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A description of $\mathfrak{p}$-homomorphisms between harmonic modules in projective contact geometry

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Abstract. In this paper we present the first information needed in the study of first order invariant differential operators acting between sections of harmonic vector bundles associated to projective contact geometries. We prove a necessary and sufficient condition for certain $\mathfrak{g}_0$-homomorphism acting between a first jet prolongation of a harmonic module and another harmonic module to be a $\mathfrak{p}$-homomorphism.

1. Introduction

The theory of invariant differential operators acting between sections of associated vector bundles over parabolic geometries has been studied by many authors, see, e.g., Čap, Slovák, Souček [**6**], [**7**], [**8**], Slovák, Souček [**18**], Calderbank, Diemer [**4**] and Calderbank, Diemer, Souček [**5**]. In all of these cited articles the fibres under consideration are finite dimensional. In this text, we present a generalization of this theory into the infinite dimensional case. This case is important in some applications, like in the theory of Dirac operators on symplectic manifolds.

1.1. First order invariant differential operators. Let $\mathfrak{g}$ be a $|k|$-graded simple Lie algebra and let us denote by $\mathfrak{p}$ the parabolic, by $\mathfrak{g}_-$ the negative and by $\mathfrak{p}_+$ the positive part of $\mathfrak{g}$. Consider a Lie group G the Lie algebra of which equals $\mathfrak{g}$ and let P be the the parabolic subgroup of G associated to the $|k|$-grading of $\mathfrak{g}$. Further let us consider two manifolds $\mathcal{G}$ and M, together with a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) on the manifold M, ω being a Cartan connection, i.e., a P-equivariant absolute parallelism reproducing fundamental vector fields on $\mathcal{G}$.

Let $(\rho, \mathbb{V})$ be a representation of the parabolic group P. For $s : \mathcal{G} \to \mathbb{V}$ being a P-equivariant map, let us consider the value of the absolutely invariant derivative ∇^ω on it

$$\nabla^\omega s : \mathcal{G} \to \mathfrak{g}_-^* \otimes \mathbb{V}.$$

This absolute invariant derivative is associated to the parabolic geometry $(p : \mathcal{G} \to M, \omega)$ and the representation $(\rho, \mathbb{V})$. For the definition, see Slovák, Souček [**18**]. Let us suppose a representation $(\sigma, \mathbb{W})$ of the group P is given and consider two associated vector bundles $VM := \mathcal{G} \times_\rho \mathbb{V}$ and $WM := \mathcal{G} \times_\sigma \mathbb{W}$. The vector space

$$J^1\mathbb{V} := \mathbb{V} \oplus (\mathfrak{g}_-^* \otimes \mathbb{V})$$

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is called the first jet prolongation of $\mathbb{V}$. This vector space comes up with a natural action of P, making it into a P-module, see again Slovák, Souček [**18**]. In the realm of parabolic geometries, one defines a first order invariant differential operator $D : \Gamma(M, VM) \to \Gamma(M, WM)$ to be a homomorphism of vector spaces for which there is a (nontrivial) P-module homomorphism $\Phi : J^1\mathbb{V} \to \mathbb{W}$ such that

$$Ds(u) = \Phi(s(u), \nabla^\omega s(u))$$

for $u \in \mathcal{G}$ and $s \in \Gamma(M, VM)$ considered as a P-equivariant $\mathbb{V}$-valued function on $\mathcal{G}$, i.e., $s \in \mathcal{C}^\infty(\mathcal{G}, \mathbb{V})^P \simeq \Gamma(M, VM)$ (a vector space isomorphism).

1.2. Projective contact geometry. There is a common property appearing in some well known parabolic geometries of certain type, say (G, P), which leads to a definition of a special grading of the Lie algebra $\mathfrak{g}$ of the group G.

Definition 1. Let $\mathfrak{g}$ be a real or complex $|k|$-graded semisimple Lie algebra. We call this grading contact if it is a depth two grading, i.e.,

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

satisfying the following two properties

(1) $[,] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is nondegenerate and
(2) $\dim \mathfrak{g}_{-2} = 1$.

A classification of contact gradings of complex and real simple Lie algebras can be found in Yamaguchi [**19**]. Let us remark, that up to an isomorphism each complex simple Lie algebra possesses a unique contact grading. This fact is no more true for real simple Lie algebras, see again Yamaguchi [**19**], where the list of all real forms of complex simple Lie algebras, which possesses no contact grading, together with an information on contact gradings of simple real Lie algebras, which are not real forms of complex simple Lie algebras are presented.

Let us consider a real simple Lie algebra $\mathfrak{g} = \mathfrak{sp}(2k, \mathbb{R})$ equipped by the contact grading given in the matrix form as displayed bellow

$$A = \left(\begin{array}{c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right),$$

where

(1) $\mathfrak{g}_0 = \mathfrak{sp}(2k-2, \mathbb{R}) \oplus \mathbb{R}E$, where E is the grading element uniquely associated to the grading,
(2) $\mathfrak{g}_{-1} = \mathbb{R}^{2k-2}$,
(3) $\mathfrak{g}_{-2} = \mathbb{R}$.

Let $G = Sp(2k, \mathbb{R})$ be a symplectic group. Consider the tautological action of G on the (unique) linear symplectic form $\mathbb{R}^{2k}$. This action factors to an action on the open rays of $\mathbb{R}^{2k}$. Let us denote by P the isotropy subgroup of an open ray in $\mathbb{R}^{2k}$. Obviously, P is a parabolic subgroup of the group G. It can be proved (see Krysl [**15**]) that P is the parabolic subgroup of G associated to the contact grading of $\mathfrak{g}$ given above.

Definition 2. Let $\mathcal{G}$ and M^{2k-1} be smooth manifolds. We call a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) on the manifold M projective contact geometry if $G = Sp(2k, \mathbb{R})$ is a symplectic group and P its parabolic subgroup defined above.

For a comprehensive treatment on contact geometries, see Blair [**1**]; for some connections of projective contact geometries to Cartan geometries see Fox [**10**] and Krysl [**15**].

1.3. Spinor representations of $\mathfrak{sp}(2k, \mathbb{C})$. Symplectic spinor representations were introduced by Bertram Kostant when he was seeking for an analogy of the Dirac operator on Riemannian or Lorentzian manifolds for the case of symplectic manifolds, i.e., an arena for Hamiltonian mechanics, see Kostant [**14**]. This Dirac operator was then studied by many authors, see, e.g., Habermann [**11**], Klein [**13**].

Consider the symplectic algebra $\mathfrak{g} = \mathfrak{sp}(2k, \mathbb{C})$ together with the set of its fundamental weights denoted by $\{\varpi_i\}_{i=1}^{k}$. Let $L(\nu)$ be the irreducible highest weight module with the highest weight ν. The irreducible highest weight modules $\mathbb{S}_+ := L(-\frac{1}{2}\varpi_k)$ and $\mathbb{S}_- := L(\varpi_{k-1} - \frac{3}{2}\varpi_k)$ are called spinor modules.

Let $\mathfrak{g}$ be an arbitrary semisimple complex Lie algebra and $\mathfrak{h}$ its Cartan subalgebra. An $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module $\mathbb{V}$ is called module with bounded multiplicities, if there is some $k \in \mathbb{N}_0$ such that the dimension of each weight space $\mathbb{V}(\nu)$ is smaller or equal to k. The minimal k with this property is called degree of $\mathbb{V}$. A module with bounded multiplicities is called completely pointed provided its degree is 1.

It can be proved (see Britten, Lemire [**3**]) that an infinite dimensional $\mathfrak{sp}(2k, \mathbb{C})$-module $\mathbb{V}$ is completely pointed iff either $\mathbb{V} = \mathbb{S}_+$ or $\mathbb{V} = \mathbb{S}_-$. Moreover, Britten, Hooper and Lemire [**2**] proved that the tensor product $L(\nu) \otimes \mathbb{S}_\pm$ for ν an integral dominant weight for $\mathfrak{g}$ is completely reducible and decomposes into a finite direct sum of irreducible highest weight $\mathfrak{g}$-modules.

Definition 3. We call a $\mathfrak{sp}(2k, \mathbb{C})$-module $\mathbb{V}$ harmonic module if there is an integral dominant weight ν s.t. $\mathbb{V}$ is a direct irreducible summand in $L(\nu) \otimes \mathbb{S}_\pm$. Further let us define a set

$$\mathbb{A} := \{\sum_{i=1}^{n} \lambda_i \varpi_i; \lambda_i \geqslant 0, \lambda_i \in \mathbb{Z},\, i = 1, \dots, n-1;\; \lambda_n \in \mathbb{Z} + \frac{1}{2}, \lambda_{n-1} + 2\lambda_n + 3 > 0\}.$$

The following fact is well known.

Theorem 1. The following are equivalent

(1) $\mathbb{V}$ a is harmonic module,
(2) the highest weight of $\mathbb{V}$ is in $\mathbb{A}$,
(3) $\mathbb{V}$ is a $\mathfrak{sp}(2k, \mathbb{C})$-module with bounded multiplicities.

Proof. See Britten, Lemire [**3**].□

In Krysl [**15**], it was proved that the tensor product of the defining representation $L(\varpi_1)$ and a harmonic module decomposes as folows.

Theorem 2. Let $\lambda \in \mathbb{A}$. Then

$$L(\varpi_1) \otimes L(\lambda) = \bigoplus_{\kappa \in \mathbb{A}_\lambda} L(\kappa),$$

where $A_\lambda \subseteq \{\kappa = \lambda + \mu, \mu \in \Pi(\varpi_1)\}$, $\Pi(\varpi_1)$ being the set of all weights of $L(\varpi_1)$.

Proof. See Krysl [**15**], [**16**]. □

2. Invariant differential operators for fields with values in some standard cyclic modules

The aim of this section is to rewrite the theory of first order invariant differential operators in parabolic geometries for the case of certain infinite dimensional standard cyclic modules. At first we will consider arbitrary modules and then we restrict our attention to the case of irreducible highest weight modules.

Let $\mathfrak{g}$ be a complex $|k|$-graded semisimple Lie algebra, $\mathfrak{p}$, $\mathfrak{p}_+$ and $\mathfrak{g}_-$ as in the first section of this article. Let $\mathbb{V}$, $\mathbb{W}$ be $\mathfrak{p}$-modules. Denote the action of $\mathfrak{p}$ on $\mathbb{V}$ by λ, $\lambda : \mathfrak{p} \to \mathrm{End}(\mathbb{V})$. Further, let $J^1\mathbb{V}$ be the first jet prolongation of the $\mathfrak{p}$-module $\mathbb{V}$ associated to the $|k|$-graded Lie algebra $\mathfrak{g}$. Let us fix some dual bases $\{\xi_\alpha\}$, $\{\eta^\alpha\}$ of $\mathfrak{g}_-$ and $\mathfrak{p}_+$. The induced action of $\mathfrak{p}$ on $J^1\mathbb{V}$ is given by

$$(1)\quad Z.(s', Y \otimes s) = (\lambda(Z)s', Y \otimes \lambda(Z)s + [Z,Y] \otimes s + \sum_\alpha \eta^\alpha \otimes [Z, \xi_\alpha]_{\mathfrak{p}} s')$$

for $Z \in \mathfrak{p}$, $Y \in \mathfrak{g}_-^*$, and $s, s' \in \mathbb{V}$, see Slovák [**17**], where this formula is derived.

Let $\mathfrak{p}_+^2$ denote the space $\mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k$. We call the space $J^1_R\mathbb{V} = \mathbb{V} \oplus (\mathfrak{g}_-^* \otimes \mathbb{V})/(\{0\} \oplus (\mathfrak{p}_+^2 \otimes \mathbb{V})) \simeq \mathbb{V} \oplus (\mathfrak{g}_1 \otimes \mathbb{V})$ space of restricted jets. This space carries a structure of a $\mathfrak{p}$-module inherited by factorization. Let us denote by $\{\eta^{\alpha'}\}, \{\xi_{\alpha'}\}$ some mutually dual bases of $\mathfrak{g}_{\pm 1}$. Finally, let $\Phi : \mathfrak{g}_1 \otimes \mathbb{V} \to \mathfrak{g}_1 \otimes \mathbb{V}$ denote the following endomorphism

$$\Phi(Z \otimes s) := \sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}].s$$

for $Z \in \mathfrak{g}_1, s \in \mathbb{V}$.

Theorem 3. Let $\mathbb{V}, \mathbb{W}$ be irreducible $\mathfrak{p}$-modules with the action of $\mathfrak{p}_+$ being trivial. Let $\Psi : J^1\mathbb{V} \to \mathbb{W}$ be a $\mathfrak{g}_0$-module homomorphism. Then Ψ is a $\mathfrak{p}$-module homomorphism if and only if Ψ factors through the restricted jets $J^1_R\mathbb{V}$ and $\Psi_{|Im(\Phi)} = 0$.

Proof. Let Ψ be a $\mathfrak{g}_0$-homomorphism. If we suppose that Ψ is a $\mathfrak{p}_+$-homomorphism, it follows immediately that Ψ vanishes on the image of the action of $\mathfrak{p}_+$.

Now, we prove that Ψ factors through the restricted jets. Inserting $s' = 0$ into the equation 1, we obtain $Z.(0, Y \otimes s) = [Z, Y] \otimes s$, for $Y \in \mathfrak{g}_-^*$, $Z \in \mathfrak{p}_+$ and $s' \in \mathbb{V}$. Let $Z_i \in \mathfrak{g}_i$, $s_i \in \mathbb{V}$, $i = 1, \ldots, k$, Because $\mathfrak{g}_1$ generates $\mathfrak{p}_+$ there are $X_i \in \mathfrak{g}_1$, $Y_i \in \mathfrak{g}_{i-1}$ for $i = 2, \ldots, k$ such that $Z_i = [X_i, Y_i]$. Thus we can write $\Psi(\sum_{i=1}^k Z_i \otimes s_i) = \Psi(Z_1 \otimes s_1 + \sum_{i=2}^k [X_i, Y_i] \otimes s_i) = \Psi(Z_1 \otimes s_1) + \Psi(X_2.(0, Y_2 \otimes s_2)) + \ldots + \Psi(X_k.(0, Y_k \otimes s_k)) = \Psi(Z_1 \otimes s_1)$. The terms $\Psi(X_k.(0, Y_k \otimes s_k)) = 0$ for $k = 2, \ldots, k$ because Ψ vanishes on the image of the $\mathfrak{p}_+$ action on $J^1\mathbb{V}$. Thus we have proved that Ψ factors through the restricted jets.

Looking at the induced action of $\mathfrak{p}_+$ on $J^1\mathbb{V}$ we derive the condition

$$\Psi(\sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}].s') = 0,$$

which means that $\Psi_{|Im(\Phi)} = 0$ due to the definition of the mapping Φ.

The opposite direction of the implication in the statement of this theorem is obvious. □

In what comes, we would like to compute the mapping Φ with help of the universal Casimir element of $\mathfrak{g}$. First, let us recall a well known theorem on the action of the universal Casimir element on a highest weight module over a simple complex Lie algebra. It is well known (see Humphreys [**12**], pp. 143) that if $\mathfrak{g}$ is a complex simple Lie algebra and $\mathfrak{h}$ its Cartan subalgebra and $\lambda \in \mathfrak{h}^*$, then the action of the universal Casimir element c on a standard cyclic module of the highest weight λ is by a scalar

$$c_\lambda = (\lambda + 2\delta, \lambda), \tag{2}$$

where δ is the sum of all fundamental weights of $\mathfrak{h}$ on $\mathfrak{g}$.

Second, let us make some assumptions on the Lie algebra $\mathfrak{g}$. Suppose that the subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ has a one dimensional center. This center is necessarily generated by the grading element E of the $|k|$-graded Lie algebra $\mathfrak{g}$. Thus we can decompose $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathbb{C}E$ where $\mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$ denotes the semisimple part of $\mathfrak{g}$. It is a well known fact that the Killing form B of the Lie algebra $\mathfrak{g}$, when restricted to $\mathfrak{g}_0$ is nondegenerate, too. Let us normalize the Killing form B by the condition $B(E,E) = 1$ and denote this resulting nondegenerate invariant form on $\mathfrak{g}_0$ by $(,) : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbb{C}$. It is easy to compute that the decomposition $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathbb{C}E$ is an orthogonal decomposition. Indeed, take an arbitrary $X \in \mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$ in the form $X = [U, V]$ for a $U, V \in \mathfrak{g}_0$ and compute $(E, X) = (E, [U, V]) = ([E, U], V) = 0$ because E is the grading element and $U \in \mathfrak{g}_0$.

Let us denote by $\{Y_a\}_{a=1}^k$, $\{Y'_a\}_{a=1}^k$ some mutually dual bases of $\mathfrak{g}_0^{ss}$ with respect to $(,)$. Sometimes we will denote the element E by Y_{k+1}. Now, we derive the following generalization of a result in Slovák, Souček [**18**].

Lemma 1. Let $\mathbb{V}$ be a representation of a semisimple $|k|$-graded Lie algebra $\mathfrak{g}$, then

$$\Phi(Z \otimes s) = \sum_{a=1}^{k} Y'_a.Z \otimes Y_a.s$$

for each $Z \in \mathfrak{g}_1$ and $s \in \mathbb{V}$.

Proof. We use the invariance of the Killing form $[Z, \xi_{\alpha'}] = \sum_a (Y'_a, [Z, \xi_{\alpha'}]) Y_a = \sum_a ([Y'_a, Z], \xi_{\alpha'}) Y_a$ in order to compute the value $\Phi(Z \otimes s)$.

$$\begin{aligned}
\Phi(Z \otimes s) &= \sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}].s \\
&= \sum_{\alpha'} \eta^{\alpha'} \otimes \sum_a (Y'_a, [Z, \xi_{\alpha'}]) Y_a.s \\
&= \sum_{\alpha'} \eta^{\alpha'} \otimes \sum_a ([Y'_a, Z], \xi_{\alpha'}) Y_a.s \\
&= \sum_{\alpha'} \sum_a ([Y'_a, Z], \xi_{\alpha'}) \eta^{\alpha'} \otimes Y_a.s \\
&= \sum_a Y'_a.Z \otimes Y_a.s.
\end{aligned}$$

□

Now, we make some assumptions on the representations we shall be dealing with. We will consider that $\mathbb{V}_\lambda$ is an irreducible $\mathfrak{p}$-module which is irreducible with the highest weight λ when considered as a $\mathfrak{g}_0^{ss}$-module. Further, we assume that $\mathfrak{g}_1 \otimes \mathbb{V}_\lambda$ decomposes into a finite direct sum of irreducible $\mathfrak{g}_0^{ss}$-modules without multiplicities and denote by π_μ the projection $\pi_\mu : \mathfrak{g}_1 \otimes \mathbb{V}_\lambda \to \mathbb{V}_\mu$ where $\mathbb{V}_\mu$ is the representation with highest weight μ which occurs in the decomposition of the completely reducible tensor product $\mathfrak{g}_1 \otimes \mathbb{V}_\lambda$. Let us suppose that the representation of the center $\mathbb{C}E$ of $\mathfrak{g}_0$ is given by $E.v := wv$ for each $v \in \mathbb{V}_\lambda$ and a $w \in \mathbb{C}$. So we are given a representation of the whole $\mathfrak{g}_0$ which is characterized by the tuple (λ, w). The complex number w is often called generalized conformal weight. Finally, we assume that $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0^{ss}$-module with the highest weight α. In order to compute the mapping Φ let us evaluate the following expression $\sum_{a=1}^{k+1}(Y_a'Y_a)(Z \otimes s)$ for $s \in \mathbb{V}_\lambda$ and $Z \in \mathfrak{g}_1$.

$$
\begin{aligned}
\sum_{a=1}^{k+1}(Y_a'Y_a).(Z \otimes s) &= \sum_{a=1}^{k+1}(Y_a'Y_a).Z \otimes s + Z \otimes \sum_{a=1}^{k+1}(Y_a'Y_a).s + \\
&\quad 2\Phi(Z \otimes s),
\end{aligned} \tag{3}
$$

where we have used the lemma 1 above. Now, we would like to compute the first two terms of the last written equation using the universal Casimir element of $\mathfrak{g}_0^{ss}$, see the equation 2.

$$
\sum_{a=1}^{k+1}(Y_a'Y_a).Z \otimes s = (\alpha, \alpha + 2\delta)Z \otimes s + Z \otimes s \tag{4}
$$

$$
Z \otimes \sum_{a=1}^{k+1}(Y_a'Y_a).s = (\lambda, \lambda + 2\delta)Z \otimes s + w^2 Z \otimes s \tag{5}
$$

After a straightforwars computation we derive that the L.H.S. of 3 equals

$$
\begin{aligned}
\sum_{a=1}^{k+1}(Y_a'Y_a).(Z \otimes s) &= \sum_{\mu}(\mu, \mu + 2\delta)\pi_\mu(Z \otimes s) \\
&\quad + \sum_{\mu} \pi_\mu[Z \otimes s + 2wZ \otimes s + w^2 Z \otimes s]
\end{aligned} \tag{6}
$$

Substituting the equations 4, 5 and 6 into the equation 3 we obtain

$$
\Phi(Z \otimes s) = \sum_{\mu}(w - c_{\lambda\alpha}^{\mu})\pi_\mu(Z \otimes s),
$$

where

$$
c_{\lambda\alpha}^{\mu} = \frac{1}{2}[(\lambda, \lambda + 2\delta) + (\alpha, \alpha + 2\delta) - (\mu, \mu + 2\delta)].
$$

We state our result as a theorem formulating explicitly the assumptions we have made.

Theorem 4. Let $\mathfrak{g}$ be a $|k|$-graded simple Lie algebra such that the subalgebra $\mathfrak{g}_0$ has a one dimensional center $\mathbb{C}E$. Let $\mathbb{V}_\lambda$ be an irreducible $\mathfrak{p}$-module with the highest weight λ if considered as a $\mathfrak{g}_0^{ss}$-module. Let the grading element E acts by the complex number w (conformal weight). Further, let $\mathfrak{g}_1$ be an irreducible $\mathfrak{g}_0^{ss}$-module with the highest weight α and consider the action of $\mathfrak{p}_+$ being trivial. Assume that the tensor product $\mathfrak{g}_1 \otimes \mathbb{V}_\lambda$ decomposes into a finite direct sum of irreducible $\mathfrak{g}_0^{ss}$-modules and has no multiplicities then

$$\Phi(Z \otimes s) = \sum_\mu (w - c^\mu_{\lambda\alpha})\pi_\mu(Z \otimes s),$$

where

$$c^\mu_{\lambda\alpha} = \frac{1}{2}[(\lambda, \lambda + 2\delta) + (\alpha, \alpha + 2\delta) - (\mu, \mu + 2\delta)].$$

Proof. See the analyzes above this theorem. □

Due to the theorem 3 we can state a corollary of the above written theorem.

Corollary 1. In the setting of the preceding theorem, let $\widetilde{\pi}_\mu$ be the trivial extension of π_μ to $J^1\mathbb{V}_\lambda = \mathbb{V}_\lambda \oplus (\mathfrak{g}_-^* \otimes \mathbb{V}_\lambda)$. Then $\widetilde{\pi}_\mu$ is a $\mathfrak{p}$-homomorphism if and only if

$$c^\mu_{\lambda\alpha} = w.$$

Proof. Due to the theorem 3 it is sufficient to show that $\widetilde{\pi}_\mu$ factors through the restricted jets and vanishes on the image of Φ. The first is from the definition and the second is a consequence of the theorem 4. The opposite implication is easy, too: look at the formula for Φ and use the fact that π_μ is onto $\mathbb{V}_\mu$. □

A special case of the corollary is the following statement.

Corollary 2. Let $\mathfrak{g} = \mathfrak{sp}(2k, \mathbb{C})$ be equipped by the contact grading, and let $\mathbb{V}_\lambda$ be a harmonic module over $\mathfrak{g}_0^{ss} = \mathfrak{sp}(2k-2, \mathbb{C})$ with the highest weight λ. Then $\widetilde{\pi}_\mu$ is a $\mathfrak{p}$-module homomorphism iff $c^\mu_{\lambda\varpi_1} = w$.

Proof. We have already shown (see theorem 2) that $\mathfrak{g}_1 \otimes \mathbb{V} = L(\varpi_1) \otimes \mathbb{V}$ decomposes into a finite multiplicity free direct sum of irreducible submodules. Thus we can apply the Corollary 1. □

In the future, we would like to use the last written theorem in the case of first order invariant differential operators in the setting of projective contact geometries and harmonic modules. This can be done if one knows a relationship between infinite dimensional representations of the Lie group G_0^{ss} (the semisimple part of the reductive group G_0 the Lie algebra of which equals $\mathfrak{g}_0$) and those of the Lie algebra $\mathfrak{g}_0^{ss}$.

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