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Adjoint symmetries in non-holonomic mechanics

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Abstract. The type of non-holonomic mechanical systems we have in mind is quite general: the system can be time-dependent and the non-holonomic constraints need not be linear or affine. The constraints are simply modelled by a given subbundle C of the first-jet bundle $J^1\tau$ of some evolution space $\tau : E \to \mathbb{R}$, and the dynamical system is considered to be a second-order differential equation field Γ, living directly on C. We discuss how the fact that this Γ comes from non-holonomic mechanics in the sense of the d'Alembert-Chetaev principle, is essentially encoded in the availability of a projection operator P, which maps arbitrary vector fields along the projection $\pi_C : C \to E$ onto those having the property that their vertical lift is tangent to C. The geometrical benefit coming from P is that it gives rise to an inherited vertical endomorphism-type tensor field on the constraint submanifold C, which in turn leads to a natural construction of a non-linear connection, associated to the dynamics Γ. The main purpose of the talk is to show how the theory of adjoint symmetries can be developed in this fairly general set-up, with the aid of the basic tools referred to above. We shall discuss the general mechanism by which all first integrals of the system can be obtained, in principle, through an algorithmic search for adjoint symmetries.

1. Non-holonomic systems: a few recent models

It is not my intention to give a review here of the many different (though related) differential geometric models which have been developed to describe mechanical systems with non-holonomic constraints; for some general references on the subject, see the books by Bloch [**1**] and Cortés [**2**]. Instead, I wish to refer to just a few of these, with the purpose of explaining the motivation and the sources of inspiration which have led to this work.

The first contribution I want to bring into focus is one by Giachetta [**3**]. In fact, Giachetta's paper is primarily devoted to a method for generating first integrals of non-holonomic systems and my purpose here is to advocate what I believe to be a better way of doing this. But before getting there, I need to make a digression on the essential geometric tools I need to develop my approach in all generality.

The basic data in Giachetta's model are: (i) a configuration space, which is a fibred manifold $\tau : E \to \mathbb{R}$, and its 1st-jet extension $\pi : J^1\tau \to E$; (ii) a fibre metric g on τ, which has a natural lift to a fibre metric on π; (iii) an 'unconstrained' second-order differential equation field (Sode) $\tilde{\Gamma}$ on $J^1\tau$; (iv) a constraint submanifold $C \subset J^1\tau$, which is a subbundle, not necessarily affine. A number of further constructions follow from these data. The Sode $\tilde{\Gamma}$, for a start, gives rise to a canonically defined Ehresmann connection on π, which in turn is used by Giachetta to construct a 2-form

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Ω on $J^1\tau$ whose kernel is spanned by $\tilde{\Gamma}$. The fibre metric g on π further induces a fibre metric $\widehat{g}$ on $\pi_C : C \to E$ by restriction, and $\widehat{g}$ is used to construct an essential tool in the whole approach, namely a projection from vertical vectors at points of C to vertical vectors tangent to C, i.e. $P : V\pi|_C \to V\pi_C$, which is defined as follows:

$$\widehat{g}(PX, Y) = g(X, Y), \quad \forall X \in V_z\pi, \quad Y \in V_z\pi_C\,.$$

Finally, P and Ω play a role in the construction of a reduced dynamics Γ, which is a SODE living on C.

A few remarks are in order at this point. I am unaware of other papers in which a projector of exactly this nature is playing a central role, but there is always some kind of projector in the model and a point I wish to make is that the P we are talking about here then is always available. For example, the original fibre metric g in Giachetta's approach can easily be allowed to depend on all coordinates of $J^1\tau$, meaning that it becomes a metric along the projection π; now, almost all authors start from a given (unconstrained) Lagrangian L on $J^1\tau$, and then the corresponding Poincaré-Cartan 2-form ω_L in fact is completely determined by such a metric g along π (the Hessian of L); the above construction of P then carries over to this slightly more general situation. An exception to this rule would seem to be the approach presented by Krupková [**4**]. There, a general unconstrained dynamics is taken as the starting point and it is represented by a so-called dynamical form, which is a 2-form of the following type: $E = (A_\alpha(t, q, \dot{q}) + B_{\alpha\beta}(t, q, \dot{q})\ddot{q}^\beta)dq^\alpha \wedge dt$. But the matrix $B_{\alpha\beta}$ then in fact can be thought of as component matrix of a type (0,2) tensor field along the projection π, not necessarily symmetric, and provided it is non-degenerate, the construction of a projection P still carries over to that situation. Having realized that tensor fields along π are at the heart of constructions which perhaps, for most authors, take place on the full jet space $J^1\tau$, yet another observation leaps to the eye: vertical vectors on $J^1\tau$ can be identified with vectors along π and with this identification, *P can be thought of as projecting vector fields along π_C to the subset of those whose vertical lift is tangent to C*. It is in this sense that I will make use of a P further on.

Let me show now that such a P in fact is also present, in a very natural way, in the quite different model for non-holonomic systems which was described in [**6**]. An additional reason for picking out this paper as second example is that it is the source of inspiration for the adapted connection I shall use later on.

$$\begin{tikzcd}
& J^1\tau \supset C \arrow[d, shift right] & \\
& \rho^* J^1\tau_0 \arrow[u, shift right, "\sigma"'] \arrow[r] \arrow[d] & J^1\tau_0 \arrow[d] \\
J^1\rho \arrow[r, shift right] & E \arrow[l, shift right, "\tilde{\sigma}"'] \arrow[r, "\rho"] \arrow[d, "\tau"] & M \arrow[d, "\tau_0"] \\
& \mathbb{R} \arrow[r, "id"] & \mathbb{R}
\end{tikzcd}$$

It is generally assumed that, as a kind of regularity requirement, the non-holonomic constraint equations can (at least locally) be solved for some of the velocity components; if we consider the case of affine constraints, for simplicity, this means that $C \subset J^1\tau$ is defined by equations of the form

$$\varphi^\mu = \dot{q}^\mu - (B_a^\mu \dot{q}^a + B^\mu) = 0.$$

The starting point for the model in [**6**] was the observation that this strongly suggests assuming that the q^a and q^μ belong to two different sets of coordinates on E, meaning geometrically that E has an extra fibration $\rho : E \to M$ (with q^μ as coordinates for the fibres), where M then is still fibred over $\mathbb{R}$, say $M \xrightarrow{\tau_0} \mathbb{R}$. With this assumption, C becomes simply the image of a section σ of the bundle $J^1\tau \to \rho^* J^1\tau_0$, and the additional elegance of this model is that the functions B^μ, B_a^μ then transform exactly as connection coefficients for a connection on the bundle ρ. In other words, giving a section σ of $J^1\tau \to \rho^* J^1\tau_0$ is the same as giving a connection $\tilde{\sigma}$ on $\rho : E \to M$. The situation is summarised in the diagram above.

What is the projector P in this picture? Observe first that a local basis for the set of vector fields on E which are vertical for τ is now dictated by the connection $\tilde{\sigma}$, i.e. we can put

$$Z_\mu = \frac{\partial}{\partial q^\mu}, \quad Z_a = \frac{\partial}{\partial q^a} + B_a^\mu \frac{\partial}{\partial q^\mu},$$

where the Z_a span (part of) the horizontal distribution for the connection $\tilde{\sigma}$. Let me introduce the notations $\overline{\mathcal{X}}_C = \langle Z_a \rangle$ and $\widetilde{\mathcal{X}}_C = \langle Z_\mu \rangle$, for the modules over $C^\infty(C)$ which these vector fields span. Then, if π_C denotes the projection $C \to E$ as before, and $\mathcal{X}(\pi_C)$ is the notation for the $C^\infty(C)$-module of vector fields along π_C, we have the decomposition

$$\mathcal{X}(\pi_C) = \langle \mathbf{T}_C \rangle \oplus \overline{\mathcal{X}}_C \oplus \widetilde{\mathcal{X}}_C,$$

where

$$\mathbf{T}_C = \frac{\partial}{\partial t} + \dot{q}^a \frac{\partial}{\partial q^a} + (B_a^\mu \dot{q}^a + B^\mu) \frac{\partial}{\partial q^\mu}$$

is a canonically defined element of $\mathcal{X}(\pi_C)$. Observe now that $\mathcal{V}Z_a(\varphi^\mu) = 0$, meaning that the vertical fields $\mathcal{V}Z_a$ on $J^1\tau$ are tangent to C. Then, it becomes obvious that $P : \mathcal{X}(\pi_C) \to \overline{\mathcal{X}}_C$ is simply the horizontal projector of $\tilde{\sigma}$ in this scheme.

A further element of relevance in this model is the fact that there is a vertical endomorphism on C. Indeed,

$$\widehat{S} = \frac{\partial}{\partial \dot{q}^a} \otimes (dq^a - \dot{q}^a dt),$$

which is essentially the canonical vertical endomorphism on $J^1\tau_0$, carries over to the pullback bundle $\rho^* J^1\tau_0$, and thus also to $C = \operatorname{Im}\sigma$. Let then Γ be some constrained SODE on C, its construction out of some original unconstrained dynamics being irrelevant for our present purposes. Putting

$$N = I - (\mathcal{L}_\Gamma \widehat{S})^2 - \Gamma \otimes dt,$$

it was shown in [**6**] that

$$P_H = \tfrac{1}{2}(I - \mathcal{L}_\Gamma \widehat{S} + \Gamma \otimes dt + N)$$

is the horizontal projector of a connection on the bundle $\pi_C : C \to E$. This connection in turn was a key element in developing the theory of symmetries and adjoint symmetries of Γ, as a theory about certain vector fields and 1-forms along the projection π_C.

The purpose here is to extend this theory to the context of general non-holonomic systems. The idea is to take a SODE Γ on C and a projection $P : \mathcal{X}(\pi_C) \to \overline{\mathcal{X}}_C$ as basic data, irrespective of the model which has generated them. The further claim is that generating first integrals of a non-holonomic system Γ through adjoint symmetries, is more transparent and efficient then following the procedure suggested by Giachetta, who worked with generating vector fields which are, however, generally not symmetries of Γ (they were called pseudo-symmetries in some of our previous work, see [**5**], [**7**]). The details for most of what follows can be found in [**8**].

2. General framework and basic constructions

Let me repeat now the basic setup for our current approach, together with the minimal ingredients which are needed for developing a consistent theory of symmetries and adjoint symmetries. We have a fibre bundle $\tau : E \to \mathbb{R}$ and a subbundle C of $J^1\tau$. With π denoting the projection $J^1\tau \to E$, and $\pi_C = \pi|_C$, let $\mathcal{X}(\pi_C) := \{X : C \to TE\}$ denote the $C^\infty(C)$-module of vector fields along π_C, and let $\mathbf{T}_C$ be its canonical element: roughly $\mathbf{T}_C$ is the total time derivative along C.

Definition: *$\overline{\mathcal{X}}_C \subset \mathcal{X}(\pi_C)$, called the set of virtual displacements, consists of all $Z \in \mathcal{X}(\pi_C)$, for which $\mathcal{V}Z$ belongs to $\mathcal{X}(C)$.*

Assume that a projector $P : \mathcal{X}(\pi_C) \to \overline{\mathcal{X}}_C$ is given, with $P(\mathbf{T}_C) = 0$. Then, there exists a complement $\widetilde{\mathcal{X}}_C$, such that

$$\mathcal{X}(\pi_C) = \langle \mathbf{T}_C \rangle \oplus \overline{\mathcal{X}}_C \oplus \widetilde{\mathcal{X}}_C, \tag{1}$$

and we let Q denote the projector on $\widetilde{\mathcal{X}}_C$. Dually, the set of 1-forms along π_C will have a corresponding decomposition

$$\mathcal{X}^*(\pi_C) = \langle dt \rangle \oplus \overline{\mathbb{C}}_C \oplus \widetilde{\mathbb{C}}_C, \tag{2}$$

whereby $\overline{\mathbb{C}}_C \oplus \widetilde{\mathbb{C}}_C$ consists of all contact forms on C (i.e. pullbacks of contact forms on $J^1\tau$ under the injection $\iota : C \hookrightarrow J^1\tau$), regarded as 1-forms along π_C.

Assume finally that Γ is a given SODE on C.

Before proceeding, I shall introduce appropriate coordinate representations for these ingredients. With (t, q^i) denoting the coordinates on E, and $(t, q^i, \dot{q}^i)$ induced coordinates on $J^1\tau$, let (t, q^i, z^a) be coordinates on $C \subset J^1\tau$ and denote the injection $\iota : C \hookrightarrow J^1\tau$ by $\dot{q}^i = \psi^i(t, q, z)$. C can alternatively be defined by some constraint equations $\varphi^\mu(t, q, \dot{q}) = 0$, so that $\varphi^\mu(t, q, \psi(t, q, z)) \equiv 0$, from which it follows that

$$\frac{\partial \psi^i}{\partial z^a} \frac{\partial \varphi^\mu}{\partial \dot{q}^i} \equiv 0. \tag{3}$$

It is clear from these identities that the

$$Z_a = \frac{\partial \psi^i}{\partial z^a} \frac{\partial}{\partial q^i} \tag{4}$$

constitute a local basis for $\overline{\mathcal{X}}_C$, because their vertical lifts $\mathcal{V}Z_a = \partial/\partial z^a$ are tangent to C. Hence, the projector P will be represented by relations of the form

$$P\left(\frac{\partial}{\partial q^j}\right) = P^a_j Z_a, \quad \text{with} \quad \frac{\partial \psi^i}{\partial z^b} P^a_i = \delta^a_b. \tag{5}$$

Denote a local basis for $\widetilde{\mathcal{X}}_C$ by

$$Z_\mu = Z^j_\mu \frac{\partial}{\partial q^j}. \tag{6}$$

Then, one easily verifies that $\partial/\partial q^i$, regarded as vector field along π_C, decomposes as

$$\frac{\partial}{\partial q^i} = P^a_i Z_a + \frac{\partial \varphi^\mu}{\partial \dot{q}^i} Z_\mu. \tag{7}$$

$\mathcal{X}(\pi_C)$ is further spanned by

$$\mathbf{T}_C = \frac{\partial}{\partial t} + \psi^i \frac{\partial}{\partial q^i}. \tag{8}$$

Dually, for the contact forms $\widehat{\theta}^i = \iota^*\theta^i = dq^i - \psi^i dt$, we have

$$\widehat{\theta}^i = \frac{\partial \psi^i}{\partial z^a}\theta^a + Z^i_\mu \eta^\mu, \tag{9}$$

where $\theta^a = P^a_i \widehat{\theta}^i$ and the η^μ span the so-called Chetaev bundle and are defined by

$$\eta^\mu = \iota^* S^*(d\varphi^\mu) = \frac{\partial \varphi^\mu}{\partial \dot{q}^j}\widehat{\theta}^j, \tag{10}$$

S denoting the canonical vertical endomorphism on $J^1\tau$:

$$S = \frac{\partial}{\partial \dot{q}^j} \otimes \theta^j. \tag{11}$$

Finally, the given SODE Γ on C is of the form

$$\Gamma = \frac{\partial}{\partial t} + \psi^i \frac{\partial}{\partial q^i} + f^a \frac{\partial}{\partial z^a}. \tag{12}$$

The given data will now be used in the first place to construct an associated connection on $\pi_C : C \to E$. Let $\widehat{S}$ on C be defined by

$$\forall X \in \mathcal{X}(C): \quad \widehat{S}(X) = \mathcal{V}(P(T\pi_C \circ X)), \quad \rightsquigarrow \quad \widehat{S} = \frac{\partial}{\partial z^a} \otimes \theta^a. \tag{13}$$

So, the availability of a projector P on virtual displacements suffices to have a reduced vertical endomorphism on C. In the second model discussed in the previous section, this would simply be the canonical $\widehat{S}$ coming from the additional fibration of E. Hence, one may expect in the present more general scheme that there will again exist a *natural connection determined by the pair* (Γ, P).

Theorem: *With* $N = I - (\mathcal{L}_\Gamma \widehat{S})^2 - \Gamma \otimes dt$,

$$P_H = \tfrac{1}{2}(I - \mathcal{L}_\Gamma \widehat{S} + \Gamma \otimes dt + N) \tag{14}$$

is the horizontal projector of an Ehresmann connection on $\pi_C : C \to E$.

As was shown in [**8**], writing the local basis of horizontal vector fields as

$$H_0 = \frac{\partial}{\partial t} - \Gamma^a_0 \frac{\partial}{\partial z^a}, \qquad H_i = \frac{\partial}{\partial q^i} - \Gamma^a_i \frac{\partial}{\partial z^a}, \tag{15}$$

the connection coefficients (Γ^a_0, Γ^a_i) for this connection are given by

$$\Gamma^a_0 = -(f^a + \psi^i \Gamma^a_i), \quad \Gamma^a_i = R^a_i - \tfrac{1}{2} P^b_i \frac{\partial \psi^j}{\partial z^b} R^a_j, \tag{16}$$

where auxiliary functions R_i^a have been introduced, defined as

$$R_i^a = \Gamma(P_i^a) + P_j^a \frac{\partial \psi^j}{\partial q^i} - P_i^b \frac{\partial f^a}{\partial z^b}. \tag{17}$$

It may be worth saying that this is not the same connection as the one used in [**3**]. We dare claim that ours is more canonical, in a sense, and this would seem to be illustrated by the fact that it gives rise to significant simplifications in the calculations which will be explained in the next section.

To finish this section, the local basis for vector fields and 1-forms along π_C, which was dictated by the projectors P and Q, can now be used to produce local frames and co-frames on C by the usual procedure of horizontal and vertical lifts. Let us put

$$\Gamma = \mathbf{T}_C{}^H, \quad X_a = Z_a{}^H, \quad X_\mu = Z_\mu{}^H, \quad V_a = \mathcal{V}Z_a = \frac{\partial}{\partial z^a}, \tag{18}$$

and observe that the dual frame is given by

$$dt, \quad \theta^a, \quad \eta^\mu, \quad \eta^a = dz^a + \Gamma_i^a \left(\frac{\partial \psi^i}{\partial z^b} \theta^b + Z_\mu^i \eta^\mu \right) - f^a dt, \tag{19}$$

where no notational distinction is made for forms which are well defined, both as elements of $\mathcal{X}^*(\pi_C)$ and as elements of $\mathcal{X}^*(C)$.

3. Symmetries and adjoint symmetries

The basic ideas underlying the concepts of symmetries and adjoint symmetries of some dynamical system Γ are very simple: they are essentially vector fields and 1-forms which are invariant under the flow of Γ. In practical applications, one will often encounter the somewhat more general concept of 'dynamical symmetry', but dynamical symmetries are equivalent modulo multiples of Γ and each equivalence class contains a representative with zero time-component, which is a symmetry in the above strict sense. For our purposes, there is no loss of generality if we work exclusively with this representative. So, in the present context: $X \in \mathcal{X}(C)$, which can be taken to satisfy $\langle X, dt \rangle = 0$ without loss of generality, is a *symmetry* of Γ if $\mathcal{L}_\Gamma X = 0$. Likewise, $\omega \in \mathcal{X}^*(C)$, with $\langle \Gamma, \omega \rangle = 0$, is an *adjoint symmetry* of Γ if $\mathcal{L}_\Gamma \omega = 0$. The adjective 'adjoint' comes from the property $\langle \mathcal{L}_\Gamma X, \omega \rangle = -\langle X, \mathcal{L}_\Gamma \omega \rangle + \mathcal{L}_\Gamma \langle X, \omega \rangle$ (see later for more details).

To understand the motivation for our approach to the more practical issues of this subject, let us go back for a moment to the case of unconstrained second-order dynamics. Think, for example, of the determination of point symmetries of second-order equations; the so-called determining equations for such symmetries, the way they arise e.g. within Lie's original method, are second-order partial differential equations. The link with a condition such as $\mathcal{L}_\Gamma X = 0$, which in coordinates is a set of first-order pdes, is the following: half of the components of X are fully determined by the other half, and it is the elimination of the 'redundant' components which gives rise to second-order conditions for the remaining ones. Our objective in such a situation is to obtain a coordinate free description of these determining equations. This can be achieved in a very natural way by making use of the available connection, which gives rise to a canonical splitting of every vector field (and 1-form) into a horizontal and vertical part.

I shall now sketch how this all works out in the present context of general non-holonomic systems, once we have arrived at the identification of a reduced dynamics Γ on C and have the projector P at our disposal (together with the natural connection associated to the pair (Γ, P)). The splitting of a vector field by means of the connection identifies the horizontal and vertical lift of some vector fields along π_C, as usual, but here the horizontal part will split again in view of the decomposition induced by the projectors P and Q. Explicitly, every $X \in \mathcal{X}(C)$, with zero time component, has a representation of the form

$$X = \overline{Z}^{H} + \widetilde{Z}^{H} + \mathcal{V}\overline{Y}, \qquad \overline{Z}, \overline{Y} \in \overline{\mathcal{X}}_C,\ \widetilde{Z} \in \widetilde{\mathcal{X}}_C. \tag{20}$$

When computing $\mathcal{L}_\Gamma X$, every term coming from this decomposition will have its own decomposition into three parts and all of these are lifts of certain elements of $\overline{\mathcal{X}}_C$ and $\widetilde{\mathcal{X}}_C$. Hence, studying the decomposition of $\mathcal{L}_\Gamma \overline{Z}^{H}$, $\mathcal{L}_\Gamma \widetilde{Z}^{H}$ and $\mathcal{L}_\Gamma \mathcal{V}\overline{Y}$ necessarily must bring out all intrinsic operations of interest on the set of vector fields along π_C. The same must be true for the dual picture of adjoint symmetries, and having identified all interesting geometrical tools, the final stage in our programme will be to use these tools in studying the subclass of adjoint symmetries which produce first integrals of the constrained dynamics Γ.

As said before, details of this programme can be found in [**8**]. Here is what we get for the decomposition of the Lie derivatives of the composing parts of X:

$$\mathcal{L}_\Gamma \overline{Z}^{H} = (\nabla\overline{Z})^{H} + \mathcal{V}(\Phi\overline{Z}), \tag{21}$$

$$\mathcal{L}_\Gamma \widetilde{Z}^{H} = (\nabla\widetilde{Z})^{H} + \mathcal{V}(\Lambda\widetilde{Z}), \tag{22}$$

$$\mathcal{L}_\Gamma \mathcal{V}\overline{Y} = -\overline{Y}^{H} + \mathcal{V}(P\nabla\overline{Y}). \tag{23}$$

So, the first operation we detect automatically is the *dynamical covariant derivative* $\nabla : \mathcal{X}(\pi_C) \to \mathcal{X}(\pi_C)$; it is a derivation of degree zero on the $C^\infty(C)$-module $\mathcal{X}(\pi_C)$, which is completely determined by the following actions:

$$\nabla \mathbf{T}_C = 0, \quad \nabla\frac{\partial}{\partial q^i} = -H_i(\psi^j)\frac{\partial}{\partial q^j}, \quad \nabla = \Gamma \ \text{ on } C^\infty(C). \tag{24}$$

The vertical parts in (21) and (22) depend tensorially on $\overline{Z}$ and $\widetilde{Z}$, respectively. In other words, what we further discover is the existence of two type (1,1) tensor fields along π_C, of the form:

$$\Phi = \Phi^a_b\,\theta^b \otimes Z_a : \qquad \overline{\mathcal{X}}_C \to \overline{\mathcal{X}}_C, \tag{25}$$

$$\Lambda = \Lambda^a_\mu\,\eta^\mu \otimes Z_a : \qquad \widetilde{\mathcal{X}}_C \to \overline{\mathcal{X}}_C. \tag{26}$$

In fact there is a third such tensor field which is a bit hidden in the horizontal part of (21). Indeed, while $P\nabla$ turns out to be a derivation, $Q\nabla|_{\overline{\mathcal{X}}_C}$ appears to be a tensor. More precisely, one can prove the following.

Lemma: *We have that $\nabla\widetilde{\mathcal{X}}_C \subset \widetilde{\mathcal{X}}_C$, but $\nabla\overline{\mathcal{X}}_C \subset \overline{\mathcal{X}}_C \oplus \widetilde{\mathcal{X}}_C$ and $Q\nabla|_{\overline{\mathcal{X}}_C} : \overline{\mathcal{X}}_C \to \widetilde{\mathcal{X}}_C$ is a tensor, Ψ say, locally of the form $\Psi = \Psi^\mu_a\,\theta^a \otimes Z_\mu$.*

Remark: it is the sum $\Phi + \Lambda + \Psi : \mathcal{X}(\pi_C) \to \overline{\mathcal{X}}_C \oplus \widetilde{\mathcal{X}}_C$ which is the analogue of what is called the *Jacobi endomorphism* in the standard theory of (unconstrained) SODEs on $J^1\tau$.

Putting the results (21-23) back together, and knowing that for a vector field to vanish, its horizontal and vertical part must vanish separately, we get the following result.

Proposition: *For a vector field X on C, of the form $X = \overline{Z}^{H} + \widetilde{Z}^{H} + \mathcal{V}\overline{Y}$, we have*

$$\mathcal{L}_\Gamma X = 0 \quad \Leftrightarrow \quad \begin{cases} \nabla\overline{Z} + \nabla\widetilde{Z} - \overline{Y} = 0, \\ \Phi\overline{Z} + \Lambda\widetilde{Z} + P\nabla\overline{Y} = 0. \end{cases}$$

The condition coming from the horizontal part involves terms in $\overline{\mathcal{X}}_C$ and $\widetilde{\mathcal{X}}_C$, respectively. Hence, taking the results about the range of ∇ into account (see the above Lemma), $\mathcal{L}_\Gamma X = 0$ is further equivalent to:

$$\begin{cases} \overline{Y} = P\nabla\overline{Z}, \\ \nabla\widetilde{Z} + \Psi\overline{Z} = 0, \\ \Phi\overline{Z} + \Lambda\widetilde{Z} + P\nabla(P\nabla\overline{Z}) = 0. \end{cases} \tag{27}$$

It is at this stage that the 'redundant components' of a symmetry show up: to construct symmetries, we need to find solutions $\overline{Z}$ and $\widetilde{Z}$ of the last two equations in (27) and then the remaining part coming from $\overline{Y}$ is determined automatically. So the true determining equations for symmetries of a non-holonomically constrained system are a coupled set of first and second-order partial differential equations for a $Z = \overline{Z} + \widetilde{Z} \in \mathcal{X}(\pi_C)$. Since these two conditions in fact live on disjunct spaces, one can formally take their sum to arrive finally at an equivalent single determining condition

$$P\nabla(P\nabla\overline{Z}) + \nabla\widetilde{Z} + (\Phi + \Psi)\overline{Z} + \Lambda\widetilde{Z} = 0, \tag{28}$$

to which we will from now on refer to as *the symmetry condition*.

Taking the adjoint of a partial differential equation is a process which is well known, for example, in the context of the calculus of variations. One can formally adopt this process with respect to any partial differential operator, and I shall do this here for the dynamical covariant derivative ∇. The general principle then can be formulated as follows: think of ∇ as being the total-time derivative in a process of partial integration, whereby boundary terms are ignored. Then, the adjoint of an equation for a vector field such as (28) is an equation for a 1-form, obtained as follows. Hooking (28) with a 1-form α, use the duality rule

$$\langle \nabla\cdot, \cdot\rangle = \nabla\langle\cdot,\cdot\rangle - \langle\cdot, \nabla\cdot\rangle$$

to transfer ∇ from vector fields to 1-forms, thereby omitting the first term on the right; and for the algebraic terms in the equation, which involve type (1,1) tensor fields such as Φ, Ψ, Λ, it is of course the adjoint linear operator which comes into the picture, according to the rule

$$\langle \Phi Z, \alpha\rangle = \langle Z, \Phi^*\alpha\rangle.$$

In the present case, some further care is needed, because (28) also involves projection operators and in fact injection operators as well. The latter are a bit hidden because it is not really worth introducing extra notations for regarding elements of $\overline{\mathcal{X}}_C$ and $\widetilde{\mathcal{X}}_C$ as elements of the bigger module $\mathcal{X}(\pi_C)$. But the situation changes when one starts taking adjoints. Indeed, the adjoints of the projectors P and Q are injections

$$P^* : \overline{\mathbb{C}}_C \to \mathcal{X}^*(\pi_C), \qquad Q^* : \widetilde{\mathbb{C}}_C \to \mathcal{X}^*(\pi_C),$$

while the adjoint operations of the injections $I : \overline{\mathcal{X}}_C \to \mathcal{X}(\pi_C)$ and $J : \widetilde{\mathcal{X}}_C \to \mathcal{X}(\pi_C)$ are projectors

$$I^* : \mathcal{X}^*(\pi_C) \to \overline{\mathbb{C}}_C, \qquad J^* : \mathcal{X}^*(\pi_C) \to \widetilde{\mathbb{C}}_C,$$

and have to be mentioned explicitly for this reason.

When all such aspects are properly taken into account, the formal adjoint equation of (28) is the following equation for an element $\alpha \in \mathcal{X}^*(\pi_C)$, of the form $\alpha = \overline{\alpha} + \widetilde{\alpha}$,

$$\nabla^2\overline{\alpha} - J^*\nabla\widetilde{\alpha} + (\Phi^* + \Lambda^*)\overline{\alpha} + \Psi^*\widetilde{\alpha} = 0, \tag{29}$$

and will be referred to as *the adjoint symmetry condition.* As in the symmetry case, (29) involves terms which live in the two disjunct spaces $\overline{\mathbb{C}}_C$ and $\widetilde{\mathbb{C}}_C$; it is therefore equivalent to the following coupled system of first and second-order partial differential equations:

$$\begin{cases} \nabla^2\overline{\alpha} + \Phi^*\overline{\alpha} + \Psi^*\widetilde{\alpha} = 0, \\ J^*\nabla\widetilde{\alpha} - \Lambda^*\overline{\alpha} = 0. \end{cases} \tag{30}$$

For the theory to be fully consistent, it should now be possible to establish a correspondence between a 1-form along π_C, satisfying the adjoint symmetry condition, and a 1-form on the constraint submanifold C which is invariant under Γ. The result in this respect is the following (see [**8**]).

Theorem: *$\alpha = \overline{\alpha} + \widetilde{\alpha} \in \mathcal{X}^*(\pi_C)$ is an* adjoint symmetry *if and only if, defining $\overline{\beta} \in \overline{\mathbb{C}}_C$ as $\overline{\beta} = -\nabla\overline{\alpha}$, the 1-form*

$$\omega = \mathcal{V}\overline{\alpha} + \widetilde{\alpha}^H + \overline{\beta}^H \in \mathcal{X}^*(C) \tag{31}$$

is invariant under Γ, i.e. $\mathcal{L}_\Gamma\omega = 0$.

As expected, this theorem illustrates that part of the components of an invariant 1-form ω are determined by the others and the essential equations to be solved are indeed the equations (29), or equivalently (30). Allow me to repeat here that introducing the calculus of forms along the projection π_C is the only way to get a coordinate free handle on the analytical problem which in the end always remains to be solved.

4. The generation of first integrals

Going back to the basic idea of adjoint symmetries, it is obvious how adjoint symmetries can generate first integrals under special circumstances: if $\mathcal{L}_\Gamma\omega = 0$ and $\omega = dF$, then $\Gamma(F) = 0$ (or at least constant). But it is interesting to investigate how the potential exactness of the form ω on C will manifest itself at the level of the determining equations, i.e. when searching for 1-forms along π_C, satisfying the equations (30). The main advantage of such an approach is that it leads to a systematic procedure for the construction of first integrals in stages (see the algorithmic procedure at the end). Besides, in the case of unconstrained systems, this approach gives a better insight also in a rather unexpected side result (see e.g. [**5**] and [**7**]), namely the identification of adjoint symmetries which generate a Lagrangian for the system (which need not be defined as being Lagrangian at the outset). The situation is of course somewhat more complicated in the constrained case. In fact, it is not even clear at this moment how one should define a pair (Γ, P) to be 'of Lagrangian type'.

The standard exterior derivative is an operation which is not defined at the level of the calculus of forms along π_C. The only derivation operator which is canonically available is a vertical exterior derivative d^V; one needs a connection to define a complementary horizontal exterior derivative d^H. For a function F on C, the decomposition of dF in the adapted coframe then reads

$$dF = \mathcal{V}(d^V F) + (d^H F)^H, \tag{32}$$

where

$$d^V F = V_a(F)\,\theta^a, \tag{33}$$
$$d^H F = \Gamma(F)dt + X_a(F)\theta^a + X_\mu(F)\eta^\mu. \tag{34}$$

To study how these exterior derivatives interact with adjoint symmetries, i.e. forms $\alpha = \overline{\alpha} + \widetilde{\alpha} \in \mathcal{X}^*(\pi_C)$ which satisfy the adjoint symmetry condition, one needs to know how they commute with ∇. We shall limit ourselves here to list the following results about such commutators, which can be verified by a coordinate calculation.

$$\nabla d^V F - d^V \nabla F = -I^* d^H F, \tag{35}$$
$$\nabla(J^* d^H F) - J^* d^H \nabla F = \Lambda^* d^V F - \Psi^* J^* d^H F, \tag{36}$$
$$\nabla(I^* d^H F) - I^* d^H \nabla F = \Phi^* d^V F + \Psi^* J^* d^H F. \tag{37}$$

With these formulae at our disposal, every further calculation can be done in a coordinate free way again; at the end of the day, the following main result should pop up.

Theorem: *A 1-form along π_C of the form*

$$\alpha = d^V F + J^* d^H F \tag{38}$$

is an adjoint symmetry of Γ if and only if the function $L = \Gamma(F)$ satisfies the equations

$$J^* d^H L = 0, \qquad I^* d^H L = \nabla d^V L. \tag{39}$$

Now, what does this result have to do with the generation of first integrals? Clearly, $L = 0$ satisfies the conditions (39). Hence, in a systematic search for adjoint symmetries of the form (38), one can be sure that all first integrals will be covered. It turns out that, in practice, most of the time also the converse will hold, i.e. if an adjoint symmetry of such a form can be found, the function $\Gamma(F)$ very likely will be zero. It is instructive to look at the structure of the corresponding invariant form on C: from (31) and (38), using (32), it easily follows that

$$\omega = dF - (d^V L + L\,dt)^H,$$

where the second term has the structure of a Poincaré-Cartan 1-form.

Now what happens if $L = \Gamma(F)$ is not zero? The equations (39) for L, in coordinates, read:

$$X_\mu(L) = 0, \tag{40}$$
$$\Gamma\left(\frac{\partial L}{\partial z^a}\right) = X_a(L) - \frac{\partial L}{\partial z^b}\left(\frac{\partial f^b}{\partial z^a} + \frac{\partial \psi^j}{\partial z^a}\Gamma^b_j\right). \tag{41}$$

As said before, if we were in the context of unconstrained SODEs on $J^1\tau$, we would be looking here at equations which express that L (provided its Hessian is non-degenerate) is a Lagrangian for Γ. To what extent we also face a 'surprise Lagrangian' in the present context of non-holonomic systems, remains to be investigated. Note in passing that this type of result in fact should not come as a total surprise: it generalizes for example the well known property that a point symmetry of a Lagrangian system which is not a Noether symmetry generates an 'alternative Lagrangian'.

Let me finally come to a description of an algorithm which will in principle lead to the generation of all first integrals of practical interest for a general non-holonomic system, and can efficiently replace the method advocated in [**3**]. In practical applications, it is reasonable to expect that the 'forces' f^a of the reduced SODE (12) will depend polynomially on the fibre coordinates z^a. So, starting with a general 1-form along π_C of the form

$$\alpha = \alpha_a \theta^a + \alpha_\mu \eta^\mu,$$

the following procedure can be followed step by step:

- make an 'ansatz' about some polynomial dependence on the z^b of the coefficients $\alpha_a,\ \alpha_\mu$;
- solve the determining equations for adjoint symmetries (30) with this ansatz;
- for each solution, test whether a function F exists such that $\alpha_a = V_a(F)$ (such an F is of course only determined to within an arbitrary function on E);
- if yes, verify whether $\widetilde{\alpha}$ is of the form $J^* d^H \widetilde{F}$, for some $\widetilde{F} = F + f(t,q)$;
- check finally whether $\Gamma(\widetilde{F}) = 0$;
- change the ansatz, if necessary, to obtain more first integrals.

Needless to say, solving the determining equations is the hardest part. But due to the polynomial structure of all functions involved, one will generally obtain an overdetermined system for the coefficients of the polynomials α_a and α_μ, so the problem will be quite tractable after all. Naturally, one will use ones favourite computer algebra package to make life easier. For a simple illustration, we take the paradigm example of non-holonomic systems: the disc which is rolling vertically without slipping. If ψ determines the position of the plane of the disc, φ the position of the disc in its internal rotation, and (x, y, R) are the coordinates of the centre of mass, the constraints are:

$$\dot{x} = (R\cos\psi)\,\dot{\varphi}, \qquad \dot{y} = (R\sin\psi)\,\dot{\varphi}.$$

$\dot{\varphi}$ and $\dot{\psi}$ can be taken as the z-coordinates on C, and the reduced SODE Γ yields the trivial equations

$$\ddot{\varphi} = 0, \qquad \ddot{\psi} = 0.$$

So, we have $f^a = 0$ and one can verify that the components of the projection P are: $P^a_\mu = 0, P^a_b = \delta^a_b$. It follows that the R^a_i as defined by (17) are zero, hence the connection coefficients all vanish. I shall not compute the determining equations (30) in detail, but mainly mention that the simplest ansatz to start the algorithm here is to take the α_a to be independent of the z^b, and the α_μ to be linear. Particular solutions are obtained when one of the α_μ is zero and the other one is $\dot{\psi}$. With the corresponding particular solutions for the α_a, we find two adjoint symmetries which happen to be of the form (38). They lead to two first integrals, given by

$$F_1 = -R\sin\psi\,\dot{\varphi} + x\,\dot{\psi}, \qquad F_2 = R\cos\psi\,\dot{\varphi} + y\,\dot{\psi}.$$

Another interesting particular solution (as reported in [**6**]) is to take both $\alpha_\mu = 0$, with the α_a equal to φ and ψ, respectively. Again, this corresponds to an adjoint symmetry of the form (38). But this time, the function F is not a first integral. Instead, we have $L = \Gamma(F) = \dot{\varphi}^2 + \dot{\psi}^2$, which can indeed be regarded as being a Lagrangian for the reduced SODE Γ.

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