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Local invariants of weakly pseudoconvex manifolds in $\mathbb{C}^2$

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Abstract. The subject of this paper is local geometry of weakly pseudoconvex hypersurfaces in $\mathbb{C}^2$. In the first part we define a set of real valued local biholomorphic invariants. Then we survey some previous results, reformulated in terms of these invariants, which show that they carry important information on local geometry. In the last part we introduce generalized model hypersurfaces, which reveal another invariant, with rational values.

1. Introduction

Our interest in this paper is in local biholomorphic invariants of three dimensional weakly pseudoconvex CR manifolds. We first use model hypersurfaces to introduce a set of real valued invariants, which can be viewed as third level local invariants, following the signature of the Levi form and the type of the point. Then we survey several results on local biholomorphic geometry, reformulated in terms of these invariants, which show that they carry important information on existence of smooth peak functions and on local convexifiability. Finally, we mention another local invariant, which is obtained by introducing generalized model hypersurfaces, which capture also the complex nontangential direction and provide a more complete characterization of locally convexifiable manifolds.

2. Local gometry and basic invariants

Let Ω be a domain in $\mathbb{C}^2$ with smooth boundary M, and let p be a point on M. The study of domains with degenerate Levi form leads to the definition of type of p, which measures the maximal order of contact of complex curves with M at p ([**6**], [**3**]). We will assume that Ω is pseudoconvex and $p \in M$ is a point of finite type k. Pseudoconvexity implies that k is an even integer.

By a standard argument ([**5**]) we find local holomorphic coordinates (z, w), $z = x + iy,\ w = u + iv$ centered at p, so that M is described by

$$v = P(z, \bar z) + O(|z|^{k+1}, u), \tag{1}$$

where $P(z, \bar z)$ is a subharmonic but not harmonic homogeneous real valued polynomial of degree k,

$$P(z, \bar z) = a_0 |z|^k + \sum_{j=2,4,\dots,k} |z|^{k-j} Re(a_j z^j) \tag{2}$$

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for some $a_j \in \mathbb{C}$ and $a_0 > 0$. The coordinates are chosen so that the direction of the positive v-axis is the inner normal direction to $\bar{\Omega}$ at p. The corresponding hypersurface defined by

$$\{(z, w) \mid v = P(z, \bar{z})\}, \tag{3}$$

which approximates M at p is called the model hypersurface, or shortly the model.

We now define the Kohn-Nirenberg invariants.

Definition 1. *Let the model hypersurface to M at p be given by (2),(3), and let l be an even integer, $0 < l < k$. We will call the real number $\kappa_l = \frac{|a_l|}{a_0}$ the Kohn-Nirenberg invariant of order l.*

The first appearance of such an invariant is the seminal paper [**7**]. The originally mysterious constant $\frac{15}{7}$, which appears in the Kohn-Nirenberg example is an invariant of order six.

We can view these invariants as 3-rd level local biholomorphic invariants, the first level being the signature of the Levi form, with values in $\{-1, 0, 1\}$, the second level being the type of p, taking integer values.

We now show that κ_l is indeed a biholomorphic invariant.

Lemma 1. *The numbers κ_l are local biholomorphic invariants of M.*

Proof. The proof is based on the following fact. The polynomial P in (2) is determined uniquely up to biholomorphic transformations

$$z^* = \alpha z, \quad w^* = w + \beta z^k, \tag{4}$$

where $\alpha, \beta \in \mathbb{C}$. This was proved in Lemma 2 in [**9**]. For the reader's convenience, we give the argument here. Let (z, w), where $z = x + iy$, $w = u + iv$, be local holomorphic coordinates centered at p, such that the hyperplane $v = 0$ is tangent to M at p, and the positive v-axis points inside of Ω. Near p, the boundary of Ω is described by

$$v = F(z, \bar{z}, u), \tag{5}$$

where F is a smooth function which vanishes together with its first order derivatives at the origin. Such coordinates can be obtained from the general case by an affine transformation.

Consider holomorphic transformations of the form

$$\begin{aligned} z^* &= z + g(z, w) \\ w^* &= w + f(z, w), \end{aligned} \tag{6}$$

which preserve the above description of M. Let F^* be the function describing M in new coordinates. Substituting (6) into $v^* = F^*(z^*, \bar{z}^*, u^*)$ and restricting the variables z, w to M, we get a change of variables formula:

$$F^*(z + g(z, u + iF(z, \bar{z}, u)),\ \overline{z + g(z, u + iF(z, \bar{z}, u))}, u + Im\ f(z, u + iF(z, \bar{z}, u)) =$$

$$= F(z, \bar{z}, u) + Re\ f(z, u + Im\ f(z, u + iF(z, \bar{z}, u)) \tag{7}$$

We have to show that if a transformation (6) preserves the form given by (1) and (2), then the new leading polynomial P^* of F^* can be obtained from P by a transformation (4). We assign weights to the variables in (6) and (7). Weight 1 is given to $z, \bar{z}$ and weight k to u and w. F starts with terms of weight k, so the terms of weight ν

for $\nu \leqslant k$ in $Im\ f(z, u + iF(z, \bar{z}, u))$ come from corresponding terms of weight ν in $f(z, w)$. It follows that f does not contain $z^2, \ldots, z^{k-1}$, for all other entries in (7) are of weight $\geqslant k$. So $Re\ f$ has also weight $\geqslant k$, and (7) becomes

$$P^*(z + g, \bar{z} + \bar{g}) + \cdots = P(z, \bar{z}) + Im\ f(z, u + iF(z, \bar{z}, u)) + \ldots$$

where dots stand for terms of weight $> k$. Now we compare terms of weight k on both sides, which shows that P^* depends only on terms of weight 1 in g and weight k in f. In other words, it is obtained from P by a transformation (4). Now we finish the proof by observing that the numbers κ_l are preserved by this transformation. □

The Kohn-Nirenberg invariants carry a lot of information about local geometry of M near p, as the following propositions demonstrate. We first consider an important local property of M, local convexifiability.

Definition 2. *M is called locally convexifiable at $p \in M$ if there exist local holomorphic coordinates in a neighbourhood of p such that M is convex in the underlying linear structure.*

In order to simplify notation, let us denote

$$\gamma_{lk} = \frac{k}{l^2 - k}$$

if $l^2 \geqslant 3k - 2$ and

$$\gamma_{lk} = \sqrt{\frac{(4k - l^2 - 4)k^2}{(4k - 4)(k^2 - l^2)}}$$

if $l^2 \leqslant 3k - 2$. The first proposition, in which we consider domains with only one nonzero Kohn-Nirenberg invariant, explains the origin of the constants γ_{lk}.

Proposition 1. *([**8**]) If M has only one nonzero Kohn-Nirenberg invariant, of order l, then its model is convex if and only if $\kappa_l \leqslant \gamma_{lk}$. If, moreover, l is not a divisor of k, then the model is locally convexifiable at p if and only if $\kappa_l \leqslant \gamma_{lk}$.*

In the general case we have a sufficient and a necessary condition for local convexifiability.

Proposition 2. *([**8**]) Let the model at $p \in M$ be given by (2),(3). If*

$$\sum_{j=2,4,\ldots,k-2} \gamma_{jk}^{-1} \kappa_j < 1,$$

then M is locally convexifiable at p.

Proposition 3. *([**9**]) Let the model at $p \in M$ be given by (2),(3). If M is locally convexifiable at p, then*
(i) $\kappa_j \leqslant \gamma_{jk}$ for all $j > \frac{k}{2}$
and
(ii) $\kappa_j \leqslant 2\gamma_{jk}$ for all $j \leqslant \frac{k}{2}$.

Next we consider the question of existence of smooth peak functions. First we recall the definition. Let $A(\Omega)$ be the space of holomorphic functions on Ω.

Definition 3. *A holomorphic function $f \in A(\Omega) \cap C^k(\bar{\Omega})$, where $k = 0, 1, \ldots, \infty$, is called a C^k-peak function at p if $f(p) = 1$ and $|f(q)| < 1$ for $q \in \bar{\Omega} \setminus \{p\}$.*

We consider again domains with only one nonzero invariant κ_l. In this case P has only one other nonzero entry beside the first summand in (2). After a normalization, it has form

$$P(z, \bar{z}) = |z|^k + \kappa_l |z|^{k-l} Rez^l, \tag{8}$$

where k,l are even integers. Let M denote the corresponding hypersurface in $\mathbb{C}^2$ defined by $\{(z, w) \in \mathbb{C}^2 \mid v = P(z, \bar{z})\}$, where P is given by (8).

Evaluating the Levi form shows that M is pseudoconvex if and only if

$$\kappa_l \leqslant \frac{k^2}{k^2 - l^2}.$$

If $\kappa_l < 1$, then $P(z, \bar{z}) > \epsilon |z|^k$ and it is easy to verify that $f(z, w) = e^{iw}$ is a smooth peak function at p.

For $\kappa_l > 1$ the situation depends on whether l is a divisor of k, or not. First we consider the case when $l \nmid k$. The following proposition was obtained in the thesis [**10**]. It is a generalization of a result of Fornaess ([**4**]). We give here the proof.

Proposition 4. *Suppose that the model at $p \in M$ is given by (3), where P has form (8). If $l \nmid k$ and $\kappa_l > 1$, then there is no C^1-peak function at p.*

Proof. Suppose, to get a contradiction, that there exists a local peak function $f \in C^1(U \cap \bar{\Omega})$. Let $g = Ref - 1$. We have $g(0) = 0$ and $g < 0$ in $\bar{\Omega} \cap U \setminus \{p\}$. By the Hopf lemma, $g_v(0) = b > 0$. For a small $\epsilon > 0$ we choose a neighbourhood of 0 such that

$$b - \epsilon < g_v(z, w) < b + \epsilon$$

in this neighbourhood. We will consider g restricted to the hyperplane $u = 0$. The intersection of M with this hyperplane is described by

$$v = |z|^k + \kappa_l |z|^{k-l} Rez^l + o(|z|^k). \tag{9}$$

Let $\delta > 0$ be sufficiently small and let z be a complex number with $|z| = \frac{1}{k} \delta^{\frac{1}{k}} \tau$ for some $\tau \in (0, 1)$. Since $\kappa_l < k$, it follows from (9) that the point $(z, -\delta)$ lies in Ω.

Also the point $(z, -\frac{1}{k^k} \tau^k (1 + a + \epsilon)\delta)$ and the whole segment joining these two points lies inside of Ω. Since $u < 0$ in Ω, it follows that

$$g(z, -\delta) < -(1 - (1 + a + \epsilon)\frac{1}{k^k}\tau^k)\delta(b - \epsilon).$$

If moreover $argz = \pi \frac{(2j+1)}{l}$, $j = 0, ..., l-1$, then also the point $(z, (a - 1 - \epsilon)\frac{1}{k^k}\tau^k \delta)$ is in Ω and we get

$$g(z, -\delta) < -(1 + (a - 1 - \epsilon)\frac{1}{k^k}\tau^k)\delta(b - \epsilon).$$

Also, since $g(0) = 0$, we get $g(0, -\delta) > -(b + \epsilon)\delta$. Using the disc $\{(z, w) = (\eta \frac{1}{k} \delta^{\frac{1}{k}}, -\delta)\} \subseteq \Omega$, where $\eta \in \mathbb{C}, |\eta| \leqslant 1$, we define a harmonic function on the unit disc by

$$f_\delta(\eta) = \frac{1}{\delta}\{g(\eta \frac{1}{k} \delta^{\frac{1}{k}}, -\delta) - g(0, -\delta)\}.$$

The functions f_δ are uniformly bounded and $f_\delta(0) = 0$. Let f be the limit of some convergent sequence of these functions with $\delta \to 0$. From the estimates on g we get

$$f(\tau e^{i\theta}) \leqslant b(1 + a)\frac{1}{k^k}\tau^k \tag{10}$$

and

$$f(\tau e^{i\pi\frac{(2j+1)}{l}}) < -b(a-1)\frac{1}{k^k}\tau^k. \tag{11}$$

As δ tends to 0, we let also ϵ tend to 0, therefore it does not appear in the last inequalities.
Since f is harmonic, it is the real part of a holomorphic function, and from (10) it follows that it vanishes to order k at 0. So $f(\eta) = c\eta^k + o(|\eta|^k)$. But then

$$\sum_{j=0}^{l-1} f(\tau e^{i\pi\frac{(2j+1)}{l}}) = 0 + o(\tau^k)$$

because $l \nmid k$. That contradicts (11). □

If $l \mid k$, there is always a smooth peak function.

Proposition 5. *(**[8]**) Let M be pseudoconvex, and let the model at $p \in M$ be given by (3),(8). If $l \mid k$, then there is a C^∞ peak function at p.*

3. Higher order invariants

The polynomial P in (1) captures local behaviour of M in the complex tangential direction. In order to study convexifiability of M near p we wish to take into account also the behaviour in complex nontangential directions, involving the variable u. A natural tool for this is provided by weighted coordinates. We will assign weights to the variables $z, \bar{z}, w$ and u.

Definition 4. *A weight is a pair of rational numbers $\lambda = (\lambda_1, \lambda_2)$ such that $\lambda_1 = \frac{1}{n}$ for some $n \in N$ and there exist integers k_1, k_2, with $k_2 > 0$ such that $k_1\lambda_1 + k_2\lambda_2 = 1$.*

The weighted degree of monomials $c_{ijk}z^i\bar{z}^ju^k$ and $d_{ij}z^iw^k$ is defined to be $(i+j)\lambda_1 + k\lambda_2$ and $i\lambda_1 + k\lambda_2$, respectively. A real valued polynomial $P(z, \bar{z}, u)$ is λ-homogeneous of weighted degree γ if it is a sum of monomials of weighted degree γ, and similarly for a holomorphic polynomial $P(z, w)$.

First we assign weight to the variables z and $\bar{z}$, equal to the reciprocal of the type of the point, i.e., $\lambda_1 = \frac{1}{k}$. The weight μ of w and u is determined as follows. We will call a weight $(\frac{1}{k}, \lambda_2)$ allowable, if there exist local holomorphic coordinates such that the defining equation can be written in the form $v = P(z, \bar{z}, u) + o_{wt}(1)$, where

$$P(z, \bar{z}, u) = \sum_{\frac{m}{k}+\lambda_2 l=1} \sum_{j=0}^{m} a_{mjl} z^j \bar{z}^{m-j} u^l, \tag{12}$$

and $o_{wt}(1)$ denotes terms of weighted degree bigger than one with respect to the weight $(\frac{1}{k}, \lambda_2)$.

Clearly, for any real $\delta > 0$ there are only finitely many rational numbers λ_2 bigger than δ such that $(\frac{1}{k}, \lambda_2)$ is a weight. When the set of allowable values for λ_2 is finite, we set μ to be the smallest of them. If the set is not bounded away from 0, we set $\mu = 0$. It is shown in **[12]** that in this case, $(\frac{1}{k}, 0)$ is allowable. By definition, μ is an invariant with values in $\mathbb{Q}$.

Now we fix local holomorphic coordinates (z, w) which correspond to the weight $(\frac{1}{k}, \mu)$. Hence the defining equation has form

$$v = \sum_{\frac{m}{k}+\mu l=1} \sum_{j=0}^{m} a_{mjl} z^j \bar{z}^{m-j} u^l + o_{wt}(1).$$

In [**12**] we show how this polynomial can be used to give stronger results on local convexifiability.

Let $\mathbb{H}_{k,\mu}$ denote the set of all real valued homogeneous polynomials in z, u of weighted degree one which are harmonic in the z variable. Clearly, $h(z, u) \in \mathbb{H}_{k,\mu}$ if and only if $h(z, u) = Re(\sum_{\frac{m}{k}+\mu l=1} \alpha_m z^m u^l)$ for some $\alpha_m \in \mathbb{C}$.

For $(z, u) \in \mathbb{C} \times \mathbb{R}$ and $\zeta \in \mathbb{C}$ we will use the following notation for the value of the real Hessian restricted to the z-direction:

$$D_z^2 P(z, u; \zeta) = \sum_{i,j=1}^{2} \frac{\partial^2 P}{\partial x_i \partial x_j}(z, u)\zeta_i \zeta_j,$$

where $z = x_1 + ix_2$ and $\zeta = \zeta_1 + i\zeta_2$. Since P is weighted homogeneous, positivity of $D_z^2 P$ is determined by its restriction to the set $S_2 \times S_1$, where $S_2 = \{(z, u) \in \mathbb{C} \times \mathbb{R} : |z|^2 + u^2 = 1\}$ and $S_1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. The following pair of conditions for local convexifiability of M is contained in [**12**].

PROPOSITION 6. *If there exists $h \in \mathbb{H}_{k,\mu}$ such that $D_z^2(P + h) > 0$ on $S_2 \times S_1$, then M is locally convexifiable.*

Proposition 6 is complemented by the following

PROPOSITION 7. *If M is locally convexifiable, then there exists $h \in \mathbb{H}_{k,\mu}$ such that $D_z^2(P + h) \geqslant 0$ on $S_2 \times S_1$.*

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