DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 629 – 637

Canonical formalism for quasi-classical particle "Zitterbewegung" in Ostrohrads'kyj mechanics

Roman Matsyuk

Abstract. The homogeneous canonical formalism of Rund is applied to the second-order Lagrangian model of the self-interacting particle of Bopp. The quasi-classical free spinning particle of Mathisson appears then as a constrained subsystem of the previous. Differential-geometrical mechanisms offered are formulated in a fairy general manner, although revealed here in a particular example of physical meaning.

1. A brief overview of Rund prescription

In his book on the Hamilton-Jacobi theory [**1**] Hanno Rund proposed a parameter-homogeneous formulation of the variational problem with higher derivatives. An appropriate prescription was given there for the transition to Hamiltonian formalism which is best suited to the needs of relativistic mechanics, and, generally speaking, especially convenient in all those cases, where an invariance with respect to some transformation group of all variables (dependent and independent ones) is imposed by the very prerequisites of the theory.

Let

$$pr\colon T^rM \setminus \{0\} \to C^r(1,M) \tag{1}$$

denote the quotient projection of the manifold of non zero Ehresmann velocities to the manifold of contact elements of r-th order with respect to the action the (local) reparametrization group $\mathrm{Gl}^r(1,\mathbb{R})$ on T^rM. Every time a Lagrange function $\mathcal{L} : T^rM \mapsto \mathbb{R}$ satisfies the so-called Zermelo conditions, it defines a parameter-invariant variational problem on T^rM. Every such problem passes to the above mentioned quotient and defines certain sheaf of equivalent semi-basic 1-forms (or Lagrangian densities) on the fibred manifold $C^r(1,M)$ over M. The general setting for this mechanism was discussed in details in [**2**], the usage of sheaf theory concepts was justified by Paul Dedecker [**3**]. In present contribution we shall limit ourselves to the case of order 2 ($r = 2$) variational problem and, moreover, shall work in local coordinate representation to touch with the physical model as announced. The convenient commonly accepted coordinates in the manifolds introduced as far read x^α, u^α, $\dot u^\alpha$, $\ddot u^\alpha$, $\dddot u^\alpha$ for T^rM and x^0, x^i, v^i, v'^i, v''^i, v'''^i for $C^r(1,M)$. As soon as in our

2000 *Mathematics Subject Classification.* 70H50.

Key words and phrases. Keywords: Generalized homogeneous Hamiltonian formalism, Ostrohrads'kyj mechanics, Quasi-classical spin, Relativistic top
Research supported by grant GACR 201/03/0512 of the Czech Grant Agency .
This paper is in final form and no version of it will be published elswhere.

application M becomes the space-time of special relativity with the diagonal metrics $(1,-1,-1,-1)$, we shall put to use vector notations of the pattern $u = (u_0, \mathbf{u})$, $u \cdot u = u_0^2 + \mathbf{u^2}$, $\mathbf{u^2} = \mathbf{u} \cdot \mathbf{u} = u_\alpha u^\alpha$.

Let $\mathcal{L}(x, u, \dot{u})$ be a Lagrange function on T^2M that satisfies Zermelo conditions:

$$
\begin{aligned}
u^\alpha \frac{\partial \mathcal{L}}{\partial \dot{u}^\alpha} &\equiv 0 \\
u^\alpha \frac{\partial \mathcal{L}}{\partial u^\alpha} + 2\,\dot{u}^\alpha \frac{\partial \mathcal{L}}{\partial \dot{u}^\alpha} &\equiv 0\,.
\end{aligned}
\tag{2}
$$

As common, let us introduce the Legendre transformation $Le\colon (x, u, \dot{u}, \ddot{u}) \mapsto (x, u, \wp, \wp')$,

$$
\begin{aligned}
\wp' &= \frac{\partial \mathcal{L}}{\partial \dot{u}} \\
\wp &= \frac{\partial \mathcal{L}}{\partial u} - \mathcal{D}_\tau \wp'\,,
\end{aligned}
\tag{3}
$$

where

$$
\mathcal{D}_\tau = u \frac{\partial}{\partial x} + \dot{u} \frac{\partial}{\partial u} + \ddot{u} \frac{\partial}{\partial \dot{u}}
\tag{4}
$$

denotes the operator of total derivative. We also mention that in forthcoming application nothing depends on space-time variables x, since everything obeys the pseudo-Euclidean symmetry.

It may be seen that the Zermelo conditions, if fulfilled, are now equivalent to the following:

$$
u^\alpha \wp'_\alpha \equiv 0
\tag{5.1}
$$

$$
u^\alpha \wp_\alpha + \dot{u}^\alpha \wp'_\alpha \equiv \mathcal{L}\,.
\tag{5.2}
$$

According to H. Rund, we assume that there exists a C^2 function $\mathcal{H}$ of the four variables $(x, u, \wp, \wp')$ which is not trivially constant along each of the last two variables, which is nevertheless constant along the Legendre transformation, and we chose that constant to be equal to 1 without any essential loss of generality:

$$
\mathcal{H} \circ Le \equiv 1\,.
\tag{6}
$$

As proved in [**1**] (see also [**5**]), under the assumption that

$$
\operatorname{rank} \left\| \frac{\partial^2 \mathcal{L}}{\partial \dot{u}^\alpha \partial \dot{u}^\beta} \right\| = \dim M - 1\,,
\tag{7}
$$

there exist proportionality factors λ and μ, in general dependent on $x, u, \dot{u}, \ddot{u}$, such that the following *canonical system* of differential equations of the first order with respect to the variables $x, u, \wp, \wp'$ is satisfied along all the extremals of the variational problem with the Lagrange function $\mathcal{L}$:

$$
\frac{dx}{d\tau} = \lambda \frac{\partial \mathcal{H}}{\partial \wp}
\tag{8.1}
$$

$$
\frac{du}{d\tau} = \lambda \frac{\partial \mathcal{H}}{\partial \wp'} + \mu u
\tag{8.2}
$$

$$
\frac{d\wp}{d\tau} = -\lambda \frac{\partial \mathcal{H}}{\partial x}
\tag{8.3}
$$

$$
\frac{d\wp'}{d\tau} = -\lambda \frac{\partial \mathcal{H}}{\partial u} - \mu \wp'\,.
\tag{8.4}
$$

Now the evolution of an arbitrary function f of the phase space variables x, u, $\wp$, $\wp'$ is given by the famous Poisson bracket

$$\{f,\mathcal{H}\} \stackrel{\text{def}}{=} \frac{\partial f}{\partial x^\alpha}\frac{\partial \mathcal{H}}{\partial \wp_\alpha} + \frac{\partial f}{\partial u^\alpha}\frac{\partial \mathcal{H}}{\partial \wp'_\alpha} - \frac{\partial f}{\partial \wp_\alpha}\frac{\partial \mathcal{H}}{\partial x^\alpha} - \frac{\partial f}{\partial \wp'_\alpha}\frac{\partial \mathcal{H}}{\partial u^\alpha}$$

as follows [**5**]

$$\frac{df}{d\tau} = \lambda\{f,\mathcal{H}\} + \mu\left[u^\alpha\frac{\partial f}{\partial u^\alpha} - \wp'_\alpha\frac{\partial f}{\partial \wp'_\alpha}\right]. \tag{9}$$

2. How to obtain the $\mathcal{H}$

The scope of possible functions $\mathcal{H}$ who satisfy (6) is rather large. But, since every parameter-independent variational problem, posed on T^rM, generates a corresponding formulation on $C^r(1,M)$, and vice versa, one may effectively try a pull-back of the Hamiltonian formulation of the problem on $C^r(1,M)$ to T^rM.

Let a variational problem on $\mathbb{R}\times T^rM$ be given in terms of the semi-basic (relative to $\mathbb{R}$) differential 1-form $\mathcal{L}\,d\tau$, where $\mathcal{L}$ is defined on T^rM solely and satisfies the Zermelo conditions. And let $L\,dx^0$ be that representative of the corresponding sheaf of equivalent semi-basic (relative to M) differential 1-forms on the fibred manifold $C^r(1,M)$, who in the above described coordinates is given by the following relation,

$$\mathcal{L}\,d\tau - (L\circ pr)\,dx^0 = -(L\circ pr)\,\vartheta\,,$$

where

$$\vartheta = dx^0 - u^0 d\tau \tag{10}$$

is one of the contact forms on $J^1(\mathbb{R},M)\approx \mathbb{R}\times TM$. Hence

$$\mathcal{L} = u^0\,L\circ pr. \tag{11}$$

The canonical momenta are being introduced here as usual:

$$\begin{aligned} \mathbf{p}' &= \frac{\partial L}{\partial \mathbf{v}'} \\ \mathbf{p} &= \frac{\partial L}{\partial \mathbf{v}} - D_t\mathbf{p}'\,, \end{aligned} \tag{12}$$

where

$$D_t = \mathsf{v}^i\frac{\partial}{\partial x^i} + \mathsf{v}'^i\frac{\partial}{\partial \mathsf{v}^i} + \mathsf{v}''^i\frac{\partial}{\partial \mathsf{v}'^i} \tag{13}$$

denotes the operator of total derivative with respect to x^0.

The correspondence between the operators (4) and (13) of total derivatives on relevant jet spaces, $J^2(\mathbb{R},M)$ and (locally) $J^2(\mathbb{R},\mathbb{R}^{\dim M-1})$ seems evident, as in fact it is: whenever f is a local function on $C^2(1,M)$, then

$$\mathcal{D}_\tau(\mathsf{f}\circ pr) = u^0\,D_t\mathsf{f}\circ pr \tag{14}$$

It would, however, be of some instructive good to obtain (14) by direct differentiation of the projection (1), which, in the 3^d order, reads in our coordinates:

$$
\begin{aligned}
\mathbf{v} \circ pr &= \frac{\mathbf{u}}{u_0} \\
\mathbf{v}' \circ pr &= \frac{\dot{\mathbf{u}}}{u_0^2} - \frac{\dot{u}_0}{u_0^3}\,\mathbf{u} \\
\mathbf{v}'' \circ pr &= \frac{\ddot{\mathbf{u}}}{u_0^3} - 3\,\frac{\dot{u}_0}{u_0^4}\,\dot{\mathbf{u}} + 3\left(\frac{\dot{u}_0^2}{u_0^5} - \frac{\ddot{u}_0}{u_0^4}\right)\mathbf{u}\,.
\end{aligned}
\tag{15}
$$

With relation (14) in hand, we are ready now to establish the correspondence between the pair of momenta $\wp = (\wp_0, \boldsymbol{\wp})$ and $\wp' = (\wp'_0, \boldsymbol{\wp}')$ in (3), calculated for the Lagrange function $\mathcal{L}$ given by (11), and the pull-back of the momenta in (12):

$$\wp'_0 = u_0 \frac{\partial(L\circ pr)}{\partial \dot{u}_0} = -\frac{1}{u_0^2}\,\mathbf{u}\left(\frac{\partial L}{\partial \mathbf{v}'}\circ pr\right) = -\frac{1}{u_0^2}\,\mathbf{u}\,(\mathbf{p}'\circ pr)\,; \tag{16.1}$$

$$\boldsymbol{\wp}' = u_0 \frac{\partial(L\circ pr)}{\partial \dot{\mathbf{u}}} = \frac{1}{u_0}\left(\frac{\partial L}{\partial \mathbf{v}'}\circ pr\right) = \frac{1}{u_0}(\mathbf{p}'\circ pr)\,; \tag{16.2}$$

(16.3)

$$
\begin{aligned}
\wp_0 &= L\circ pr + u_0\,\frac{\partial(L\circ pr)}{\partial u_0} - \mathcal{D}_\tau \wp'_0 \quad \text{by the reason of (15), (14) and (16.1)} \\
&= L\circ pr - u_0\left[\frac{1}{u_0^2}\,\mathbf{u}\left(\frac{\partial L}{\partial \mathbf{v}}\circ pr\right) + \frac{2}{u_0^3}\,\dot{\mathbf{u}}\left(\frac{\partial L}{\partial \mathbf{v}'}\circ pr\right) - \frac{3\dot{u}_0}{u_0^4}\,\mathbf{u}\left(\frac{\partial L}{\partial \mathbf{v}'}\circ pr\right)\right] \\
&\qquad - 2\frac{\dot{u}_0}{u_0^3}\,\mathbf{u}(\mathbf{p}'\circ pr) + \frac{1}{u_0^2}\,\dot{\mathbf{u}}(\mathbf{p}'\circ pr) + \frac{1}{u_0}\,\mathbf{u}(D_t\mathbf{p}' \circ pr) \\
&= L\circ pr - \frac{1}{u_0}\,\mathbf{u}\left(\frac{\partial L}{\partial \mathbf{v}}\circ pr\right) + \frac{\dot{u}_0}{u_0^3}\,\mathbf{u}(\mathbf{p}'\circ pr) - \frac{1}{u_0^2}\,\dot{\mathbf{u}}(\mathbf{p}'\circ pr) + \frac{1}{u_0}\,\mathbf{u}(D_t\mathbf{p}' \circ pr) \\
&= L\circ pr - \mathbf{v}\mathbf{p}\circ pr - \mathbf{v}'\mathbf{p}'\circ pr\,;
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\wp} &= u_0\,\frac{\partial(L\circ pr)}{\partial \mathbf{u}} - \mathcal{D}_\tau \boldsymbol{\wp}' \quad \text{by the reason of (15), (14) and (16.2)} \\
&= u_0\left[\frac{1}{u_0}\left(\frac{\partial L}{\partial \mathbf{v}}\circ pr\right) - \frac{\dot{u}_0}{u_0^3}\left(\frac{\partial L}{\partial \mathbf{v}'}\circ pr\right)\right] + \frac{\dot{u}_0}{u_0^2}\,(\mathbf{p}'\circ pr) - D_t\mathbf{p}'\circ pr) \\
&= \frac{\partial L}{\partial \mathbf{v}}\circ pr - D_t\mathbf{p}'\circ pr = \mathbf{p}\circ pr\,.
\end{aligned}
\tag{16.4}
$$

From (16.3) and (16.4) it follows that

$$\wp\, u = u_0\, L\circ pr - u_0\,\mathbf{v}'\mathbf{p}'\circ pr\,, \tag{17.1}$$

whereas from (16.1) and (16.2) in view of (15) it follows that

$$\wp'\dot{u} = u_0\,\mathbf{v}'\mathbf{p}'\circ pr\,, \tag{17.2}$$

and (5.2) keeps true immediately.

In our further considerations we choose the approach of the generalized Hamiltonian theory as exposed in [**4**]. Then, in our coordinates, it is best to describe the system evolution by the kernel of the differential two form

$$\omega = -dH \wedge dx^0 + d\mathbf{p}\wedge d\mathbf{x} + d\mathbf{p}'\wedge d\mathbf{v}\,, \tag{18}$$

where the exterior product sign $\wedge$ comprises the contraction of vector differential forms, if necessary. Now, it is tentative that on the manifold $\mathbb{R} \times T^3 M$ the evolution of this same system be described by a differential two form of the same shape,

$$\Omega = -d\mathcal{H} \wedge d\tau + d\wp \wedge dx + d\wp' \wedge du\,, \tag{19}$$

where the momenta $\wp$ and $\wp'$ due to the Lagrange function (11).

Conventionally one puts $H = \mathbf{pv} + \mathbf{p'v'} - L$. Under this assumption it is straightforward to calculate the difference between (19) and (18), taking into account the relations (16.2, 16.4) and the Zermelo condition (5.1):

$$\Omega - pr^*\omega = d(pr^* H + \wp_0) \wedge dx^0 - d\mathcal{H} \wedge d\tau\,. \tag{20}$$

We wish that this difference be proportional to the contact form (10), namely,

$$\Omega - pr^*\omega = \alpha \wedge \vartheta\,. \tag{21}$$

The simplest reasonable way to comply in (20) with (21) is to put

$$d\mathcal{H} = u^0 d(pr^* H + \wp_0) \tag{22}$$

and

$$\mathcal{H} = u_0 pr^* H + \Psi\,. \tag{23}$$

Now proceed to determine this deviating function Ψ. From (23) we have:

$$pr^* dH = \frac{d\mathcal{H}}{u_0} + (\Psi - \mathcal{H})\,\frac{du_0}{u_0^2} - \frac{d\Psi}{u_0}\,. \tag{24}$$

It suffices now to substitute (24) into (22) to obtain the relation

$$\frac{\mathcal{H} - \Psi}{u_0}\,du_0 - u_0\,d\wp_0 = \; - d\Psi\,,$$

from where it becomes clear that

$$\begin{cases} \Psi = u_0 \wp_0 + c \\ \mathcal{H} = c \end{cases}$$

and also, by the reason of (6), $c = 1$.

Hence

$$\mathcal{H} = u_0 pr^* H + u_0 \wp_0 + 1 \tag{25}$$

3. *Zitterbewegung* of quasiclassical relativistic particle

As far back as 1946 Fritz Bopp developed a second-order Lagrange function for the description of classical particle motion from the second step approximation with respect to the parameter of retard interaction [**6**]. It seems prominent that the Bopp Lagrangian may be cast into a simple shape in terms of the first curvature of the particle's world line,

$$k = \frac{\|\dot{u} \wedge u\|}{\|u\|^3}\,,$$

as follows:

$$\mathcal{L} \overset{\text{def}}{=} a\mathcal{L}_r + A\mathcal{L}_e = \frac{a}{2}\|u\|k^2 + \frac{A}{2}\|u\|\,, \tag{26}$$

where we assume $a \neq 0$ to confine with (7). This Lagrange function satisfies the Zermelo conditions (2). The first addend in (26), $\mathcal{L}_r$, turns out to be of the type considered by H. Rund in [**1**] (see also [**5**]). The second addend, $\mathcal{L}_e$, is the free particle Lagrange function. According to (11), the corresponding local Lagrange density on $C^2(1, M)$ may be expressed in coordinates x^0, $\mathbf{v}$ and $\mathbf{v}'$:

$$\begin{aligned} L\,dx^0 &\overset{\text{def}}{=} aL_r dx^0 + AL_e dx^0 \\ &= \frac{a}{2}\sqrt{(1+\mathbf{v^2})}\left(\frac{\mathbf{v'^2}}{(1+\mathbf{v^2})^2} - \frac{(\mathbf{v}\cdot\mathbf{v}')^2}{(1+\mathbf{v^2})^3}\right)dx^0 + \frac{A}{2}\sqrt{(1+\mathbf{v^2})}dx^0\,. \end{aligned} \tag{27}$$

The momenta (12) for this Lagrangian read:

$$\mathbf{p}'_r = \frac{\mathbf{v}'}{(1+\mathbf{v^2})^{3/2}} - \frac{\mathbf{v}\cdot\mathbf{v}'}{(1+\mathbf{v^2})^{5/2}}\,\mathbf{v}$$

$$\begin{aligned} \mathbf{p}_r = -\frac{\mathbf{v}''}{(1+\mathbf{v^2})^{3/2}} &+ 3\,\frac{\mathbf{v}\cdot\mathbf{v}'}{(1+\mathbf{v^2})^{5/2}}\,\mathbf{v}' \\ &+ \frac{\mathbf{v}\cdot\mathbf{v}''}{(1+\mathbf{v^2})^{5/2}}\,\mathbf{v} - \frac{1}{2}\frac{\mathbf{v'^2}}{(1+\mathbf{v^2})^{5/2}}\,\mathbf{v} - \frac{5}{2}\frac{(\mathbf{v}\cdot\mathbf{v}')^2}{(1+\mathbf{v^2})^{7/2}}\,\mathbf{v}\,. \end{aligned}$$

We introduce the standard Hamilton function

$$\begin{aligned} H &= \mathbf{p}\mathbf{v} + \mathbf{p}'\mathbf{v}' - L \\ &\overset{\text{def}}{=} aH_r + AH_e = a\mathbf{p}_r\mathbf{v} + a\mathbf{p}'_r\mathbf{v}' - aL_r + A\mathbf{p}_e\mathbf{v} - AL_e\,, \end{aligned} \tag{28}$$

because $\mathbf{p}'_e = 0$. It is necessary to exclude the variable $\mathbf{v}'$ in (28). We calculate:

$$\begin{cases} \mathbf{p}'_r\mathbf{v}' = 2L_r \\ \mathbf{p}'^{\mathbf{2}}_r + (\mathbf{p}'_r\mathbf{v})^2 = 2\,\dfrac{L_r}{(1+\mathbf{v^2})^{3/2}}\,, \end{cases}$$

and finally the Hamilton function reads

$$H = \mathbf{p}\mathbf{v} + \frac{1}{2a}\left(1+\mathbf{v^2}\right)^{3/2}\left(\mathbf{p'^2} + (\mathbf{p}'\mathbf{v})^2\right) - \frac{A}{2}\sqrt{1+\mathbf{v^2}}\,. \tag{29}$$

In his paper [**6**], Fritz Bopp asserted: “Der klassischen Bewegung überlagert sich eine Zitterbewegung, die durch die neuen Variabeln $\mathbf{v}$ und $\mathbf{p}'$ beschrieben wird. Sie führt zu spinartigen Effekten…” *

Now the Hamilton function on T^3M may be obtained from(25):

$$\mathcal{H} = \wp u + \frac{1}{2a}\|u\|^3\wp'^2 - \frac{A}{2}\|u\| + 1\,. \tag{30}$$

Alternatively, one could get the same expression directly from the assertion

$$\mathcal{H} = \wp u + \wp'\dot{u} - \mathcal{L} + 1\,, \tag{31}$$

*Upon the classical motion some vibrational one superimposes itself that is described by the new variables $\mathbf{v}$ and $\mathbf{p}'$. It leads to the effects of spin type…

assuming $\mathcal{L}$ be taken from (26). In view of (7), one cannot resolve the Legendre transformation (3) in full, but nevertheless it is possible to eliminate the variable $\dot{u}$ from (31). First we calculate the momenta for (26)

$$\wp' = \frac{a}{\|u\|^5}\left[u^{\mathbf{2}}\dot{u} - (u\cdot\dot{u})u\right]$$

$$\wp = \frac{Au}{2\|u\|} - a\left[\frac{\ddot{u}}{\|u\|^3} - 3\,\frac{u\cdot\dot{u}}{\|u\|^5}\,\dot{u} - \frac{u\cdot\ddot{u}}{\|u\|^5}\,u + \frac{\dot{u}^{\mathbf{2}}}{2\|u\|^5}\,u + \frac{5}{2}\frac{(u\dot{u})^2}{\|u\|^7}\,u\right].$$

In the next step we express all those quantities in (31) wherein the $\dot{u}$ enters, in terms of $\wp'$ and u alone

$$\begin{cases} \wp'\dot{u} = \dfrac{\|u\|^3}{a}\,\wp'^{\mathbf{2}} \\ \mathcal{L}_r = \dfrac{\|u\|^3}{2a^2}\,\wp'^{\mathbf{2}}\,, \end{cases} \tag{32}$$

and substitute into (31) to finally achieve the Hamilton function (30).

The approach to building up the Hamilton function in present paper differs from that of H. S. P. Grässer. I was inspired by his treatment of general Lagrange function, quadratic in velocities, in the framework of Finsler space, of which ours is a very special case. But the physical model herein considered demands to include also the free particle term $\mathcal{L}_e$.

It is not of much labour now to calculate the fourth-order Euler-Poisson equation of the variational problem with the Lagrange function (26) from the starting point of the Hamilton system (8) with the expression (30) in hand. For (8) we have:

$$\begin{cases} \dfrac{dx}{d\tau} = u \\ \dfrac{du}{d\tau} = \dfrac{\|u\|^3}{a}\,\wp' \\ \dfrac{d\wp}{d\tau} = 0 \\ \dfrac{d\wp'}{d\tau} = \dfrac{A}{2}\dfrac{u}{\|u\|} - \wp - \dfrac{3\|u\|}{2a}\,\wp'^{\mathbf{2}}u\,. \end{cases}$$

Only at this stage one has the right to put some constraints on the choice of the parameter τ. We put $u^{\mathbf{2}} = 1$ to obtain

$$\frac{du}{d\tau} = \frac{\wp'}{a} \tag{33.1}$$

$$\frac{d\wp'}{d\tau} = \frac{A}{2}\,u - \wp - \frac{3}{2a}\wp'^{\mathbf{2}}u\,, \tag{33.2}$$

and it is clear that $\lambda = 1$ and $\mu = 0$ in (8), so that the evolution equation (9) regains the traditional shape now.

Next we differentiate the equation (33.1) and substitute the equation (33.2) therein, to obtain

$$\ddot{u} = \frac{A}{2a}\,u - \frac{\wp}{a} - \frac{3}{2a^2}\wp'^{\mathbf{2}}u\,, \tag{34}$$

$$\frac{\wp\dot{u}}{a} = -\,\ddot{u}\cdot\dot{u}\,, \tag{35}$$

and, on the other hand, the contraction of (33.1) with (33.2) gives

$$\wp' \cdot \dot{\wp}' = -a\,\wp\dot{u}\,. \tag{36}$$

Differentiating (34) once again produces

$$\dddot{u} = \frac{A}{2a}\,\dot{u} - \frac{3}{a^2}\,(\wp' \cdot \dot{\wp}')\,u - \frac{3}{2a^2}\,\wp'^{\mathbf{2}}\dot{u}\,, \tag{37}$$

in where we substitute (36), (33.1), and, sequentially, (35), to finish at the resulting fourth-order equation of motion

$$\dddot{u} + \left(\frac{3}{2}\,\dot{u}^{\mathbf{2}} - \frac{A}{2a}\right)\,\dot{u} + 3\,(\dot{u}\cdot\ddot{u})\,u = 0\,. \tag{38}$$

As soon as in the actual parametrization $k^2 = \dot{u}^{\mathbf{2}}$, on the constrained manifold of constant relativistic acceleration k_0 the equation (38) *reduces to the equation of helical motion of relativistic spinning particle considered first by F. Riewe* [**7**] *and then by G. C. Costantelos* [**8**]*:*

$$\frac{d^2}{ds^2}\,\ddot{x}^\alpha + \varpi^2\ddot{x}^\alpha = 0\,, \tag{39}$$

where we have put $\varpi = \frac{3}{2}\,k_0^2 - \frac{A}{2a}$.

In the previous paper [**10**], I proved that *this fourth order equation of motion (39) may be rigorously developed from the third order general equation of motion of classical dipole particle proposed by Mathisson in 1937 in* [**9**].

References

[1] H. Rund, *The Hamilton-Jacobi theory in the calculus of variations.* D. Van Nostrand Company Ltd., London e.a., 1966

[2] R. Ya. Matsyuk, *Autoparallel variational description of the free relativistic top third order dynamics*, in: Differential Geometry and Its Applications, Proc. Conf. 2001, Opava (Czech Republic). Silesian University, Opava, 2002, 447–459.

[3] P. Dedecker, *On the generalization of symplectic geometry to multiple integrals in the calculus of variations*, in: Lecture Notes in Mathematics, vol. 570. Springer, Berlin, 1977, p. 395–456.

[4] O. Krupková, *The Geometry of ordinary variational equations*, in: Lecture Notes in Mathematics, vol. 1678. Springer, Berlin, 1997.

[5] H. S. P. Grässer, *On the canonical formalisms for higher order problems in the calculus of variations of single intergrals*, Istit. Lombardo Accad. Sci. Lett., Rend. A **122** (1988), 105–125.

[6] F. Bopp, *Quantentheorie der Feldmechanik*, Zf. für Naturf, **1** (1946), 196–203.

[7] F. Riewe, *Relativistic classical spinning-particle mechanics*, Il Nuovo Cim., **8 B** (1972), no. 1, 271–277.

[8] G. C. Costantelos, *On the Hamilton-jacobi theory with derivatives of higher order*, Il Nuovo Cim., **84 B** (1984), no. 1, 91–101.

[9] M. Mathisson, *Neue Mechanik materieller Systeme*, Acta Phys. Polon. **6** (1937), fasc. 3, 163–200.

[10] R. Ya. Matsyuk, *A covering second-order Lagrangian for the relativistic top without forces*, Proc. Inst. Math. of NAS of Ukraine, **43** (2002), Part 2, 741–745.

Institute for Applied Problems in Mechanics and Mathematics
15 Dudayev St., 290005 L'viv, Ukraine

e-mail: `matsyuk@lms.lviv.ua`