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# Leafwise holonomy in reduction of non-holonomic systems

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Abstract. In this paper we recall the concept of generalised connections and show how these connections are encountered in the theory of non-holonomic systems with symmetry. Using results from a recent paper in which the definition of holonomy for generalised connections was introduced, we show amongst others that these holonomy elements are encountered in the reconstruction process for reduced non-holonomic systems with symmetry. This is illustrated with the well-known snakeboard example.

## 1. Introduction

The study of non-holonomic mechanical systems has experienced a growing interest during the last decades . There are many approaches to non-holonomic systems, and it is not our intention to give an overview of these different geometric formulations. Instead, for a detailed overview of the literature, we refer to two recent books [**1, 5**]. The purpose of this paper is to show that the theory of generalised connections, introduced in [**3**], is useful in the study of non-holonomic systems, especially for those non-holonomic systems which exhibit some symmetry. The main result in this direction states that the well-known property from the reconstruction theory in mechanical systems with symmetry, saying that geometric phases are holonomy elements of an appropriately chosen connection [**13**], has its analogue in the non-holonomic case. In particular, the non-holonomic geometric phases will appear as holonomy elements of a specific generalised connections which is involved in the reduction process.

The structure of this paper is as follows. In Section 2 we introduce the notion of a generalised connection and of an associated covariant derivative cf. [**3**]. Next we recall the concept of holonomy for generalised connections, and we state some results proven in [**10**], which are generalisations of theorems in standard connection theory [**8**]. Next, in Section 3 we apply the theory of generalised connections to simple non-holonomic mechanical control systems. By 'simple' we mean systems for which the Lagrangian equals kinetic minus potential energy and where the generalised forces are determined by some control inputs. The main point in this section is that a unique generalised connection is associated with a non-holonomic dynamical system and that the 'geodesic spray' of this generalised connection is used to give a geometric description of the equations of motion. We then study the reduction of non-holonomic system with symmetry, again using a specific generalised connection.

This paper is in final form and no version of it will be published elswhere.

It should be mentioned that these results were partly discussed in a previous paper [**9**]. We conclude the paper by showing that elements in the holonomy group are encountered as specific motions (also called gaits) of the mechanical systems under consideration and by showing the applicability of these results by applying it to the snakeboard example.

Throughout this paper we assume that the reader is familiar with all natural concepts associated to a connection on a principal fibre bundle. All manifolds are smooth, finite dimensional and without boundary. By smooth we will always mean of class $C^\infty$.

## 2. Generalised connections and holonomy groups

The concept of a generalised connection finds its origin in the work of R.L. Fernandes [**6, 7**] involving the study of the so-called contravariant connection on a Poisson manifold $P$. Roughly speaking, the key point in the definition of a contravariant connection, is that one no longer defines 'parallel transport' along an arbitrary curve in $P$, but rather along curves belonging to a specific subclass, determined by an admissibility condition. In the case of contravariant connections, this admissibility condition implies that the curve entirely lies in a symplectic leaf of the foliation induced by the Poisson structure.

An appropriate mathematical formulation for this admissibility condition follows from the definition of an *anchored bundle* (cf. [**3, 14**]) namely: an anchored bundle over a manifold $M$ is defined by a linear bundle $\nu : N \to M$ and a linear bundle map $\rho : N \to TM$, fibred over the identity. An *admissible curve* in this setting is a curve $c : I = [t_0, t_1] \to N$ such that $d/dt(\nu(c(t))) = \rho(c(t))$.

There are many systems in differential geometry that have the underlying structure of an anchored bundle. We only mention the most important examples:

- A Poisson structure $\Lambda$ on a manifold $P$ defines a mapping $\sharp_\Lambda : T^*P \to TP$ such that the image is precisely the generalised integrable distribution whose leafs are symplectic submanifolds of $P$.
- A Lie algebroid is, by definition, an anchored bundle with the additional property that the module of sections of $N \to M$ is equipped with a real Lie algebra, which satisfies a Leibniz condition with respect to multiplication by functions on $M$.
- Any regular distribution $D$ on $M$ is a subbundle of $TM$. The natural injection into $TM$ makes $D$ into an anchored bundle, whose admissible curves are precisely the set of curves tangent to the distribution
- A sub-Riemannian structure on a manifold $M$ is a regular distribution $i : D \hookrightarrow TM$ which is equipped with a Riemannian bundle metric, say $h$. The mapping $g : T^*M \to TM$, defined by $g = i \circ \sharp_h \circ i^*$ makes $T^*M$ into an anchored bundle.
- Sub-Finsler geometry is a generalisation of sub-Riemannian geometry in that the constraint distribution $D$ is equipped with a Finsler metric, rather than a Riemannian bundle metric.
- Linear control systems are typically modeled by a system of differential equations of the form $\dot{x}^i = f_a^i(x)u^a$, where $x \in \mathbb{R}^n$ represents the configuration of the system and $u \in \mathbb{R}^k$ represents the external (typically human) input to the system, steering the system in the direction given by

$f^i_a(x)u^a\partial/\partial x^i$. It is not difficult to see that, in a local coordinate chart, the admissible curves associated with an anchored bundle $\rho : N \to TM$ are precisely pairs $(x(t), u(t))$ such that $\dot{x}(t) = f^i_a(x(t))u^a(t)$, where $f^i_a$ is the local expression for the anchor map $\rho$. In this sense, the structure of an anchored bundle has been put forward as a differential geometric setting for studying control theory (see also [**15**]).

Assume in the following that an anchored bundle $\nu : N \to M$ with anchor map $\rho : N \to TM$ is kept fixed. As mentioned in the introduction, the definition of a *generalised connection on a principal fibre bundle* $\pi : P \to M$ *with structure group* $G$ has to involve the definition of the lift of an admissible curve to a curve in $P$. Such a 'lifting procedure' is provided by means of a mapping $h : P \times_M N \to TP$, fibred over the identity on $P$ and such that the following three conditions hold for all $(p, n) \in P \times_M N$:

(1) $TR_g(h(p, n)) = h(pg, n)$ for all $g \in G$, ;
(2) $T\pi(h(p, n)) = \rho(n)$;
(3) $h$ is linear in its second argument, i.e. $h(p, n + n') = h(p, n) + h(p, n')$.

Such a mapping $h$ is called a *generalised connection on* $P$. The lift of an admissible curve is then defined as follows. Assume that $c : I = [t_0, t_1] \to N$ is an admissible curve, with base curve $\widetilde{c}$ in $M$ and $\widetilde{c}(t_i) = m_i$ for $i = 0, 1$. The *lift of* $c$ *through a point* $p_0$ *in* $\pi^{-1}(m_0)$ is defined as the unique curve $c^h$ in $P$ such that $\dot{c}^h(t) = h(c^h(t), c(t))$ and $c^h(t_0) = p_0$. It is proven in [**3**] that there always exists a solution to this differential equation, defined over the entire time interval $I$. The lift of an admissible curve $c$ projects under $\pi$ onto the base curve $\widetilde{c}$. It is straightforward to see that the map $h$ also induces a lifting of sections of $\nu$ to vector fields on $P$, i.e. for $\eta \in \Gamma(\nu)$, then $\eta^h \in \mathcal{X}(P)$ is defined by $\eta^h(p) = h(p, \eta(\pi(p)))$, for $p \in P$.

It is well-known in standard connection theory that any connection on a principal fibre bundle, induces a connection (i.e. a covariant derivative operator) on every bundle associated with $P$. This property carries over to generalised connections. Let $\epsilon : E \to M$ denote a linear bundle associated with $P$. It can be proven that the generalised connection $h$ defines a 'covariant derivative' $\nabla$ such that for any admissible curve $c$, the operator $\nabla_c$ acts on sections of $\epsilon$ along the base curve of $c$. A detailed construction of $\nabla$ can be found in [**3**]. The derivative operator satisfies the following properties, where $f, f' : I \to \mathbb{R}$ are smooth functions, $\sigma, \sigma'$ are sections of $\epsilon$ along $\widetilde{c}$, and where $c'$ is an admissible curve with $\widetilde{c} = \widetilde{c}'$:

(1) $\nabla_c(\sigma + \sigma')(t) = \nabla_c\sigma(t) + \nabla_c\sigma'(t)$;
(2) $\nabla_{(fc+f'c')}\sigma(t) = f(t)\nabla_c\sigma(t) + f'(t)\nabla_{c'}\sigma(t)$;
(3) $\nabla_c f\sigma(t) = \dot{f}(t)\sigma(t) + \nabla_c\sigma(t)$.

>From the above properties, it follows that $\nabla_c\sigma(t)$ only depends on the value of $c$ at $t$. We say that a section $\sigma$ is parallel transported along $c$ if $\nabla_c\sigma(t) = 0$ for all $t \in I$.

It should be noted that any covariant derivative with the above properties is induced by a unique generalised connection, say $h^F$ on the principle fibre bundle of linear frames of $\epsilon : E \to M$. The parallel transport operator associated to the derivative operator and to the generalised connection $h^F$ are related as follows: a curve $\{\sigma_1(t), \ldots, \sigma_{\dim(E)}(t)\}$ in the frame bundle is the lift by $h^F$ of an admissible curve iff every element of the frame $\sigma_i(t)$ satisfies the equation $\nabla_c\sigma_i(t) = 0$.

In this paragraph we consider some coordinate expressions. Consider a coordinate chart adapted to both fibrations $\nu$ and $\epsilon$ simultaneously, with coordinate functions $(x^i, u^a)$ on $N$ and $(x^i, \xi^A)$ on $E$. The fibre coordinates are assumed to be linear, i.e. they are determined by a local basis for the sections of $\nu$ and $\epsilon$, say $\mathbf{e}_a$ on $N$ and $\mathbf{e}_A$ on $E$, respectively. The covariant derivative operator is completely defined by the set of local functions $\Gamma^B_{aA}$ on $M$ such that

$$(\nabla_{\mathbf{e}_a}\mathbf{e}_A)^B = \Gamma^B_{aA},$$

which are called the *connection coefficients*. Let us now consider local expressions for $c$ and $\sigma$, namely $c(t) = (x^i(t), u^a(t))$ and $\sigma = (x^i(t), \xi^A(t))$, then the following expression holds:

$$\nabla_c\sigma(t) = \big(\dot{\xi}^A(t) + \Gamma^A_{aB}(x(t))u^a(t)\xi^B(t)\big)\mathbf{e}_A(\widetilde{c}(t)).$$

The key idea underlying generalised connections is well represented in the above equation. In standard connection theory, the connection coefficients are contracted with the velocities of the base curve, i.e. $\Gamma^A_{iB}\dot{x}^i(t)\xi^B(t)$. For generalised connections, however, these coefficients are contracted with the fibre components of the admissible curve. More intuitively, one could say that these fibre components, when thinking in terms of admissible curves, carry 'more information' in comparison with the velocities $\dot{x}^i(t)$ and should therefore be contracted with the connection coefficients.

Let us now pass to the definition of holonomy. For that purpose we return to the definition of generalised connections on a principal fibre bundle $\pi : P \to M$. An *admissible loop* is an admissible curve for which the base curve is a loop curve in the usual sense, i.e. $c : [t_0, t_1] = I \to N$ is an admissible loop iff $\widetilde{c}$ is a loop in $M$ with base point $\widetilde{c}(t_0) = m_0 = \widetilde{c}(t_1)$. Similar to the standard definition of holonomy elements, we note that the lift of an admissible loop defines an automorphism of the fibre $\pi^{-1}(m_0)$ (a bijective map is called an automorphism if it commutes with the right action). The set of all automorphisms associated with admissible loops is a subgroup of the automorphism group denoted by $\Phi(m_0)$. Assume that a point $p_0 \in \pi^{-1}(m_0)$ is given, then every such automorphism can be evaluated at $p_0$. The image corresponds to a unique element in the structure group $G$, and the collection of all these elements is a Lie subgroup $\Phi(p_0)$ of $G$, called the holonomy group at $p_0$. It can be proven that a generalised version of the Reduction Theorem holds for generalised connections. For a more detailed treatment, we refer to [**10**]. It is our goal to show that these holonomy elements are encountered in the reconstruction of reduced non-holonomic mechanical systems with symmetry.

## 3. Generalised connections in non-holonomic mechanical systems

In this section we wish to describe how generalised connections naturally appear in non-holonomic mechanics. Assume in the following that $M$ is the configuration manifold of a mechanical system, described by a Lagrangian function $L$ on $TM$. We assume that it $L$ of the type $L = T - \tau^*_M V$, where $V$ is a function on $M$ representing the potential function of the conservative forces acting on the system and where $T(v) = \frac{1}{2}g(v, v)$ is the kinetic energy, determined by a Riemannian metric $g$ on $M$. We assume, in addition, that the solution curves have to satisfy some linear non-holonomic constraints, characterised by a regular (non-integrable) constraint distribution $D$ on $TM$. The generalised forces $Q \in \mathfrak{X}^*(M)$ depend on some parameters that represent certain external control functions, i.e. they represent the steering of the motion by external influences changing the generalised forces. These control functions are denoted by $u^i$, $i = 1, \dots, \ell$ (see also [**11**]), i.e. for simplicity we assume that $Q : \mathbb{R}^k \to \mathfrak{X}^*(M)$. >From d'Alembert's principle, a motion of the system is a curve $x : I \to M$ tangent to $D$ and satisfying

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} - Q_i(u(t))\right) dx^i \in D^0_{x(t)} \qquad \forall t \in I, \tag{1}$$

with $D^0$ the annihilator of $D$. The control input $u(t)$ is assumed to be given explicitly and therefore determines a time dependent force $Q$.

If we apply the map $\sharp_g$ to the one form in the above condition then, after some straightforward calculations, we can show that the above condition is equivalent to saying that:

$$\pi_D\big(\nabla^g_{\dot{x}}\dot{x}(t)\big) = \pi_D\big(-\operatorname{grad} V + \sharp_g(Q_{u(t)})\big)(x(t)),$$

where $\pi_D$ is the orthogonal projection onto $D$ w.r.t the metric $g$ and where $\nabla^g$ is the Levi-Civita connection associated with $g$.

Consider the anchored bundle $i : D \hookrightarrow TM$, and define the following generalised connection on the bundle $D$ by means of its 'covariant derivative' $\nabla^{nh}$ by:

$$\nabla^{nh}_{\dot{x}} Y(t) = \pi_D\big(\nabla^g_{\dot{x}} Y(t)\big),$$

where $\dot{x}$ is an admissible curve in $i : D \hookrightarrow TM$ with base curve $x$ (i.e. $x$ is tangent to $D$) and where $Y$ is a vector field along $x$ lying in $D$. The derivative $\nabla^{nh}$ determines a unique generalised connection, which we call the non-holonomic connection: see [**9**]. In that paper we proved that this connection only depends on the projection map $\pi_D$ and on the restriction of the metric $g$ to the subbundle $D$. Using these new notations, (1) can now be equivalently expressed by

$$\nabla^{nh}_{\dot{x}}\dot{x}(t) = \pi_D\big(-\operatorname{grad} V + \sharp_g(Q_{u(t)})\big)(x(t)). \tag{2}$$

In the remaining of this section we describe how the above equation can be 'reduced' if the system is invariant under symmetry. The main ideas behind this reduction process have been published in [**9**], where only free non-holonomic systems where studied (i.e. $V = 0$ and $Q = 0$). For further details, we refer the reader to that paper.

We now consider the case where the given non-holonomic system is invariant under the action of a symmetry group. This invariance condition means that the following assumptions hold. We assume that a Lie group $G$ acts on the right on the configuration manifold $M$ in such a way that, when taking the quotient, the bundle $\pi : M \to \overline{M} = M/G$ is a principal fibre bundle with structure group $G$. The non-holonomic constraint distribution $D$ on $M$ is assumed to be invariant under this action, i.e. $TR_h(D_m) = D_{mh}$ for $h \in G$ and, similarly, the kinetic energy metric, the potential and the generalised constraint forces are all assumed to be $G$-invariant, i.e. $R^*_h g = g$, $R^*_h V = V$ and $R^*_h Q_u = Q_u$ for all $h \in G$ and $u \in \mathbb{R}^k$. Our goal is to construct a reduction for the dynamical system described by (2). It should be noted that other approaches to reduce non-holonomic systems with symmetry have been followed (cf. [**2, 4**]). The main advantage of the approach described below is the fact that the only assumption we impose on the constraint distribution $D$, apart from being invariant, is its regularity.

Since Equation (2) is an equation in the distribution $D$, it seems natural to consider the quotient $D/G$ as the space on which the reduced equations of motion can be described. Note that $D/G$ is a bundle over $\overline{M}$, and can be made into an anchored bundle by defining $\rho : D/G \to T\overline{M}$ as $\rho([Y_m]) = T\pi(Y_m)$, where $[\cdot]$ stands for the equivalence class of an element $Y_m$ under the group action $TR_h$. It is easily seen that $\rho$ is well-defined. The sections of $\nu : D/G \to \overline{M}$ are precisely the right invariant vector fields on $M$ contained in $D$. Since $M$ is a principal fibre bundle over $\overline{M}$, we can define a generalised connection $h : M \times_{\overline{M}} D/G \to TM$ by $h(m, [Y_m]) = Y_m$, i.e. the image of $(m, [Y_m])$ is the unique representative of the equivalence class $[Y_m]$ in $D_m$. The connection $\nabla^{nh}$ over the bundle map $i : D \hookrightarrow TM$ is reducible (by restricting the action to right invariant vector fields) to a connection $\overline{\nabla}^{nh}$ on $D/G$ along $\rho : D/G \to T\overline{M}$. By assumption we have that the vector fields $\pi_D(\operatorname{grad} V)$ and $\pi_D(\sharp_g Q_u)$ are right invariant and, therefore, induce sections of $\nu$, denoted by $\overline{\operatorname{grad}} V$ and $\overline{F}_u$, respectively. The reduced equation then reads

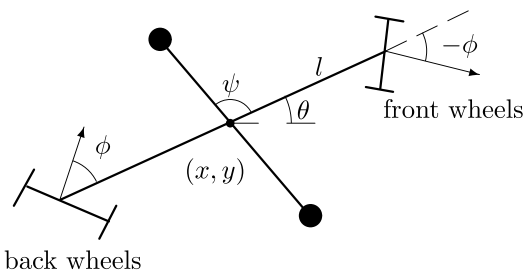
$$\overline{\nabla}^{nh}_c c(t) = \big( - \overline{\operatorname{grad}} V + \overline{F}_{u(t)} \big)(\widetilde{c}(t)), \tag{3}$$

for some $\rho$-admissible curve $c$ with base $\widetilde{c}$ in $\overline{M}$. A straightforward generalisation of a result in [**9**] states that the projection onto $D/G$ of any solution of (2) is a solution of (3). In turn, the lift of $h$ of any solution of (3) is a solution of (2). The latter operation is also called *reconstruction*. It is precisely in this reconstruction procedure that the holonomy elements of $h$ are encountered.

Assume that a $\rho$-admissible loop $c$ is a solution of (3). Then, keeping in mind the definition of the holonomy group $\Phi(m_0)$, the endpoint of the reconstructed curve $c^h$ lies in the same fibre as the starting point $m_0$, up to an element of the holonomy group $\Phi(m_0)$. Therefore, although there is no net movement in the reduced space, after reconstruction, the motion in the total space can generate a net movement for the system. This net movement then corresponds to an element in the holonomy group $\Phi(m_0)$ of $h$. A necessary condition for such a phenomena is that the lift $h$ has nontrivial holonomy. These ideas are made precise in the following section, where we consider the snakeboard example.

## 4. Holonomy elements as gaits for the snakeboard example

In this section, we heavily rely on the results described in [**12**], where the snakeboard example was analysed in detail. The snakeboard is a variant of the skateboard in which the passive wheel assemblies can pivot freely about a vertical axis. A peculiar characteristic of the snakeboard is the fact that the rider can generate a snake-like locomotion without having to kick off the ground. The picture below sketches a simplified model. The human torso is simulated by a momentum wheel, rotating about the vertical axis through the centre of mass. The picture below was taken from [**5**].



The configuration space can be identified with $M = SE(2) \times S^1 \times S^1$, with local coordinates denoted by $(x, y, \theta, \psi, \varphi)$. The two copies of $S^1$ describe the internal variables $(\psi, \varphi)$, representing the internal state of the snakeboard, while the Euclidean group $SE(2)$ represents the state of the snakeboard in the plane (centre of mass $(x, y)$ and orientation in the plane $(\theta)$). The requirement that the wheels do not slip in the direction of their axis imposes two non-holonomic constraints:

$$\begin{aligned} -\sin(\theta+\varphi)\dot{x} + \cos(\theta+\varphi)\dot{y} - l\cos\varphi\,\dot{\theta} = 0;\\ -\sin(\theta-\varphi)\dot{x} + \cos(\theta-\varphi)\dot{y} + l\cos\varphi\,\dot{\theta} = 0; \end{aligned}$$

which, in turn, determine the distribution $D$, spanned by:

$$\frac{\partial}{\partial\psi}, \frac{\partial}{\partial\varphi} \text{ and } -l\cos\varphi\cos\theta\frac{\partial}{\partial x} - l\cos\varphi\sin\theta\frac{\partial}{\partial y} + \sin\varphi\frac{\partial}{\partial\theta}. \tag{4}$$

The kinetic energy Lagrangian determining the motion of the snakeboard takes the form

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}(J + J_r + 2J_w)\dot{\theta}^2 + J_r\dot{\theta}\dot{\psi} + \tfrac{1}{2}J_r\dot{\psi}^2 + J_w\dot{\varphi}^2,$$

where $m$ is the mass of the snakeboard, $J$ is the moment of inertia of the board, $J_w$ the moment of inertia of the wheels about the vertical axis and $J_r$ the moment of inertia of the rotor. Following **[5]** we make the additional simplifying assumption $ml^2 = J + J_r + 2J_w$, which keeps the inertias on similar scales. The metric $g$ on $M$ has the following non-trivial components:

$$\begin{array}{ll} g_{xx} = m & g_{yy} = m \\ g_{\theta\theta} = J + J_r + 2J_w & g_{\theta\psi} = J_r = g_{\psi\theta} \\ g_{\psi\psi} = J_r & g_{\varphi\varphi} = 2J_w; \end{array}$$

(all other components are zero). There is no conservative force acting on the system (i.e. $V = 0$), however, the rider of the snakeboard is able to control the torque forces by changing the orientation of his torso (in the direction of $\psi$) and or feet (in the direction of $\varphi$). This control force is assumed to take the following simple form $\sharp_g Q_u = \sharp_g(u^1 d\psi + u^2 d\varphi)$.

The constraints as well as the metric $g$ are invariant under the right action of $SE(2)$. Denoting the elements of $SE(2)$ by $h = (a, b, \alpha)$, this action is given by

$$R_h(x, y, \theta, \psi, \varphi) = (x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b, \theta + \alpha, \psi, \varphi).$$

The configuration space $M$ thus inherits the structure of a principal fibre bundle with structure group $SE(2)$ over the base space $S^1 \times S^1$. The three vector fields from (4) form a basis for $D$ which is invariant under this action and therefore, since these vector fields correspond to sections of $D/SE(2) \to S^1 \times S^1$, they determine a basis for the sections of the bundle $D/SE(2)$, which will be denoted by $\{e_1, e_2, e_3\}$, where $e_i$ corresponds to $X_i$ for $i = 1, 2, 3$.

Since, eventually, we have to find the coordinate expression for the equation $\overline{\nabla}^{nh}_c c(t) = \overline{F}_{u(t)}$, it will be profitable to work with the following basis of $\mathcal{X}(M)$

$$\left\{ \begin{aligned} &X_1 = \frac{\partial}{\partial\psi},\ X_2 = \frac{\partial}{\partial\varphi}, \\ &X_3 = -l\cos\varphi\cos\theta\frac{\partial}{\partial x} - l\cos\varphi\sin\theta\frac{\partial}{\partial y} + \sin\varphi\frac{\partial}{\partial\theta}, \\ &X_4 = \sin\theta\frac{\partial}{\partial x} - \cos\theta\frac{\partial}{\partial y}, \\ &X_5 = l\sin\varphi\cos\theta\frac{\partial}{\partial x} + l\sin\varphi\sin\theta\frac{\partial}{\partial y} + \cos\varphi\frac{ml^2}{ml^2 - J_r}\left(\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\psi}\right) \end{aligned} \right\},$$

where $X_4, X_5$ determine a basis for $D^\perp$, the orthogonal complement of $D$ w.r.t. the metric $g$. Recalling the definition of a lift of a section of $D/SE(2) \to S^1 \times S^1$ to a vector field on $M$, then we can write $e_i^h = X_i$ for $i = 1, 2, 3$. A local expression for the anchor map $\rho : D/SE(2) \to TM$ can now be given: $\rho(\omega^i e_i) = \omega^1\partial/\partial\psi + \omega^2\partial/\partial\varphi$.

In order to obtain the local equations for $\overline{\nabla}^{nh}$ one might compute the connection coefficients for this connection. These are non-trivial functions and it would require a long and tedious calculation to derive them. Therefore, we shall follow a different route. Let $c(t) = \omega^i(t)e_i(\psi(t), \varphi(t))$ be a $\rho$-admissible curve. We now derive the coordinate expression for $\pi_D(\nabla^g_{\dot{c}^h}\dot{c}^h)$ (this makes sense since, by definition, $\pi_D(\nabla^g_{\dot{c}^h}\dot{c}^h) = (\overline{\nabla}^{nh}_c c)^h$). Using the right invariant basis $\{X_1, X_2, X_3\}$ for $D$, we can write

$$\dot{c}^h = \omega^1 X_1 + \omega^2 X_2 + \omega^3 X_3.$$

Since $g$ has only constant coefficients (i.e. all connection coefficients are zero), we have that

$$\nabla^g_{\dot{c}^h}\dot{c}^h(t) = \dot{\omega}^1(t)X_1(c(t)) + \dot{\omega}^2(t)X_2(c(t)) + \dot{\omega}^3(t)X_3(c(t)) + \omega^3(t)\dot{X}_3(t),$$

where $\dot{X}_3(t)$ is the tangent vector, defined by:

$$\frac{d}{dt}\left(-l\cos\varphi\cos\theta\right)\frac{\partial}{\partial x} - \frac{d}{dt}\left(l\cos\varphi\sin\theta\right)\frac{\partial}{\partial y} + \frac{d}{dt}\left(\sin\varphi\right)\frac{\partial}{\partial\theta},$$

where $d/dt$ represents the time derivation along $c^h(t)$ at $t$. The orthogonal projection of this tangent vector on $D$, gives us the coefficients of $\pi_D(\nabla^g_{\dot{c}^h}\dot{c}^h(t))$ with respect to the basis $\{X_1, X_2, X_3\}$ of $D$ and, in turn, the coefficients of $\overline{\nabla}^{nh}_c c(t)$ with respect to $\{e_1, e_2, e_3\}$, where $a = ml^2/(ml^2 - J_r)$:

$$\begin{aligned} \overline{\nabla}^{nh}_c c(t) \quad = \quad & \left(\dot{\omega}^1 + \frac{a\cos\varphi}{a\cos^2\varphi + \sin^2\varphi}\dot{\varphi}\omega^3\right)e_1 + \dot{\omega}^2 e_2 + \\ & \left(\dot{\omega}^3 + \frac{(1-a)\cos\varphi\sin\varphi}{a\cos^2\varphi + \sin^2\varphi}\dot{\varphi}\omega^3\right)e_3. \end{aligned}$$

If we substitute $\omega^2 = \dot{\varphi}$ and calculate $\pi_D(\sharp_g(Q_u))$, then the equations for $\omega(t) = (\omega^1(t), \omega^2(t), \omega^3(t))$ to be a solution of the reduced equations are precisely (up to a constant rescaling of the control parameters $(u_1, u_2)$):

$$\begin{aligned}
\dot{\omega}^1 &= -\frac{a\cos\varphi}{a\cos^2\varphi+\sin^2\varphi}\omega^2\omega^3+\left(\frac{m\ell^2}{J_r}-\frac{a\cos^2\varphi}{a\cos^2\varphi+\sin^2\varphi}\right)u_1(t);\\
\dot{\omega}^2 &= u_2(t);\\
\dot{\omega}^3 &= -\frac{(1-a)\cos\varphi\sin\varphi}{a\cos^2\varphi+\sin^2\varphi}\omega^2\omega^3-\frac{\sin\varphi}{a\cos^2\varphi+\sin^2\varphi}u_1(t).
\end{aligned}$$

It was proven in [**12**] that, given any curve $(\psi(t), \varphi(t))$ in $S^1 \times S^1$, there exists a control $(u^1(t), u^2(t))$, such that $(\psi(t), \varphi(t))$ is the projection of a solution to the above equations. Thus, there exist loops on the torus $S^1 \times S^1$ that are the base curve of a $\rho$-admissible curve that, at the same time, is a solution of the reduced non-holonomic equation. In particular, curves that are reconstructed from such $\rho$-admissible loops correspond to the gait motions studied in [**12**]. Therefore, these gaits are holonomy elements of the lift $h$.

To conclude this paper, we would like to make the following two remarks. First, in view of the above, it seems natural to wonder what holonomy elements of $h$ are encountered as physical motions. It is not in general the case that every admissible loop is the solution to the non-holonomic equations of motion on $D/G$ and, therefore, not every holonomy element has to be a gait motion. We leave this for future work. Secondly, it is well-known that in the standard theory of geometric phases it is possible, under some additional assumptions, to compute the holonomy elements by making use of the curvature tensor. This follows, roughly speaking, from the fact that in standard connection theory the holonomy groups are generated by the curvature. However, for generalised connections, the concept of a curvature tensor was only introduced in the case that the anchored bundle has the structure of a Lie algebroid. It therefore remains an open problem to find a suitable characterisation of the holonomy elements in terms of the generalised connection involved.

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