DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 323 – 332

Curvature bounds for asymptotically flat manifolds and the spinor operator

Margarita Kraus

1. Introduction

This article is based on a joint work with F. Finster. The motivation for it comes of general relativity. We consider a Lorentzian manifold (N^{n+1}, g) of signature $(-,+,\ldots,+)$ of dimension $n+1$ and a spacelike hypersurface $M \subset N$. We denote the normal unit vector field by ν and the second fundamental form by $h = -\nabla\nu$. We suppose that N satisfies the Einstein equation

$$\mathrm{Ric} - \frac{1}{2} s g = 8\pi T$$

where Ric denotes the Ricci curvature, s the scalar curvature and T the energy momentum tensor, which describes the distribution of matter in the space time N. The energy momentum tensor should satisfy the ***dominant energy condition,*** which means

(i) $T(u,u) \leqslant 0$
(ii) $T(u,\cdot)$ non spacelike

for every timelike vectorfield u. Note that in the time symmetric case, that means $h = 0$, the dominant energy condition is equivalent to $s \geqslant 0$.

The spacelike hypersurface $M^n \subset N^{n+1}$ is supposed to be ***asymptotically flat***.

Definition 1. *M is* **asymptotically flat** *if there is a compact set $K \subset M$ and a diffeomorphism φ which maps $M \setminus K$ to the region $\mathbb{R}^n \setminus B_r(0)$ outside a ball of radius r. Under this diffeomorphism, the metric and second fundamental form should be of the form*

$$\begin{aligned} (\varphi_* g)_{ij} &= \delta_{ij} + \mathcal{O}(r^{2-n})\,, \quad & \partial_k(\varphi_* g)_{ij} &= \mathcal{O}(r^{1-n})\,, \quad & \partial_k\partial_l(\varphi_* g)_{ij} &= \mathcal{O}(r^{-n}) \\ (\varphi_* h)_{ij} &= \mathcal{O}(r^{1-n})\,, & \partial_k(\varphi_* h)_{ij} &= \mathcal{O}(r^{-n})\,. \end{aligned}$$

Besides of mathematical reasons this definition has also a physical motivation: In case $n = 3$ the metric differs from the flat euclidean metric by a term, which behaves at infinity just like the Newton potential of a point particle $V \sim \frac{1}{r}$.

In asymptotically flat manifolds it is possible to introduce the ADM-energy and momentum, which have an interpretation as the total energy and momentum of space time.

This paper is in final form and no version of it will be published elswhere.

Definition 2. *([**1**]) The **ADM-energy and momentum** are defined by* $(E,P)\in\mathbb{R}^{n+1}$

$$\begin{aligned}
E &= c_1(n)\lim_{r\to\infty}\int_{S_r}\big(div(\varphi_* g(n_r,\cdot)) - (d\mathrm{tr}\ \varphi_* g)(n_r)\big)d\mu\\
<P,X> &= c_2(n)\lim_{r\to\infty}\int_{S_r}\big(\varphi_* h(n_r,X) - <X,n_r> \mathrm{tr}\ h\big)d\mu
\end{aligned}$$

where $c_1(n)$ *and* $c_2(n)$ *are constants and* n_r *is the normal unit vectorfield on the sphere* S_r *of radius* r. *In the case* $h\equiv 0$ *the total energy* $E =: m$ *is also called the* ***mass*** *of the asymptotically flat Riemannian manifold* M.

The energy and momentum are well defined and $E-|P|$ depends according to [**2**] only on the metric g.

Theorem 1. *(Positive energy theorem). If* N *satisfies the dominant energy condition, then*

(i) $E-|P|\geqslant 0$

(ii) $E=|P|$ *if and only if* N *is flat along* M.

This has been proven by Schoen and Yau ([**11**]) in dimension $n\leqslant 7$, and recently by minimal surface techniques by Christ and Lohkamp in the case $h\equiv 0$. There is another proof based on an idea of Witten [**13**] which works only in the case of spin manifolds. We will come to this proof in more detail in section 2. The second part of the positive energy theorem suggests, that there is a upper bound of the curvature tensor by the mass. Unfortunately as easily is seen from examples it is not possible to give an L^2-bound for the curvature tensor by the mass but one has to remove an exceptional set. More exactly the following holds:

Theorem 2. [**5, 4**] *Let* $M^n\subset N^{n+1}$ *an asymptotically flat hypersurface of a Lorentzian manifold, which satisfies the dominant energy condition. Let* $h\equiv 0$ *or* $n=3$. *In case* $h\equiv 0$ *let* $L=1$, *otherwise choose* $L\geqslant 3$ *such that*

$$(L^\alpha-1)^2 \ \geqslant\ C\,\frac{4\pi E+\|h\|_2}{k^2\,(k+24\,\|h\|_3)^2}\,\|\rho\|_3$$

where

$$\begin{aligned}
\alpha &= \left(1+24\,\frac{\|h\|_3}{k}\right)^{-1}\\
\rho &= |h|^2+|\nabla h|.
\end{aligned}$$

Then there is a set U *with measure bounded by*

$$\mu(U)^{\frac{n-2}{n}}\ \leqslant\ c_1\,\frac{L^6}{k^2}\,(4\pi E+\|h\|_2^2)$$

such that on $M\setminus U$ *the following inequality holds,*

$$\int_{M\setminus U}\eta\,|R_M|^2\,d\mu\ \leqslant\ c_2\,\sup_M\big(|\Delta\eta|+|\nabla\eta||h|+\eta\,(|s|+\rho)\big)E \tag{1.1}$$

$$+c_3\,L\,\sup_M\big(\eta\,(|\nabla R_M|+|h||R_M|)\big)\sqrt{E} \tag{1.2}$$

$$+c_4\,\frac{\sqrt{L+1}}{k}\,(\sup_M\eta)\,\sqrt{\|\rho\|_{6/5}}\ \||\nabla R_M|+|h||R_M\|_{5/12}\ \sqrt{E}\ . \tag{1.3}$$

Here $c_1, \dots, c_4$ are numerical constants (independent of L and the geometry), $\eta \in C^2(M)$ is a positive test function, R_M is the Riemannian curvature tensor of N restricted to M, and k is the isoperimetric constant, $k = \inf A/V^{\frac{n-1}{n}}$. In the proof of the theorem in case $h \equiv 0$ the ***Witten spinor,*** a harmonic spinor which satisfies certain boundary conditions, plays an important part similar as in the proof of the positive mass theorem by Witten . We will introduce this in section 2 and give an overview of the proof of the theorem in section 3. In case $h \not\equiv 0$ the role of the Witten spinor is taken over from the so called ***spinor operator***. In the last section we study the role of the spinor operator in more detail. For this purpose we restrict to a certain class of manifolds, which is at infinity conformally equivalent to the sphere without the north pole. We will show that the spinor operator is closely related to the Green's function of the Dirac operator on the compactified manifold. This is especially interesting as the mass also has an interpretation as a positive multiple of the constant term in the Green's function of the Yamabe operator [**12**].

2. The Hypersurface Diarc Operator

We assume in the following that (N, g) is a Lorentzian spin manifold with fixed spinorbundle $\Sigma_N \to N$. On the hypersurface $M \subset N$ we consider the restriction of the spinorbundle $\Sigma := \Sigma_N|M$ and the restriction $D\!\!\!/ : \Gamma\Sigma \to \Gamma\Sigma$ of the Dirac operator $D\!\!\!/_N$ on Σ_N to Σ, i.e. locally $D\!\!\!/ = \Sigma e_i \cdot \nabla_{e_i}$ with an orthonormal frame $(e_1, \dots, e_n)$ of M and the Levi Civita connection ∇ on N.

On Σ there is a indefinite fiber scalar product, which we denote by $< \cdot, \cdot >$ and a positive definite scalar product $(\cdot, \cdot) :=< \cdot, \nu \cdot >$. For the Levi Civita connection it holds

$$X(\Psi_1, \Psi_2) = (\nabla_X \Psi_1, \Psi_2) - (\Psi_1, \nabla^*_X \Psi_2)$$

with $\nabla^*_X = -\nabla_X - \nu \cdot h(X, .)$

According to [**10**] the hypersurface Dirac operator satisfies

$$D\!\!\!/^2 = \Delta + \frac{\mathcal{R}}{4}$$

Here Δ is the Laplacian $\Delta = \mathrm{tr}\,(\nabla^* \nabla)$ in the spinor bundle and $\mathcal{R}$ is the curvature expression $\mathcal{R} = \frac{1}{4}(s + 2\,\mathrm{Ric}(\nu, \nu) + 2 \sum_{i=1}^n \mathrm{Ric}(\nu, e_i)(\nu \cdot e_i)$. The dominant energy condition yields $\mathcal{R} \geqslant 0$.

In the coordinates induced by the diffeomorphism Φ of Definition 1.1, we choose a constant spinor ψ_0 of norm one in the asymptotic end and consider the boundary value problem

$$D\!\!\!/\psi \;=\; 0\,, \qquad \lim_{|x| \to \infty} \psi(x) \;=\; \psi_0 \text{ with } |\psi_0| = 1\,. \tag{2.1}$$

The existence and uniqueness of a solution of (2.1) is proven in [**10**]. The solution decays at infinity as

$$\psi \;=\; \psi_0 + \mathcal{O}(r^{2-n}),\;\; \partial_j \psi \;=\; \mathcal{O}(r^{1-n}),\;\; \partial_{kl} \psi \;=\; \mathcal{O}(r^{-n}).$$

Using the Weitzenböck formular (2), it is shown in [**10**] that for a solution of (2.1),

(2.2)
$$\|\nabla \psi\|^2_{L^2(M)} \;=\; 4\pi \left(E\,|\psi_0|^2 + < \psi_0 | P \cdot \psi_0 >\right) - (\psi | \mathcal{R} \psi) \;\leqslant\; 4\pi \left(E + < \psi_0 | P \cdot \psi_0 >\right)\,,$$

where P is the momentum as defined by (1.2). If one chooses ψ_0 such that

$$< \psi_0 | P \cdot \psi_0 >= -|P|,$$

one obtains the positive mass theorem [**13, 10**]

$$0 \leqslant 4\pi \left(E - |P|\right) . \tag{2.3}$$

For general ψ_0, (2.2) and (2.3) give rise to an L^2-bound of $\nabla\psi$,

$$\|\nabla\psi\|^2_{L^2(M)} \leqslant 4\pi \left(E + |P|\right) \leqslant 8\pi E . \tag{2.4}$$

3. The Proof of the Theorem

In this section we will give the steps of the proof of the theorem and point out the role of the spinor operator. First we will remind of the proof of the theorem in the case $h \equiv 0$, and $n = 3$. For details see [**3**]. We only treat here the case $\eta \equiv 1$. With R^Σ we denote the curvature tensor of the spinorbundle:

$$(R^\Sigma \Psi)(v, w) = \nabla^2_{v,w}\Psi - \nabla^2_{w,v}\Psi$$

with $\nabla^2_{v,w} := \nabla_v \nabla_w - \nabla_{\nabla_v w}$. Then

$$|R^\Sigma \Psi|^2 \leqslant 4|\nabla^2\Psi|^2 \tag{3.1}$$

In case $n = 3, h \equiv 0$ due to the commutation relations in the Clifford algebra it holds

$$|R|^2|\Psi|^2 \leqslant 2|R^\Sigma\Psi|^2 \tag{3.2}$$

On the other hand using partial derivation one gets

$$\int_M |\nabla^2\Psi|^2 d\mu \leqslant c_1 \int_M |R|^2|\nabla\Psi|^2 d\mu + c_2 \int_M |\nabla R|(\Psi, \nabla\Psi) d\mu \tag{3.3}$$

for constants c_1, c_2, which depend only on the dimension of the manifold M. From

$$\Delta|\Psi|^2 = \frac{s}{2}|\Psi|^2 + 2|\nabla\Psi|^2 \geqslant 0 \tag{3.4}$$

it follows

$$|\Psi|^2 \leqslant 1 \tag{3.5}$$

Inserting (2.4) and (3.5) in (3.3) and using the Hölder inequality gives the right hand side of (1.1) and (1.2). On the other hand it follows from (3.5)

$$\nabla|\Psi|^2 \leqslant 2|\nabla\Psi| \tag{3.6}$$

Now we use the Sobolev inequality, which has been proofed in [**4**]:

Lemma 1. *Let M be an asymptotically flat manifold of dimension $n \geqslant 3$. Then every non-negative function $g \in C^\infty(M) \cap H^{1,2}(M)$ with $\lim_{|x|\to\infty} g(x) = 0$ satisfies the inequality*

$$\|g\|_q \leqslant \frac{q}{k}\|\nabla g\|_2 \qquad \text{with} \qquad q = \frac{2n}{n-2}$$

and k the isoperimetric constant. From this lemma and (3.6) one gets $||1 - |\Psi|^2||_6 \leqslant \frac{36m}{k^2}$ and thus for $c > 0$ it holds $|\Psi(x)| \geqslant c$ except on a set $U(c)$ of the measure

$$\mu(c) \leqslant \left(\frac{cm}{k^2(1-c^2)}\right)^3 \tag{3.7}$$

In the case $n = 3, h \neq 0$ there are four points, which have to be modified:

(i) equality (3.2) does not hold,
(ii) $|\Psi|^2$ is not longer a subharmonic function, thus (3.5) does not longer hold and consequently
(iii) (3.6) does not hold, and the Sobolev inequality can not be applied.
(iv) In the partial integration 3.3 one gets additional curvature terms.

The tool to deal with (i) and (iii) is the ***spinor operator***, which has been introduced in [**4**]. For this we use the fact that we have not only one harmonic spinor but a whole family of at infinity linearly independent harmonic spinors. We will use the

NOTATION 1. *Let $E_1 \to X$, $E_2 \to X$ be vector bundles over the manifold X. Then by*

$$E_1 \boxtimes E_2 \to X \times X$$

we denote the vector bundle

$$\pi_1^* E_1 \otimes \pi_2^* E_2$$

with the projections $\pi_i : X \times X \to X$ to the ith factor.

We choose an orthonormal basis of constant spinors $(\psi_0^i)_{i=1,\dots,4}$, $(\psi_0^i, \psi_0^j) \equiv \delta_{ij}$ at the asymptotic end and denote the corresponding solutions of the boundary problem (2.1) by $(\psi_i)_{i=1,\dots,4}$. Then we define the spinor operator $\Pi \in \Gamma(\Sigma^* \boxtimes \Sigma)$ by

$$\Pi(x,y) : \Sigma_{N,x} \to \Sigma_{N,y}, \quad \psi \mapsto \sum_{i=1}^{4} (\psi_i(x), \psi)\psi_i(y).$$

At infinity, (ψ_i) goes over to an orthonormal basis, thus

$$\lim_{|x|\to\infty} \Pi(x,x) = \mathrm{id}$$

For $A \in \mathrm{End}\,(\Sigma_x)$ it holds

$$\sum_{i=1}^{4} (\psi_i(x), A\psi_i(x)) = \mathrm{tr}\,(A \circ \Pi(x,x))$$

and therefore we can replace (3.1) and (3.2) in case $h \neq 0$ by

$$4|\sum_i \nabla^2 \Psi_i|^2 \geqslant \sum_{i=1}^{4} |R^\Sigma \Psi_i| \geqslant \sum_{i=1}^{4} ||R^\Sigma||^2 - ||R^{\Sigma *} R^\Sigma||\,||1-\Pi|| \geqslant \frac{1}{2}(1-||1-\Pi||)|R_M|^2$$

where $||\cdot||$ denotes the Hilbert-Schmitt norm, and the last inequality is derived by using the commutating relations in the Clifford algebra.

Therefore instead of (3.7) one needs an upper bound for $||1-\Pi||$ which is deduced from the Sobolev inequality and the fact that

$$\frac{1}{4}\sum |\Psi_i(x)|^2 \leqslant |\Pi(x,x)| \leqslant \sum_{i=1}^{4} |\Psi_i(x)|^2$$

which yields $|1-\Pi(x,x)| < \varepsilon$ except on $U(\varepsilon)$ with measure

$$(\mu(\varepsilon))^{\frac{1}{3}} \leqslant \frac{c}{\varepsilon^2 k^2}(4\pi E + ||h||^2) \tag{3.8}$$

For a constant c. Finally using again partial integration and (2.4) we get

$$\int_M |\nabla^2 \Psi|^2 d\mu \leqslant c_1 \sup_M (|R| + |\rho|)E + \int_M (|\nabla R| + |h||R|)|\Psi||\nabla\Psi| d\mu$$

with $\rho = |h|^2 + |\nabla h|$. Instead of (3.4) now the inequality

$$\Delta|\Psi|^2 \geqslant -\rho|\Psi|^2 \tag{3.9}$$

holds, so $|\Psi|^2$ is not subharmonic in this case. We split the second integral $\int_M = \int_{\Omega_L} + \int_{M\setminus\Omega_L}$ with $\Omega_L = \{x \in M | |\Psi(x)| > L\}$. Using Sobolev inequality we estimate

$$\||\Psi|^2 - 1\|_{L^6(\Omega_L)} \leqslant \frac{72}{k^2}(L+1)\|\rho\|_{\frac{6}{5}}$$

for L as in the theorem. This estimates the first integral. Finally we use a barrier function constructed from (3.9) to deduce the theorem. For details see [**5**]. In case $n > 3, h \equiv 0$ the main difficulty is (i) which can be handled using the spinor operator as in case $n = 3, h = 0$. For details see [**4**].

4. The Spinor Operator

The basic problem of the theorem is, that the Sobolev constant enters the estimates for the exceptional set. So in cases where the isoperimetric constant of the manifold is small, the estimates are not useful . To get rid of this problem, we need estimates for $||1 - \Pi||$ which do not contain the Sobolev constant of the manifold. For this purpose we study the spinor operator, which plays an important role in the proof of the theorem, as we have seen in the last section in more detail. We restrict to the case $n \equiv 3$ and $h \equiv 0$ and a special class of asymptotically flat manifolds with nonnegative scalar curvature. Finally our methods will also go through in more general cases.

Definition 3. *A three-dimensional complete Riemannian manifold* (M, g) *of nonnegative scalar curvature is* **asymptotically Schwarzschild**, *if there is a compact set* $K \subset M$ *and a diffeomorphism*

$$\varphi : M \setminus K \to \mathbb{R}^3 \setminus B_R(0)$$

where $B_R(0) = \left\{x \in \mathbb{R}^3 \,\middle|\, |x|^2 \leqslant R\right\}$ *with*

$$\varphi_* g \;=\; \left(1 + \frac{2m}{|x|}\right)^4 g_0$$

where g_0 *denotes the standard scalar product on* $\mathbb{R}^3$.

Recall that the sphere with one point removed $S^n \setminus P$ is conformally equivalent to the $\mathbb{R}^n$ by the stereographic projection. The conformal factor is given by $\left(\frac{2}{1+|x|^2}\right)^2$. This motivates to consider the following compactification of an asymptotically Schwarzschild manifold M:

Choose N with

$$N^3 > 1 + 2m\frac{1+R^2}{R^3} \text{ and let } c := \sqrt{2}(1 + N^2R^2)^{\frac{1}{2}}.$$

Now we choose a function $\lambda : M \to \mathbb{R}^+$ with:

$$\begin{aligned}
&(i) && \lambda|K \;\equiv\; 1 \\
&(ii) && \lambda(x) \;=\; \left(\tfrac{2c}{1+r^2(x)}\right)\cdot\left(1 + \tfrac{2m}{r(x)}\right)^{-2} \quad \text{for } x \in \varphi^{-1}(\mathbb{R}^3 \setminus B_{NR}(0)) \\
& && r(x) \;:=\; |\varphi(x)| \\
&(iii) && \tilde{s} \;:=\; 8\lambda^{-\frac{5}{2}}\,\Delta_M\lambda^{\frac{1}{2}} \geqslant 0 \;\text{ for } x \in \varphi^{-1}(B_{NR}(0) \setminus B_R(0))
\end{aligned} \tag{4.1}$$

We will show in [**6**] that such a λ exists. Let $(M, \tilde{g})$ be the manifold M together with the metric $\tilde{g} = \lambda^2 g$. Note that (ii) implies that the manifold $(\varphi^{-1}(\mathbb{R}^3 \setminus B_{NR}(0)), \tilde{g})$ is isometric to the cap of the sphere of radius c.

$$\left\{ (x_1, x_2, x_3, x_4) \in S_c^3 \,\middle|\, \frac{N^2R^2 - 1}{N^2R^2 + 1} < \frac{x_4}{c} < 1 \right\} \tag{4.2}$$

and (iii) implies that the scalar curvature of $(\varphi^{-1}(\mathbb{R}^3 \setminus B_R(0)), \tilde{g})$ is nonnegative.

Definition 4. *Let (M, g) be an asymptotically Schwarzschild manifold, $(\tilde{M}, \tilde{g})$ be the conformal diffeomorphic manifold with $\tilde{g} = \lambda^2 g$, such that λ satisfies (4.1). We call the closure of $(M, \tilde{g})$ the* **conformal compactification** *$(\overline{M}, \tilde{g})$ of (M, g).For $R' \geqslant NR$ we denote*

$$C_{R'} := p^{-1}(\mathbb{R}^3 \backslash B_R) \subset \tilde{M} \cap S_c^3$$

the sperical cap of M. Here p denotes the stereographic projection from the north pole $P \in S_c^3$.

According to [**8, 7**] there is a fiberwise isometric map $\Sigma \to \tilde{\Sigma}$, $\psi \mapsto \tilde{\psi}$ such that

$$\tilde{D\!\!\!/}\tilde{\psi} \;=\; \lambda^{-2}\, \widetilde{D\!\!\!/\lambda\psi}.$$

Therefore we define the conformally transformed spinor operator by

Definition 5. *On a n-dimensional manifold $(M, \tilde{g})$ with $\tilde{g} = \lambda^2 g$ we define the* **conformally transformed** *spinor operator $\tilde{\Pi} \in \Gamma(\tilde{\Sigma}^* \boxtimes \tilde{\Sigma})$ as*

$$\tilde{\Pi}(x, y) \;:=\; \lambda^{-\frac{n-1}{2}}(x)\lambda^{-\frac{n-1}{2}}(y) \sum_{i=1}^{4} (\tilde{\psi}_i(x),\ .\ )\tilde{\psi}_i(y)$$

where $\psi_1, \ldots, \psi_{[\frac{n}{2}]}$ are solutions of (2.1).

In the following we will relate the conformally transformed spinor operator to the Green's function of the Dirac operator.

Definition 6. *Let X be a spin manifold. A smooth distribution S_X with values in the bundle $\Sigma^* \boxtimes \Sigma$*

$$S_X :\ X \times X \setminus \Delta \longrightarrow \Sigma_X^* \boxtimes \Sigma_X$$

is called the **Green's Function of the Dirac operator** *$D\!\!\!/$ on X iff*

$$D\!\!\!/_{X,x}\, S_X \;=\; \delta(x, y)$$

holds.

This is equivalent that

$$\int\limits_M (S_X(x, y), D\!\!\!/_x\, \psi(x)) \;=\; \psi(y)$$

for compactly supported smooth sections.

Now we proof the following important relation between the spinor operator and the Green's function of the Dirac operator:

THEOREM 3. *Let (M,g) be an asymptotically Schwarzschild manifold, $(\bar{M},\tilde{g})$the conformal compactification of M. Then for the conformal transformed spinor operator $\tilde{\Pi}$ and the Green's function $\tilde{S}_{\bar{M}}$ of the Dirac operator on $(\bar{M},\tilde{g})$ holds:*

$$\tilde{\Pi}(x,y) \;=\; 4c^2 \tilde{S}_{\bar{M}}(x,0)\, \tilde{S}_{\bar{M}}(0,y)\, \omega_2^2$$

where ω_2 is the volume of S^2

PROOF. Let $P \in C_R \subset (\bar{M},\tilde{g})$ be the north pole in the spherical cap of the compactification, and $B_\varepsilon \subset C_{NR}$ a ball of radius ε around the north pole. Then

$$\begin{aligned} \int\limits_{\bar{M}\setminus B_\varepsilon} \tilde{\Pi}(x,y)(\tilde{\not{D}}\tilde{\psi})(x)\;dx \;&=\; \sum_{i=1}^{4}\left(\int\limits_{\bar{M}\setminus B_\varepsilon} \mathrm{div} V_{\tilde{\psi},\psi_i}(x)\,dx\right)\bar{\psi}_i(y) \\ &=\; \sum_{i=1}^{4}\left(\int\limits_{\partial B_\varepsilon} \left(\tilde{\psi}, n_\varepsilon\cdot\bar{\psi}_i\right)(x)\,dx\right)\bar{\psi}_i(y) \end{aligned}$$

where $\bar{\psi}_i = \lambda^{-1}\cdot\tilde{\psi}_i$, the vectorfield V_{ψ,ψ_i} is defined by

$$\tilde{g}(V_{\psi,\psi_i}, w) \;=\; (\tilde{\psi}, w\cdot\tilde{\psi}_i)\,\lambda^{-1}(x)$$

and n_ε denotes the normal unit vector field on ∂B_ε. Similarily it follows

$$\int\limits_{\bar{M}\setminus B_\varepsilon}\int\limits_{\bar{M}\setminus B_{\varepsilon'}} \tilde{\Pi}(x,y)\left(\tilde{\not{D}}\tilde{\varphi}_1\right)(x)\left(\tilde{\not{D}}\tilde{\varphi}_2\right)(y)\,dx\,dy \tag{4.3}$$

$$= \sum_{i=1}^{4}\left(\int\limits_{\bar{M}\setminus B_\varepsilon} \left(\tilde{\varphi}_1, n_\varepsilon\cdot\bar{\psi}_i\right)(x)\,dx\right)\cdot\left(\int\limits_{\bar{M}\setminus B_{\varepsilon'}} \left(n_{\varepsilon'}\cdot\bar{\psi}_i, \tilde{\varphi}_2\right)(y)\,dy\right). \tag{4.4}$$

To investigate $\lim\limits_{\varepsilon\to 0}\int\limits_{\bar{M}\setminus B_\varepsilon} \left(n_\varepsilon\cdot\bar{\psi}_i,\tilde{\varphi}\right)(x)\,dx$ we consider the trivialization of the spinor bundle in a neighbourhood of the north pole given by stereographic projection ρ from the south pole $-P$. The change of the charts of $S^n\setminus(P\cup -P)$ given by the stereographic projections from the north and south pole is given by $\omega(x) = \frac{x}{|x|^2}$ with differential $d\omega_x(v) = \frac{1}{|x|^2}S(\frac{x}{|x|})v$ where $S(\frac{x}{|x|})$ denotes the reflection at $(\mathbb{R}\cdot x)^\perp$. The lift of $S(\frac{x}{|x|}) \in \mathrm{O}(3)$ to $\mathrm{Spin}(3)$ is given by clifford multiplication with $\frac{x}{|x|} = \frac{2c}{1+|x|^2}d\rho^{-1}(n_\varepsilon(x))$ for $x\in\rho^{-1}(\partial B_\varepsilon)$. and therefore $(n_\varepsilon\cdot\tilde{\psi}_i)_{i=1,\dots,4}$ is extendable to the north pole as orthonormal basis $\tilde{\psi}_{0,1}(0),\dots,\tilde{\psi}_{0,4}(0)$.
From the asymptotic behaviour of the solutions of (2.1) it follows that

$$\bar{\psi}_i(x) \;=\; \frac{1}{2c}r'(x)^{-2}\,\tilde{\psi}_{0,i}(0) + \mathcal{O}(|r(x)|^{-1}) \qquad \text{with } (\tilde{\psi}_{0,i},\tilde{\psi}_{0,j}) = \delta_{ij}$$

where $(r',\Omega): S^3\to\mathbb{R}^3$ denotes the coordinates of the stereographic projection from the south pole.
Therefore

$$\begin{aligned} \lim_{\varepsilon\to 0}\int\limits_{\partial B_\varepsilon} \left(\tilde{\varphi}(x), n_\varepsilon\cdot\bar{\psi}_i(x)\right)dx \;&=\; \lim_{\varepsilon\to 0}\frac{2c}{\varepsilon^2}\mathrm{vol}(S^2_\varepsilon)(\tilde{\varphi}(0),\tilde{\psi}_{0,i}(0)) + \mathcal{O}(\varepsilon) \\ &=\; 2c\omega_2(\tilde{\varphi}(0),\tilde{\psi}_{0,i}(0)). \end{aligned}$$

Now taking the limits in (4.3) we get

$$\int_{\bar{M}\times\bar{M}} \left(\tilde{\Pi}(x,y)\left(\tilde{D\!\!\!/}\tilde{\varphi}_1\right)(x), \left(\tilde{D\!\!\!/}\tilde{\varphi}_2\right)(y)\right) \, dx\, dy \; = $$

$$= 4c^2 \int_{\bar{M}\times\bar{M}} \left(S_{\bar{M}}(0,x)\,\tilde{D\!\!\!/}\varphi_1(x), S_{\bar{M}}(0,y)\,\tilde{D\!\!\!/}\varphi_2(y)\right) \, dx\, dy \cdot \omega_2^2.$$

□

Our aim is to use this identity to calculate a weighted L^1-norm for $1-\Pi$. Therefore we calculate

$$\int_M \left(1 - \left|\Pi(x,x)\right|\right) \, \lambda(x)\, dx =$$

$$= \int_{\tilde{M}} \left(\lambda^{1-n}(\tilde{x}) - \left|\tilde{\Pi}(\tilde{x},\tilde{x})\right|\right) \, d\tilde{x}$$

$$= \int_{\tilde{M}} \left(\lambda^{1-n}(\tilde{x}) - (2c)^{n-1}\omega_{n-1}^2 \left|\tilde{S}(x,0)\tilde{S}(0,x)\right|\right) \, d\tilde{x}$$

$$=: c(\lambda) + g_0 \int_{\tilde{M}} \left(\lambda^{1-n}(\tilde{x}) - \left(\frac{2c}{1+r(\tilde{x})^2}\right)^{1-n} \chi_\delta(\tilde{x})\right) d\tilde{x}$$

$$-(2c)^{n-1}\omega_{n-1}^2 \int_{\tilde{M}} \left(\left|\tilde{S}(0,\tilde{x})\tilde{S}(\tilde{x},0)\right| - \frac{1}{\omega_{n-1}^2}\left(\frac{4c^2}{1+r(\tilde{x})^2}\right)^{1-n}\chi_\delta(\tilde{x})\right) \, d\tilde{x}$$

The first term is a constant depending only on λ. To evaluate the second term, we remark that for the Green's function G_M of $D\!\!\!/_M^2$ it holds

$$G_M(y,y') = \int_M S_M(y,x)S_M(x,y')dx$$

We will show in **[6]** that

(4.5)

$$\int_M \left(\left|\tilde{S}(0,\tilde{x})\tilde{S}(\tilde{x},0)\right| - \frac{1}{\omega_{n-1}^2}\left(\frac{4c^2}{1+r(\tilde{x})^2}\right)^{1-n}\chi_\delta(\tilde{x})\right) \, d\tilde{x} = \lim_{y\to 0}\left(\tilde{G}(0,y) - G_\delta(y)\right)$$

with

$$G_\delta(y) = \int_{B_\delta} \left(S_{S_c^3}(0,x)S_{S_c^3}(x,y)\right) dy$$

The advantage of the expression (4.5) is, that it takes place only on the spherical cap C_{NR}. More precisely, we have to evaluate

$$\gamma(y) := \eta G_{S^3}(0,y) - \eta\tilde{G}(0,y) \text{ at } 0,$$

where η is a smooth function with support in C_{NR} and $\eta \equiv 1$ in a neighbourhood of the north pole. Therefore it will be possible to evaluate $\int_M (1-|\Pi(x,x)|)\lambda(x)dx$ without using the isoperimetric constant k of the manifold M. This will be treated in **[6]**, and is a first step to get a curvature bound where the estimate of the exceptional set does not contain the isoperimetric constant.

References

[1] R. Arnowitt, S. Deser, C. Misner, "Energy and the criteria for radiation in general relativity," *Phys. Rev.* 118, 1100 (1960))

[2] R. Bartnik, "The Mass of an Asymptotically Flat Manifold", Commun. Pure Appl. Math. XXXIX, 661-693 (1986)

[3] H. Bray, F. Finster, "Curvature estimates and the positive mass theorem," *Comm. Anal. Geom.* 10 (2002) 291-306

[4] F. Finster, I. Kath, "Curvature estimates in asymptotically flat manifolds of positive scalar curvature," *Comm. Anal. Geom.* 10 (2002) 1017-1031

[5] F. Finster, M. Kraus, "Curvature estimates in asymptotically flat Lorentzian manifolds," ***to appear in Can. Journ. Math.***

[6] F. Finster, M. Kraus, "Weighted L^2-estimates of the Witten spinor in Asymptotically Schwarzschild Manifolds," ***Preprint 2/2005 Universität Regensburg***

[7] O. Hijazi "A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing Spinor", ***Comm. Math. Phys.*** 104, 151-162, 1986

[8] , N. Hitchin," Harmonic spinors". ***Adv. Math.***, 14, 1-55, 1974

[9] H.-B. Lawson, M.-L. Michelsohn, *Spin Geometry*, Princeton University Press, Princeton 1989.

[10] T. Parker, C. H. Taubes, "On Witten's proof of the positive energy theorem," *Commun. Math. Phys.* 84, 223-238 (1982)

[11] R. Schoen, S.-T. Yau, "On the proof of the positive mass conjecture in general relativity," *Commun. Math. Phys.* 65, 45-76 (1976)

[12] R. Schoen and S.-T. Yau, "Conformally flat manifolds, Kleinian groups and scalar curvature", ***Invent. Math.,*** 92 (1), 47-71, 1988

[13] E. Witten, "A new proof of the positive energy theorem," *Commun. Math. Phys.* 80, 381-402 (1981)

Fachbereich Physik, Mathematik und Informatik
Johannes-Gutenberg-Universität Mainz
Staudingerweg 9,
D-55099 Mainz Germany

e-mail: `MKraus@mathematik.uni-mainz.de`