{E}\,,$$

with the coordinate expressions

$$\delta = d^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i)\,, \quad \theta = (d^i - y^i_\lambda d^\lambda) \otimes \partial_i\,. \tag{23}$$

Then we have the following, by using the notation of Section 2, geometric description of $\widetilde{\Gamma}(\Lambda, K)$.

Theorem 6. *All connections on $J^1\boldsymbol{E}$ naturally given by Λ (in order s) and by K (in order r, $s \geqslant r-2$) are of the maximal order one and are of the form*

$$\widetilde{\Gamma}(\Lambda, K) = \Gamma(\Lambda, K) + \theta \circ h^K\big(S(\Lambda)\big) + L \otimes G(\Lambda, K) + e_2\, R[K](L) + h_1\, \nu_K \otimes \hat{T}\,.$$

Remark 7. The 14-parameter family $\widetilde{\Gamma}(\Lambda, K)$ from Theorem 5 can be obtained from the 15-parameter family of Theorem 3 by applying the operator χ, i.e. $\widetilde{\Gamma}(\Lambda, K) = \chi\big(\widetilde{D}(\Lambda, K)\big)$. In fact $\chi\big(\widetilde{D}(\Lambda, K)\big) = \chi\big(D(\Lambda, K)\big) + \widetilde{\chi}\big(\Phi(\Lambda, K)\big)$, where

$$\widetilde{\chi} \equiv \mathrm{id}_{T^*\boldsymbol{E}} \otimes \theta \otimes \delta : J^1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} \otimes T\boldsymbol{E} \otimes T^*\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes T^*\boldsymbol{M}\,.$$

But

$$\widetilde{\chi}\big(h^K(\hat{T} \otimes I_{T\boldsymbol{M}})\big) = -\widetilde{\chi}(\hat{T} \otimes \nu_K)$$

which implies that operators standing in the family of Lemma 1 with coefficients a_3 and h_2 admit the same operator, standing with the coefficient a_3 in the family of Lemma 2. It is the reason why $\widetilde{\Gamma}(\Lambda, K)$ is only 14-parameter family.

Remark 8. From the coordinate expression of $\Gamma(\Lambda, K)$ and $\varphi(\Lambda, K)$ it is easy to see that all connections $\widetilde{\Gamma}(\Lambda, K)$ are affine.

Corollary 5. *All natural operators transforming a general linear connection K on $\boldsymbol{E}$ and a symmetric classical connection Λ on $\boldsymbol{M}$ into connections on $J^1\boldsymbol{E}$ are of the maximal order one and form the 4-parameter family*

$$\widetilde{\Gamma}(\Lambda, K) = \Gamma(\Lambda, K) + L \otimes \big(d_1\, C^1_1 R[\widetilde{\Lambda}] + d_2\, C^1_2 R[\widetilde{\Lambda}] + e_1\, C^1_1 R[K]\big) + e_2\, R[K](L)\,.$$

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