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Nonstandard invariant operators on quaternionic geometries

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Abstract. First, the problem of strong invariance and the appropriate technique is reviewed following [**1**]. Then the nonstandard operators between exterior forms on a class of geometries including the quaternionic structures are introduced along the lines of [**2**]. In the case of a four dimensional manifold, such a nonstandard operator corresponds to the so called Paneitz operator (an analog of the second power of Laplacian) due to the equivalence of quaternionic and conformal structures in this dimension. It is well known in the conformal geometry that this operator is not strongly invariant, cf. [**1**]. We recover this result directly using the spinor notation in quaternionic geometries. Finally, we show that an analogous nonstandard operator in higher dimensions is not strongly invariant as well and we indicate how to explore the other nonstandard operators. This paper is an extension of the results in [**3**].

1. Strongly invariant operators

From the geometrical point of view, the invariant differential operators are those which respect the given structure. They are therefore the main tool which one can use to explore the geometries.

On the level of the homogenous spaces G/P, the so called "flat case", we deal with the homogenous vector bundles. The invariant differential operators are defined as the mapings between spaces of sections of the homogenous vector bundles $D : \Gamma(E) \to \Gamma(F)$ which intertwine the induced action of the given Lie group on $\Gamma(E)$, resp. $\Gamma(F)$. This means $D(g \cdot s) = g \cdot D(s)$ for all $g \in G$ and $s \in \Gamma(E)$. One can view this as a morphism $\tilde{D} : J^k(E) \to F$ of homogenous vector bundles for an appropriate k. It is well-known, that the question of the existence of such a morphism (or corresponding invariant operator) can be converted to a purely algebraical problem. One can construct an induced module $V(E_0) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} E_0$, where $\mathfrak{U}(\mathfrak{g})$ is the universal enveloping algebra to the Lie algebra $\mathfrak{g}$ corresponding to the given Lie group G. This $(\mathfrak{g}, P)$-module is called the generalized Verma module and using this, one can formulate the following proposition:

Proposition 1. *Let $P \subset G$ be a closed subgroup of a semisimple Lie group G and E_0, F_0 the finite dimensional representations of the group P. Let E and F dentote the homogenous vector bundles $G \times_P E_0$ a $G \times_P F_0$.*

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Then the space of the linear invariant differential operators $D : \Gamma(E) \to \Gamma(F)$ *is isomorphic to the space* $\mathrm{Hom}_{(\mathfrak{g},P)}(\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} F_0^*, \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} E_0^*)$ *of the homomorphisms of* $(\mathfrak{g}, P)$*-moduls.*

We can generalize our situation now and we can pass from the case of the homogenous model to the case of the general manifold M, to the so called "curved case". Cartan's generalized spaces are curved analogues of the homogenous spaces G/P defined by means of an Ad-invariant $\mathfrak{g}$-valued absolute paralelism ω on a principal P-bundle $\mathcal{G} \to M$. We refer to then as Cartan geometries. In this situation, we deal with the natural bundles $EM = \mathcal{G} \times_P E_0$, the generalizations of the homogenous bundles. In the curved case we get the natural operators $D_M : \Gamma(EM) \to \Gamma(FM)$ acting between the sections of these bundles as the generalization of the invariant operators in the flat case. As in the flat case, one has a bijection between these operators of order k and the morphisms of the natural bundles $J^k EM \to FM$. But no straightforward analog of the above proposition holds for the curved case. What we can do is to define the so called semi-holonomic jet prolongation $\bar{J}EM$ of a natural bundle EM. Now we can state a proposition:

PROPOSITION 2. *Let* E_0 *be a representation of* $P \subset G$ *and* $EM = \mathcal{G} \times_P E_0$ *the corresponding natural bundle on the category of Cartan geometries* $(\mathcal{G} \to M, \omega)$*. Then the semi-holonomic prolongation* $\bar{J}^k EM$ *is canonicaly isomorphic to the natural bundle* $\mathcal{G} \times_P (\bar{J}^k E_0)$*. Moreover, the iterated invariant derivative* ∇^ω *induces a canonical embedding of* k*th prolongation* $J^k EM$ *in the bundle of the semi-holonomic jets* $\bar{J}^k EM$*,*

$$j^k_u \sigma \mapsto (\sigma(u), \nabla^\omega \sigma(u), \dots, (\nabla^\omega)^k \sigma(u))$$

where $\sigma \in \Gamma(EM)$ *is here expressed as the equivariant function on* $\mathcal{G}$*.*

Now, we can use the duality as in the flat case again, and we can construct the so called semi-holonomic Verma module $\bar{V}(E_0) = \bar{\mathcal{U}}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} E_0^*$, where $\bar{\mathcal{U}}(\mathfrak{g})$ is semi-holonomic extension of the universal enveloping algebra of [**1**]. The difference to the flat case is that there is no bijection between the space of the homomorphisms of these modules and the space of the natural operators. The proposition 1.2 says only, that once we have found such a homomorphism which gives a nonzero symbol, we can construct a natural operator using this homomorphism and the invariant derivative. The operators that arise in this way (from a homomorphism of semi-holonomic Verma modules) are called the strongly invariant operators (indicated by s.i.o. in the scheme below). It is far from being the case that every operator is strongly invariant, as shown for example in [**1**]. Let us summarize the facts we know about the flat case and the curved case and the differences between them in order to make the situation more clear:

Flat case

Homogenous bundles
$E = G \times_P E_0$,
$F = G \times_P F_0$

Invariant differential operators
$\Gamma(E) \to \Gamma(F)$
of order k

$1{-}1 \updownarrow$

morphisms $J^k E \to F$

$1{-}1 \Updownarrow$

P-homomorphisms
$J^k E_0 \to F_0$

dually:
$F_0^* \to (J^k E_0)^* = V_k(E_0)$

Curved case

Natural bundles
$EM = \mathcal{G} \times_P E_0$,
$FM = \mathcal{G} \times_P F_0$

Natural operators
$\Gamma(EM) \to \Gamma(FM)$
of order k

$1{-}1 \updownarrow$

morphisms $J^k EM \to FM$

s.i.o. $\qquad \cancel{1{-}1 \Updownarrow}$

$\bar{J}^k EM \to FM$
$\bar{J}^k E_0 \to F_0$

dually:
$F_0^* \to (\bar{J}^k E_0)^* = \bar{V}_k(E_0)$

Given a natural operator (in the next we will deal with the curved case only and we will write invariant operator instead of natural operator) the problem of its strongly invariance can be reformulate as an algebraic lifting problem by using the facts we mentioned above. More specifically, on the level of the hmomogenous model the symbol of this invariant operator gives the corresponding homomorphism $V(F_0) \to V(E_0)$ of Verma modules and the operator is strongly invariant if and only if we can lift this homomorphism to a homomorphism $\bar{V}(F_0) \to \bar{V}(E_0)$ of the semi-holonomic Verma modules. We will work with the operators of a given order (specifically $k = 4$) in the next and so we can use the natural filtration of the (semi-holonomic) Verma module and consider the homomorphisms between V_k resp. $\bar{V}_k$ only. According to the Frobenius reciprocity, the lifting problem is then equivalent to the problem of completing of the following diagram:

$$\begin{tikzcd}
 & \bar{V}_k(E_0) \arrow[d] & \supset \otimes^k \mathfrak{g}_{-1} \otimes_{\mathcal{U}(\mathfrak{p})} E_0^* \\
F_0^* \arrow[ur, dashed, "?"] \arrow[r] & V_k(E_0) & \supset \odot^k \mathfrak{g}_{-1} \otimes_{\mathcal{U}(\mathfrak{p})} E_0^*
\end{tikzcd}$$

We can once more reformulate the lifting problem a little. We assume F_0^* to be an irreducible representation. Then the image in $V_k(E_0)$ is again an irreducible representation with an appropriate highest weight vector that lies in $\odot^k \mathfrak{g}_{-1} \otimes_{\mathcal{U}(\mathfrak{p})} E_0^*$. To lift the homomorphism means now to lift this highest weight vector to the highest weight vector in $\otimes^k \mathfrak{g}_{-1} \otimes_{\mathcal{U}(\mathfrak{p})} E_0^* \subset \bar{V}_k(E_0)$.

2. Nonstandard operators on quaternionic geometries

Quaternionic structure on a $4m$-dimensional manifold M is given by the torsion free reduction of the structure group of the tangent bundle to the group

$$G_0 = S(GL(p/2, \mathbb{H}) \times GL(q/2, \mathbb{H})) \subset GL(4m, \mathbb{R}).$$

The complexified tangent bundle enjoys then the following fundamental identification:

$$(TM)^{\mathbb{C}} = \mathcal{E}^A \otimes \mathcal{E}_{A'},$$

together with the identification of the top degree exterior powers of the complex vector bundles $\mathcal{E}_{A'}$ and $\mathcal{E}^A$ of dimensions p and q:

$$\Lambda^p \mathcal{E}_{A'} \cong \Lambda^q \mathcal{E}^A.$$

where we have used the Penrose index notation. The complexified Lie algebra in this case equals: $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(p+q, \mathbb{C})$ and the Satake diagram corresponding to the quaternionic geometry is:

$$\bullet \!-\!\!-\!\!-\, \times \,-\!\!-\!\!-\, \bullet \,-\!\!-\!\!-\quad \cdots \quad -\!\!-\!\!-\, \circ \,-\!\!-\!\!-\, \bullet$$

We define bundles of densities or weighted bundles as follows. The weight -1 line bundle $\mathcal{E}[-1]$ over M is identified with

$$\mathcal{E}^{[A'_1 A'_2 \cdots A'_p]}.$$

Then, for integral w, the weight w line bundle $\mathcal{E}[w]$ is defined to be $(\mathcal{E}[-1])^{-w}$. In view of the defining isomorphism $h : \Lambda^q \mathcal{E}^A \to^{\cong} \Lambda^p \mathcal{E}_{A'}$ we also have

$$\mathcal{E}[1] \cong \mathcal{E}^{[A_1 A_2 \cdots A_q]} \cong \mathcal{E}_{[A'_1 A'_2 \cdots A'_p]}.$$

A scale for our structure is a nowhere vanishing section ξ of $\mathcal{E}[1]$. One can show that the choice of the scale determines the unique connections on $\mathcal{E}^A$ and $\mathcal{E}_{A'}$ that belong to our quaternionic structure.

Let us write V^α for the module corresponding to the standard representation of G on $\mathbb{R}^{p+q}$ and write V_α for the dual module. The index α is a Penrose-type abstract index and we write $\mathcal{E}^\alpha$ and $\mathcal{E}_\alpha$ for the respective bundles induced by these G-modules. All finite dimensional G-modules are submodules in tensor products of the fundamental representations V^α and V_α. Thus the bundles $\mathcal{E}^\alpha$ and $\mathcal{E}_\alpha$ play a special role and we term these (local) twistor bundles. Given a scale we have $\mathcal{E}^\alpha = \mathcal{E}^A \oplus \mathcal{E}^{A'}$ and $\mathcal{E}_\alpha = \mathcal{E}_A \oplus \mathcal{E}_{A'}$, so we may write sections of these bundles as "matrices" such as

$$[u^\alpha]_\xi = \begin{pmatrix} u^A \\ u^{A'} \end{pmatrix} \in [\Gamma \mathcal{E}^\alpha]_\xi, \;\; [v_\alpha]_\xi = (v_A \; v_{A'}) \in [\Gamma \mathcal{E}_\alpha]_\xi.$$

We shall mostly deal with the weighted twistors, i.e. the tensor products of the form $\mathcal{E}^{\alpha\ldots\beta}_{\gamma\ldots\delta}[w] = \mathcal{E}^{\alpha\ldots\beta}_{\gamma\ldots\delta} \otimes \mathcal{E}[w]$.

Let us suppose that we have a fixed scale now. Using the connection determined by this choice of scale (mentioned above) we can define the so called twistor connection ∇_a on $\mathcal{E}^\alpha$ and $\mathcal{E}_\alpha$ as follows:

$$\nabla^{P'}_A \begin{pmatrix} v^B \\ v^{B'} \end{pmatrix} = \begin{pmatrix} \nabla^{P'}_A v^B + \delta^B_A v^{P'} \\ \nabla^{P'}_A v^{B'} - \mathbf{P}^{P'B'}_{AB} v^B \end{pmatrix}$$

and

$$\nabla^{P'}_A (u_B \; u_{B'}) = (\nabla^{P'}_A u_B + \mathbf{P}^{P'B'}_{AB} u_{B'} \; \nabla^{P'}_A u_{B'} - \delta^{P'}_{B'} u_A),$$

where $\mathbf{P}_{ab}$ is the usual Rho-tensor. Although we have fixed a choice of scale to present formulae for these connections, it is easily verified that the twistor connections are in fact independent of the choice of scale and so are invariant operators on our structure.

Following [**2**] we observe that if $f \in \mathcal{E}\,[w]$ then the spinor-twistor object

$$D^{A'}_{\beta} f := (\nabla^{A'}_{B} f \;\; w\delta^{A'}_{B'} f)$$

is invariant. We may regard this as an injecting part of the invariant twistor object $D^{\alpha}_{\beta} f := X^{\alpha}_{A'} D^{A'}_{\beta} f$. By regarding, in this formula for D^{α}_{β}, ∇ to be the coupled twistor-spinor connection it is easily verified that the operator D^{α}_{β} is well defined and invariant on sections of the weighted twistor bundles $\mathcal{E}^{\rho\cdots\mu}_{\alpha\cdots\gamma}[w]$. The operators $D^{\alpha}_{\beta} : \mathcal{E}^{\rho\cdots\mu}_{\delta\cdots\gamma}[w] \to \mathcal{E}^{\alpha\rho\cdots\mu}_{\beta\delta\cdots\gamma}[w]$ are called the twistor-D operators according to [**2**].

Following further [**2**] we may form the compositions of these operators and remove some redundand information by certain symmetrizations of the targets. As the result we obtain the following theorem:

PROPOSITION 3. *Let the quaternionic structure is given on M. For each integer k such that $0 \leqslant k \leqslant q-2$ there is a fourth order invariant operator,*

$$\square_{ABCD} : (\updownarrow^{k}\; \mathcal{E}_E)[-k] \to (\updownarrow^{k+2}\; \mathcal{E}_E)[-k-2]$$

which is nontrivial on flat structures.

We have used the Young symmetrizers in order to describe the individual irreducible components of the natural bundles. This notation should be clear from figure 1, where the short arrows show the decomposition of the de Rham resolution of (complex) functions into irreducible components, in the special case of 8-dimensional underlying manifold M. It means the case $p = 2$, $q = 4$, i.e. case of the lowest dimensional interesting quaternionic geometry. Now, the long arrows on the left-hand side describe the non-standard invariant operators on the flat model. One can show, that the operators from the theorem above are exactly the curved analogues of these long arrows. Our aim is to answer the question whether these operators are strongly invariant using the technique described in the previous section.

3. Results in dimension 4

The case of the 4-dimensional underlying manifold M corresponds to the case of the quaternionic geometry for $p = q = 2$. It is well-known that this quaternionic structure on M is isomorphic to the conformal structure on M. One can show this isomorphism on the Satake diagrams, which are equal for these two structures and look as:

$$\bullet \;\text{———}\; \times \;\text{———}\; \bullet$$

The decomposition of the de Rham resolution to the irreducible components in this case is demonstrated on the figure 2. In the dimension four we have in the flat model the only one nonstandard invariant operator (one long arrow) $\Delta^2 : \mathcal{E} \to \mathcal{E}[-4]$, which corresponds to the second power of Laplacian for the Euclidean metric. The appropriate homomorphism of the Verma modules is $V(\Lambda^4) \to V(\Lambda^0)$, where Λ^1 denotes $\mathfrak{g}^*_{-1}$ as a P-representation and Λ^k is its k-th exterior power. Due to the isomorphism with the conformal structure and due to the results of [**1**] we know that the above mentioned homomorphism of Verma modules does not lift to the

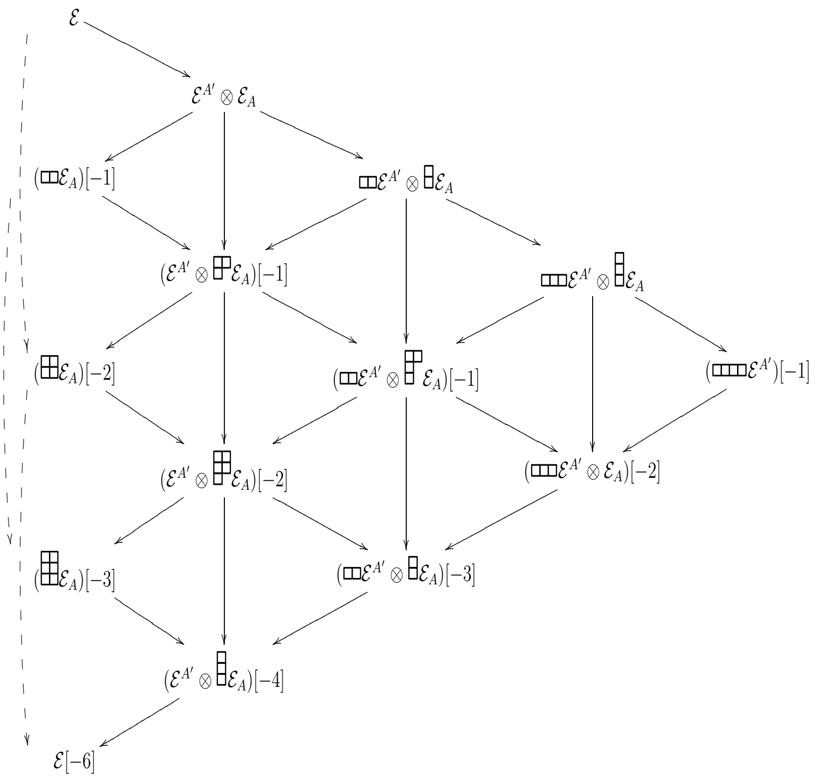

FIGURE 1

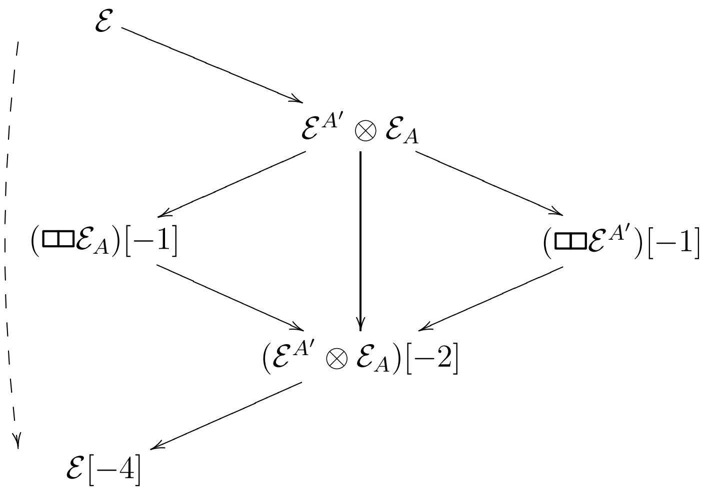

FIGURE 2

corresponding homomorphism of the semi-holonomic Verma modules and thus the operator $\Box_{ABCD}$ we deal with is not strongly invariant in this case of the lowest dimension. The first step to the further generalizations should be now to prove this directly without using any isomorphism with the conformal structure.

To advance as sketched in the first section, we first find the explicit formula for the action of $\mathfrak{g}_1$ on $\otimes^k\mathfrak{g}_{-1}\otimes E_0^*$, considered as a $\mathfrak{g}_0$-submodule of $\bar{V}(E_0)$. The elements of $\otimes^k\mathfrak{g}_{-1}\otimes E_0^*$ can be written as the linear combinations of the simple elements

$$y = y_1 \otimes y_2 \otimes \cdots \otimes y_k \otimes e \qquad \text{pro} \quad y_1, y_2, \ldots, y_k \in \mathfrak{g}_{-1} \quad \text{a} \quad e \in E_0^*.$$

Using the Poincarè-Birkhoff-Witt procedure we compute the action of $x \in \mathfrak{g}_1$ on y:

$$\left.\begin{array}{l} \sum_{1\leqslant p<q\leqslant k} y_1 \otimes \cdots \otimes [[x, y_p], y_q]\hat{y}_q \otimes \cdots \otimes y_k \otimes e \\ + \sum_{1\leqslant p\leqslant k} y_1 \otimes \cdots \otimes \hat{y}_p \otimes \cdots \otimes y_k \otimes [x, y_p]e \end{array}\right. \in \otimes^{k-1}\mathfrak{g}_{-1} \otimes E_0^* \ ,$$

where $\hat{}$ denotes omitting of the element with the appropriate index. We need to write this formula down in our abstract index formalism in quaternionic geometry. We shall write the typical elements $Y \in \mathfrak{g}_{-1}$, $Z \in \mathfrak{g}_0$ and $X \in \mathfrak{g}_1$ as

$$Y = y^A_{A'}, \quad Z = (z^{A'}_{B'}\delta^B_A + z^B_A\delta^{A'}_{B'}), \quad X = x^{A'}_A.$$

In this notation, the brackets in the matrix Lie algebra $\mathfrak{g}$ can be expressed by

$$\begin{array}{l} [Z, Y] = -z^{B'}_{A'}y^A_{B'} + z^A_B y^B_{A'} \\ [Z, X] = z^{A'}_{B'}x^{B'}_A - z^B_A x^{A'}_B \\ [X, Y] = x^{A'}_C\, y^C_{B'}\delta^B_A - y^B_{C'}x^{C'}_A\, \delta^{A'}_{B'} \end{array}$$

When we apply these relations twice we get:

$$\mathfrak{g}_{-1} \ni [[X, Y_p]\,, Y_q]^A_{A'} = -x^{C'}_C\,(y_p)^A_{C'}(y_q)^C_{A'} - x^{C'}_C\,(y_p)^C_{A'}(y_q)^A_{C'}$$

And using this we get finally the desired formula for the action of $\mathfrak{g}_1$:

$$\begin{array}{l} \otimes^{k-1}\mathfrak{g}_{-1} \otimes E_0^* \ni (x \cdot (y \otimes e))^{A_1 \ldots A_{k-1}}_{A'_1 \ldots A'_{k-1}} = \\ -\sum_{1\leqslant p<q\leqslant k} \left[\begin{array}{c} x^{C'}_C\, y^{A_1 \ldots A_{p-1} A_{q-1} A_p \ldots A_{q-2} C A_q \ldots A_{k-1}}_{A'_1 \ldots A'_{p-1} C' A'_p \ldots A'_{q-2} A'_{q-1} A'_q \ldots A'_{k-1}} \\ +x^{C'}_C\, y^{A_1 \ldots A_{p-1} C A_p \ldots A_{q-2} A_{q-1} A_q \ldots A_{k-1}}_{A'_1 \ldots A'_{p-1} A'_{q-1} A'_p \ldots A'_{q-2} C' A'_q \ldots A'_{k-1}} \end{array} \right] \otimes e \\ + \sum_{1\leqslant p\leqslant k} [\cdots]\, e. \end{array}$$

We will not need the explicit form of the second summand and that is why it is omitted here. One can see the reason immediately in the dimension four we are talking about now. We would like to find a highest weight vector in the target semi-holonomic Verma module of a homomorphism $\bar{V}(\Lambda^4) \to \bar{V}(\Lambda^0)$ and it means that the representation in our case is $E_0 = \Lambda^0$, i.e. trivial. The action on the symmetric $\omega \in \odot^4\mathfrak{g}_1 \subset V(\Lambda^0)$ is then given by the symmetrization of the above mentioned action:

$$\begin{array}{l} \odot^3\mathfrak{g}_{-1} \ni (x \cdot \omega)^{A_1 A_2 A_3}_{A'_1 A'_2 A'_3} = \\ -4\left[x^{C'}_C\, y^{A_1 C A_2 A_3}_{C' A'_1 A'_2 A'_3} + x^{C'}_C\, y^{A_2 C A_1 A_3}_{C' A'_2 A'_1 A'_3} + x^{C'}_C\, y^{A_3 C A_1 A_2}_{C' A'_3 A'_1 A'_2}\right] \end{array}$$

The nonstandard operator in the flat case corresponds to a homomorphism $V(\Lambda^4) \to V(\Lambda^0)$ of Verma modules. It is not hard to find out that the corresponding highest weight vector can be expressed by:

$$\begin{array}{l} \omega^{ABCD}_{A'B'C'D'} = \\ \epsilon_{A'B'}\epsilon_{C'D'}\epsilon^{AB}\epsilon^{CD} + \epsilon_{A'C'}\epsilon_{B'D'}\epsilon^{AC}\epsilon^{BD} + \epsilon_{A'D'}\epsilon_{B'C'}\epsilon^{AD}\epsilon^{BC}, \end{array}$$

where ϵ is the cannonical volume form on $\mathcal{E}_{A'}$ resp. $\mathcal{E}^{A}$. Really, when we compute the action on this symmetric vector we get

$$\odot^3\mathfrak{g}_{-1} \ni (x\cdot\omega)^{A_1A_2A_3}_{A'_1A'_2A'_3} = -4x^{[A_1}\epsilon^{A_2A_3]}x_{[A'_1}\epsilon_{A'_2A'_3]} = 0,$$

because the square brackets denote the antisymmetrization and we deal with the tensor products of two dimensional vector spaces and thus $x^{[A_1}\epsilon^{A_2A_3]} = x_{[A'_1}\epsilon_{A'_2A'_3]} = 0$.

The general lift of this highest weight vector can be expressed by:

$$\otimes^4\mathfrak{g}_{-1} \ni \tilde{\omega}^{ABCD}_{A'B'C'D'} = \\ K\cdot\epsilon_{A'B'}\epsilon_{C'D'}\epsilon^{AB}\epsilon^{CD} + L\cdot\epsilon_{A'C'}\epsilon_{B'D'}\epsilon^{AC}\epsilon^{BD} + M\cdot\epsilon_{A'D'}\epsilon_{B'C'}\epsilon^{AD}\epsilon^{BC},$$

where the constans K, L, M have to satisfy $K + L + M = 1$. When we compute the action of $\mathfrak{g}_1$ on this general lift and when we use again that $x^{[A_1}\epsilon^{A_2A_3]} = x_{[A'_1}\epsilon_{A'_2A'_3]} = 0$ we get:

$$\begin{aligned}
&\otimes^3\mathfrak{g}_{-1} \ni (x\cdot\tilde{\omega})^{A_1A_2A_3}_{A'_1A'_2A'_3} = \\
&(2K+2L+2M)x^{A_2}_{A'_2}\epsilon_{A'_1A'_3}\epsilon^{A_1A_3} + \\
&(-2K-2L-2M)x^{A_1}_{A'_2}\epsilon_{A'_1A'_3}\epsilon^{A_2A_3} + \\
&(-2K-2L-2M)x^{A_2}_{A'_1}\epsilon_{A'_2A'_3}\epsilon^{A_1A_3} = \\
&2x^{A_2}_{A'_2}\epsilon_{A'_1A'_3}\epsilon^{A_1A_3} - 2x^{A_1}_{A'_2}\epsilon_{A'_1A'_3}\epsilon^{A_2A_3} - 2x^{A_2}_{A'_1}\epsilon_{A'_2A'_1}\epsilon^{A_1A_3} \neq 0.
\end{aligned}$$

And so we have got the final result that the homomorphism $V(\Lambda^4) \to V(\Lambda^0)$ of Verma modules does not lift to the corresponding homomorphism $\bar{V}(\Lambda^4) \to \bar{V}(\Lambda^0)$ of the semi-holonomic Verma modules and thus we have confirmed that the operator $\Box_{ABCD}$ is not strongly invariant in dimension four.

4. Results in general dimension

In the higher dimensions we have more natural operators $\Box_{ABCD}$, each corresponding to a long arrow in the flat case. We deal with the one corresponding to the first long arrow now:

$$\Box^{\,0}_{ABCD} : \mathcal{E} \to (\boxplus\mathcal{E}_A)[-2]$$

The appropriate homomorphism is $V((\boxplus\mathcal{E}_A)[-2]) \to V(\Lambda^0)$ and so the representation of the target module is again trivial. That is why we can follow the solution in dimension four. The difference is that the q-spinor bundle has dimension bigger than two and we have no such two-form ϵ that we have used for prescribing the highest weight vector. But what we know from the representation theory is that the irreducible representation $\boxplus\mathcal{E}_A$ of $\mathfrak{sl}(q,\mathbb{C})$ is a subrepresentation of $S^2(\Lambda^2\mathbb{C}^q)$ and thus we can express the highest weight vector of this representation by $v^{ABCD} = \frac{1}{2}(v^{[AB][CD]} + v^{[CD][AB]})$. We do not need to fix a basis and to use the explicit form of this highest weight vector. It is enought to use its symmetries denoted above to prove that for such v

$$\omega^{ABCD}_{A'B'C'D'} = \\ \epsilon_{A'B'}\epsilon_{C'D'}v^{ABCD} + \epsilon_{A'C'}\epsilon_{B'D'}v^{ACBD} + \epsilon_{A'D'}\epsilon_{B'C'}v^{ADBC}$$

is the symmetric highest weight vector. When we put this expression to the formula for the action of $\mathfrak{g}_1$ and when we use the symmetries of v we really get:

$$\odot^3\mathfrak{g}_{-1} \ni (x\cdot\omega)^{A_1A_2A_3}_{A'_1A'_2A'_3} = -12x^{C'}_{C}\epsilon_{C'[A'_1}\epsilon_{A'_2A'_3]}(v^{\boldsymbol{[}[A_2A_3][A_1\boldsymbol{]}C]} - v^{[C\boldsymbol{[}A_1][A_2A_3]\boldsymbol{]}}) = 0$$

We again assume its general lift to $\otimes^4\mathfrak{g}_{-1}$ and we compute the action of $\mathfrak{g}_1$ on it. Although only $x_{[A'_1}\epsilon_{A'_2A'_3]} = 0$ holds, we get a similar term as in the dimension four which cannot be zero for a general $x \in \mathfrak{g}_1$. Thus the homomorphism $V((\boxplus\mathcal{E}_A)[-2]) \to V(\Lambda^0)$ of Verma modules does not lift to the corresponding homomorphism of the semi-holonomic Verma modules and we have shown that the natural operator $\Box^0_{ABCD}$ is not strongly invariant.

We intend to follow this proof for the other operators $\Box_{ABCD}$ corresponding to the other long arrows in the flat model but the situation is more complicated. In particular, the representations appearing in the target (semi-holonomic) Verma modules are nontrivial and thus the formula for the action of $\mathfrak{g}_1$ is more sophisticated.

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