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A geometrical hypoelliptic diffusion

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Abstract. We use Langerock's connection in order to define a geometrical hypoelliptic diffusion. By using Bismut's intrinsic way of Malliavin Calculus, we get a geometrical Hoermander's theorem.

1. Introduction

Let us consider a compact Riemannian manifold (M, g). Let Δ be the Laplace-Beltrami operator. The heat semi-group $\exp[-t\Delta]$ on M is represented by the Brownian motion on M. By using Levi-Civita connection, Eells-Elworthy-Malliavin have got an intrinsic conctruction of the Brownian motion on M, starting from x, by considering a flat Brownian motion in $T_x(M)$. By using this construction, Bismut ([**3**]) established some intrinsic integration by parts for the Brownian motion on a manifold, and has got an intrinsic Malliavin Calculus on a manifold ([**3**]).

If we consider vector fields which satisfy strong Hoermander's hypothesis, we can consider an operator written under Hoermander's shape: $L = -\sum X_i^2$. Hoermander ([**14**]) have established that the heat semi-group $\exp[-tL]$ has a smooth density. Malliavin ([**2, 20, 22**]) has got this result by using probabilistic methods.

The problem of Hoermander's type operators are that they are not geometrical.

If we consider a distribution Q of p-planes with a metric inherited from the global Riemannian metric g on the tangent bundle $T(M)$, we get a canonical connection, called Langerock's connection, on it ([**16, 17**]). This connection depends only on the Riemannian metric and of the distribution of p-planes. In the integrable case, this connection is analoguous to the connection used by Bismut ([**4**]) and Bismut-Freed ([**5**])for the study of families of operators. By using this connection, Bismut-Freed ([**5**]) are able to construct a heat semi-group where the distribution of p-planes play a distinguished role. But the semi-group of Bismut-Freed is non degenerated.

Our goal is to extend Bismut-Freed's construction at the hypoelliptic case. We use Langerock's connection, which is unitary on the tangent bundle and such that the decomposition of $T(M)$ into Q and its orthogonal is parallel. In order to construct a process starting from x, we consider the stochastic parallel transport associated to Langerock's connection and a flat Brownian motion starting from 0 in $Q(x)$. This process is Markov and defines a semi-group which is generated by a degenerated operator.

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In this hypoelliptic situation, we establish Bismut's type intrinsic integration by parts formulas, which allow us to get a geometrical Hoermander's theorem. We don't use in this geometrical set-up Malliavin's matrix.

The reader may be worried by the fact that the Sub-Riemannian structure on Q comes from a Riemannian metric on M, by restriction. But this hypothesis is traditional in the literature ([**6, 7, 16, 17**]).

2. Degenerated Eells-Elworthy-Malliavin equation

Let (M, g) be a compact Riemannian manifold. Let Q be a distribution of p-plane and Q_1 be the distibution orthogonal to Q for the metric g. Let $\pi(x)$ be the orthogonal projection from $T_x(M)$ on $Q(x)$ and $\pi_1(x)$ be the orthogonal projection on $Q_1(x)$. Let ∇^g be the Levi-Civita connection on M. We put ([**16, 17**])

$$\nabla^L = \pi\nabla^g\pi + \pi_1\nabla^g\pi_1 \tag{2.1}$$

The connection ∇^L is unitary and preserves the decomposition of $T(M)$ in the orthogonal sum of Q and Q_1.

Let γ_{flat} be a Brownian motion on $Q(x)$. We consider Eells-Elworthy-Malliavin equation starting from x:

$$d\gamma_s^L(x) = \tau_s^L(x)d\gamma_{flat,s} \tag{2.2}$$

where $s \to \tau_s^L(x)$ is the parallel transport for ∇^L along the path $s \to \gamma_s^L(x)$. This equation can be seen as a traditional Staratonovitch equation by trivializing the tangent bundle. If Γ^L are the Christoffel symbol of ∇^L, we can realize (2.2) as:

$$\begin{aligned} d\tau_s^L(x) &= -\Gamma^L_{\tau_s^L(x)}(\gamma_s^L(x))\tau_s^L(x)d\gamma_{flat,s} \\ d\gamma_s^L(x) &= \tau_s^L(x)d\gamma_{flat,s} \end{aligned} \tag{2.3}$$

which is a traditional Stratonovitch equation. Since $\tau_s^L(x)$ is a rotation, and is therefore bounded because the Riemannian manifold is compact, (2.3) has a solution which does not blow up.

Let F_t be the sigma-algebra spanned by $\gamma_{flat,s}(x)$ for $s \leqslant t$. $\tau_s^L(x)$ is an isometry between $T_x(M)$ and $T_{\gamma_s^L(x)}$ which preserved the decomposition of the tangent space in the orthogonal sum of Q and Q_1. Moreover, conditionnally to F_t, $s \to \tau_t^L(x)\gamma_{flat,s+t}$ is a Brownian motion in $Q(\gamma_t^L(x))$. Let f be a continuous function on M. Let us denote

$$P_t f(x) = E[f(\gamma_t^L(x))] \tag{2.4}$$

We get:

Theorem 2.1:P_t is a semi-group.

Proof: The fact that $P_t(f)$ is continuous comes from (2.3). We conditionate along F_t, and we use the previous remark in order to see that $\gamma_{s+t}^L(x)$ conditionnaly to F_t has the law of $\gamma_s^L(\gamma_t^L(x))$

3.A geometrical Hoermander's theorem

In the sequel, we will do the following hypothesis:

Hypothesis H: The distribution Q is completely non integrable. This means that for all x in M, we cand find a neighborhhod of x and a local basis X_i of Q such that the Lie algebra spanned by the X_i is equal to $T_e(M)$.

Let F be a cylindrical functional $\gamma_.^L(x) \to F(\gamma_{s_1}^L(x), .., \gamma_{s_r}^L(x))$ for some $s_1 < s_2 < .. < s_r$. Let $X_s = \tau_s^L H_s$ where $s \to H_s$ is a finite energy path in $Q(x)$ starting from 0.

τ_s^l realizes a section of the bundle on the path space $(e_0^*Q)^* \otimes (e_s^*Q)$ where e_s is the evaluation map $\gamma_.^L(x) \to \gamma_s^L(x)$. This bundle inherites clearly a connection from ∇^L. We call this connection still ∇^L. If e_t^*Q is the pullback bundle by e_t of Q, it inherites a connection from ∇^L still denoted by ∇^L.

Let us recall Arafeva-Bismut formula ([**1, 3, 19**]):

$$\nabla_{X_.}^L \tau_t^L = \tau_t^L \int_0^t (\tau_s^L)^{-1} R^L(d\gamma_s^L(x), X_s)\tau_s^L \tag{3.1}$$

where R^L is the curvature tensor of ∇^L. By doing as in [**18**] and in [**19**], we get:

$$\nabla_{X_.}^L d\gamma_{flat,t} = \nabla_{X_.}^L (\tau_t^L)^{-1} d\gamma_t^L(x) + (\tau_t^L)^{-1}\nabla_{X_.}^L d\gamma_t^L(x) \tag{3.2}$$

But

$$\nabla_{X_.}^L d\gamma_t^L(x) = \tau_t^L dH_t \tag{3.3}$$

such that

$$\nabla_{X_.}^L d\gamma_{flat,t} = -\int_0^T (\tau_s^L)^{-1} R^L(d\gamma_s^L(x), X_s)\tau_s^L d\gamma_{flat,t} + dH_t \tag{3.4}$$

But $(\tau_s^L)^{-1} R^L(d\gamma_s^L(x), X_s)\tau_s^L$ is an antisymmetric operator on $Q(x)$. We deduce as in [**3**] the following intrinsic by part formula (of Bismut's type):

Proposition 3.1: For all cylindrical functional,$E[< dF, X_. >] = E[F\, divX_.]$ where

$$divX_. = \int_0^1 < d/dsH_s, \delta\gamma_{flat,s} > +1/2 \int_0^1 < S_{X_s}^L, \delta\gamma_{flat,s} > \tag{3.5}$$

where δ denotes the Itô integral and where S^L denotes the partial Ricci tensor associated to ∇^L:

$$S^L = \sum R^L(e_i, .)e_i \tag{3.6}$$

where e_i is an orthonormal basis of Q.

Since we get this integration by parts formula, we can repeat word by word in this hypoelliptic situation the construction of Sobolev spaces on the path space done by Léandre ([**18, 19**]) for the path space of the Riemannian manifold in elliptic case.

We introduce first of all a connection ∇^∞ on vector fields:

$$\nabla_{Y_.}^\infty X_. = \tau_.^L < dH_., Y_. > \tag{3.7}$$

where we take the derivative of $H_.$ alone in the vector field $X_. = \tau_.^L H_.$

Let us recall that the main novelty of the Malliavin Calculus with respect of its prelimary versions (See works of Elworthy, Hida, Fomin, Albeverio, Berezanskii..)

was to complete the differential operations on the Wiener space, because we have some integration by parts formulas. The integration by part of Proposition 3.1 allows us to do the same in this hypoelliptic situation.

Let us consider a functional F: we define $d^r_{\nabla^\infty} F$ by its kernels:

$$d^r_{\nabla^\infty} F(X_1, .., X_r) = \int_{[0,1]^r} k(s_1, .., s_r) dH^1_{s_1} .. d_{s_r} H^r_{s_r} \tag{3.8}$$

where $d^r_{\nabla^\infty} F$ is defined inductively:

$$d^r_{\nabla^\infty} F(X_1, .., X_r) =$$

$$= < d(d^{r-1}_{\nabla^\infty} F(X_1, .., X_{r-1})), X_r > - \sum d^{r-1}_{\nabla^\infty} F(X_1, .., \nabla^\infty_{X_r} X_i, .., X_{r-1}) \tag{3.9}$$

The Sobolev norm $\|.\|_{r,p}$ for $r \geqslant 1$ and $p \geqslant 1$ is defined as follows:

$$\|F\|_{r,p} = E[(\int_{[0,1]^r} |k(s_1, .., s_r)|^2 ds_1 .. ds_r)^{p/2}]^{1/p} \tag{3.10}$$

The space of functionals F smooth in the Malliavin sense is the space of functionals F such that for all p and r $\|F\|_{r,p} < \infty$. It is an algebra. Moreover, in (3.5), $div X_.$ is smooth in the Malliavin sense (See **[18]** and **[19]** for an analoguous statement in the Riemannian case) and τ^L_t belongs to all the Sobolev spaces (See **[19]** and **[18]** for an intrinsic definition of this fact in the Riemannian case).

We get:

Theorem 3.2: Let F be smooth in the Malliavin sense. Let $X_.$ be a vector field $X_s = \tau^L_s H_s$ where H_s is a deterministic finite energy path starting from 0 in $Q(x)$. We get:

$$E[< d_{\nabla^\infty} F, X_. >] = E[F div X_.] \tag{3.11}$$

We deduce from Theorem 3.2 the following theorem:

Theorem 3.3: The law of $\gamma^L_1(x)$ has a smooth density with respect of the Riemannian measure on M.

Proof: Let us consider $X_s = \tau^L_s s(\tau^L_1)^{-1} Y(\gamma^L_1(x))$ where Y is a smooth section of Q. By theorem 3.2, we deduce that for any smooth function f on M:

$$E[Y f(\gamma^L_1(x))] = E[f(\gamma^L_1(x)) L(Y)] \tag{3.12}$$

where $L(Y)$ belongs to all the Sobolev spaces. By doing by induction, we deduce that for smooth sections $Y^1, .., Y^r$ of Q

$$E[Y^1 .. Y^r f(\gamma^L_1(x))] = E[f(\gamma^L_1(x)) L(Y^1, .., Y^r)] \tag{3.13}$$

where $L(Y^1, .., Y^r)$ belongs to all the Sobolev spaces.

But $[Y^1, Y^2] = Y^1 Y^2 - Y^2 Y^1$. By Hypothesis H, we deduce that for any family $Z^1, .., Z^r$ of sections of the **tangent bundle** of M, we get that:

$$E[Z^1 .. Z^r f(\gamma^L_1(x))] = E(f(\gamma^L_1(x)) L(Z^1, .. Z^r)] \tag{3.14}$$

where $L(Z^1, .., Z^r)$ is smooth in the Malliavin sense. Therefore the result.

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