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Sprays and Cartan projective connections

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Abstract. Around 80 years ago, several authors (for instance H. Weyl, T.Y. Thomas, J. Douglas and J.H.C. Whitehead) studied the projective geometry of paths, using the methods of tensor calculus. The principal object of study was a spray, namely a homogeneous second-order differential equation, or more generally a projective equivalence class of sprays. At around the same time, E. Cartan studied the same topic from a different point of view, by imagining a projective space attached to a manifold, or, more generally, attached to a 'manifold of elements'; the infinitesimal 'glue' may be interpreted in modern language as a Cartan projective connection on a principal bundle. This paper describes the geometrical relationship between these two points of view.

1. Introduction

The theory of sprays and Cartan's theory of projective connections are two different approaches to the study of the projective geometry of paths: that is, both the theories consider sets of curves in a manifold without regard to their parametrization, where there is a unique curve through each point in any given direction. In both cases there is a restricted version, the affine case, and also a more general version: in the affine case, the paths are the (unparametrized) geodesics of a symmetric affine connection. The two approaches to the subject are quite different in philosophy, but it is nevertheless possible to describe in some detail the relationships between the two theories. This paper is a report of some joint work on this topic with Mike Crampin (see [**3, 4, 5**]).

The next section below summarise the features of the two theories which are relevant to the present discussion. Section 3 recalls some elementary constructions in projective geometry which which we use to motivate our approach to these theories, and the remaining sections describe how the two theories may be related using these ideas.

2. Sprays, path spaces and Cartan projective connections

The study of path spaces in terms of symmetric affine connections, or more generally of sprays, is due to several authors, in particular H. Weyl [**12**], T.Y. Thomas [**10, 11**], J. Douglas [**6**] and J.H.C. Whitehead [**13**].

Let M be a manifold M (with $\dim M = m$), and let ∇ be a symmetric affine connection on M. Write Γ for the geodesic spray Γ defined by ∇: recall that a spray

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is a vector field on the tangent bundle TM which is second-order and homogeneous. The second-order condition is simply that each integral curve $\widetilde{\gamma}$ of Γ in TM is the lift of a curve γ, a geodesic, in M. The homogeneity condition may be written in the form $[\Delta, \Gamma] = \Gamma$, where Δ is the dilation field on TM. If we take coordinates (x^a) on M, so that $(x^a, \dot{x}^a)$ are the induced coordinates on TM, then

$$\Gamma = \dot{x}^a \frac{\partial}{\partial x^a} - \Gamma^a \frac{\partial}{\partial \dot{x}^a}$$

where the functions Γ^a are quadratic in fibre coordinates $\dot{x}^a$,

$$\Gamma^a = \Gamma^a_{bc}(x)\dot{x}^b\dot{x}^c \,.$$

Two such connections ∇, $\widetilde{\nabla}$ are said to be *projectively equivalent* if they have the same geometric geodesics, possibly with different parametrizations. The corresponding geodesic sprays are related by

$$\Gamma - \widehat{\Gamma} = \alpha\Delta$$

where the function α satisfies $\alpha = \alpha_a(x)\dot{x}^a$.

The fundamental descriptive invariant (Douglas) of a connection ∇ is given by

$$\Pi^a_{bc} = \Gamma^a_{bc} - \frac{1}{m+1}(\Gamma^d_{bd}\delta^a_c + \Gamma^d_{dc}\delta^a_b) \,,$$

and depends only on the equivalence class of connections. Such an equivalence class is called a *restricted path space.*

More generally, geodesics in M may be obtained from a spray Γ on the slit tangent bundle $T^\circ M$ (excluding the zero section) whose coefficients Γ^a satisfy

$$\Delta(\Gamma^a) = 2\Gamma^a \,,$$

so that they are positively homogeneous of degree 2 in the fibre coordinates $\dot{x}^a$ but need not be quadratic. Such a spray gives rise to a connection ∇ on the pull-back vector bundle

$$\tau^{\circ *}_M(TM) \to T^\circ M$$

called the *Berwald connection* of the spray. The connection coefficients of ∇ are given by

$$\Gamma^a_{bc}(x, \dot{x}) = \tfrac{1}{2}\,\frac{\partial^2 \Gamma^a}{\partial \dot{x}^b \partial \dot{x}^c} \,.$$

Once again, a curve γ in M is a geodesic of the connection if the lifted curve $\widetilde{\gamma}$ in $T^\circ M$ is an integral curve of the spray Γ. We shall henceforth restrict consideration to *reversible* sprays, satisfying the property that if $\gamma(t)$ is the geodesic through $x \in M$ with initial tangent vector $v \in T_x M$ then $\gamma(-t)$ is the geodesic with initial tangent vector $-v$. For such a spray the coefficients Γ^a satisfy the additional condition $\Gamma^a(-v) = \Gamma^a(v)$ and so are homogeneous of degree 2 without qualification, rather than just positively homogeneous. Projectively equivalent reversible sprays are related by

$$\Gamma - \widehat{\Gamma} = \alpha\Delta$$

where α satisfies $\Delta(\alpha) = \alpha$ and $\alpha(-v) = -\alpha(v)$, and so is homogeneous of degree 1 in the fibre coordinates, but need not be linear. For these more general sprays, the fundamental descriptive invariant becomes

$$\Pi^a_{bc} = \Gamma^a_{bc} - \frac{1}{m+1}\left(\Gamma^d_{bd}\delta^a_c + \Gamma^d_{dc}\delta^a_b + \dot{x}^a \frac{\partial \Gamma^d_{bc}}{\partial \dot{x}^d}\right)$$

with an additional term which vanishes in the affine case.

A quite different approach to the same problem was taken by Cartan, who in 1924 [**2**] studied a manifold M with projective spaces attached at each point. These projective spaces were 'infinitesimally connected', and a curve in M could be 'developed' into any one of these spaces using the connection. A curve would be a geodesic if its development was a straight line in one of these projective spaces. Each class of torsion-free connections giving the same geodesics had a distinguished representative, called the 'normal' connection. In the same work, Cartan also considered the general second order differential equation (in the case $m = 2$). For this, he used instead of M the 'manifold of elements' — a point together with a direction at that point — in modern terminology, the projective tangent bundle $\mathrm{P}TM$.

In order to describe his projective connection, Cartan considered a reference simplex in each projective space, where the first element of the simplex was always the point where the space was attached to the manifold. Given one such space with its simplex, any other space and simplex gave rise to a transformation matrix; the connection was a specification of such a matrix for any infinitesimally-close space, so that effectively it was a map from tangent vectors to matrices depending upon a local gauge.

A modern, bundle-theoretic view of Cartan connections is given by Sharpe [**9**]. Taking this point of view, Cartan connections may be compared with Ehresmann connections on a principal H-bundle $P \to M$, where H is a Lie group. For an *Ehresmann connection*, the connection form ω takes values in $\mathfrak{h}$, the Lie algebra of H. The kernel of $\omega\colon TP \to \mathfrak{h}$ contains the horizontal vectors of the connection. On the other hand, a *Cartan connection* involves a larger group G, where $\dim G = \dim P$ and $H \subset G$. The connection form ω now takes values in $\mathfrak{g}$, the Lie algebra of G, rather than in $\mathfrak{h}$; in addition, $\omega\colon TP \to \mathfrak{g}$ is required to be an isomorphism on each fibre. The curvature of the connection is $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$, and the torsion is $\kappa(\Omega)$ where $\kappa\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the natural projection. The canonical examples of Cartan connections arise when $P = G$ and when M is the homogeneous space G/H. The connection form ω is then the Maurer-Cartan form of G, and in this case of course the curvature and torsion vanish.

A Cartan connection may be used to develop a curve γ in M to a curve through any point in the homogeneous space G/H in the following way. Take any curve $\bar{\gamma}$ in P which projects to M, so that the lifted curve $\tilde{\bar{\gamma}}$ is a curve in TP. Then $\omega \circ \tilde{\bar{\gamma}}$ is a curve in the Lie algebra $\mathfrak{g}$ which may be developed into a curve through any point of the group G in the usual way, by integrating. The projection of this curve to G/H then does not depend on the particular choice of $\bar{\gamma}$. If the homogeneous space G/H contains straight lines then a curve developing in this way to straight lines is called a geodesic.

A *Cartan projective connection* is a Cartan connection ω where $G = \mathrm{PGL}(m+1)$, the quotient of $\mathrm{GL}(m+1)$ by non-zero multiples of the identity, and where H is the stabilizer of $[1, 0, \dots, 0] \in \mathrm{P}^m = G/H$. Note that $\mathrm{PGL}(m+1) \cong \mathrm{SL}(m+1)$ when m is even, whereas when m is odd $\mathrm{PGL}(m+1)$ may be identified with $\pm$-equivalence classes of matrices having determinant ± 1. In either case, the connection form takes its values in $\mathfrak{sl}(m+1)$, the Lie algebra of $\mathrm{PGL}(m+1)$.

3. Projective geometry

An indication of how spray geometry and Cartan projective connections can be related comes from a basic construction in projective geometry: the projective version of a vertical lift.

The ordinary vertical lift on a tangent bundle is a map $\tau_M^*(TM) \to V\tau_M \subset TTM$ whose restriction at each $x \in M$, a map $T_xM \times T_xM \to TT_xM$, is a particular case of the map

$$V \times V \to TV, \qquad (v, w) \mapsto \gamma'_{v,w}(0)$$

for an arbitrary vector space V: here, $\gamma'_{v,w}(0)$ denotes the tangent vector at $v \in V$ defined by the curve $\gamma_{v,w} : t \mapsto v + tw$ at $t = 0$. The existence of this map is simply a manifestation of the fact that the tangent bundle $\tau_V : TV \to V$ is trivial, and of course the map is just the inverse of the canonical trivialisation $TV \to V \times V$.

Now consider, instead of the vector space V, its projective space $\mathrm{P}V$. We wish to find a bundle $\mathrm{P}\mathcal{W}V \to \mathrm{P}V$ with a canonical trivialization $\mathrm{P}\mathcal{W}V \to \mathrm{P}V \times \mathrm{P}V$ whose inverse may be taken to define a 'projective vertical lift'. To construct this bundle, let $\varphi_s : V_\circ \to V_\circ$ be given by $\varphi_s(v) = sv$ for $s \in \mathbf{R}_\circ$, where $V_\circ = V - \{0\}$ and $\mathbf{R}_\circ = \mathbf{R} - \{0\}$, so that $\mathrm{P}V = V_\circ/\varphi_s$. The tangent map $\varphi_{s*} : TV_\circ \to TV_\circ$ may then be used to define a manifold $\mathcal{W}V = TV_\circ/\varphi_{s*}$. To see that this manifold is fibred over $\mathrm{P}V$, take a general element $(v, w) \in TV_\circ = V_\circ \times V$; a general element of $\mathcal{W}V$ is then $[v, w]$, where the brackets denote the equivalence class under the relation $(v, w) \sim (sv, sw)$ defined by φ_{s*}. The projection $\mathcal{W}V \to \mathrm{P}V$ is then $[v, w] \mapsto [v]$, and it has the structure of a vector bundle: it is in fact the Lie algebroid canonically associated to the principal $\mathbf{R}_\circ$-bundle $V_\circ \to \mathrm{P}V$. The projective bundle $\mathrm{P}\mathcal{W}V \to \mathrm{P}V$ then has a general element $[\![v, w]\!]$, where the double brackets denote the equivalence class under the relation $[v, w] \sim [v, sw]$. The required trivialization is thus

$$\mathrm{P}\mathcal{W}V \to \mathrm{P}V \times \mathrm{P}V, \qquad [\![v, w]\!] \to ([v], [w]).$$

Geometrically, we may picture a family of projective spaces $\mathrm{P}\mathcal{W}_{[v]}V$ attached to each point $[v] \in \mathrm{P}V$, and the projective vertical lift from $\mathrm{P}V$ to any particular $\mathrm{P}\mathcal{W}_{[v]}V$ is $[w] \mapsto [\![v, w]\!]$.

The symmetries of this structure are also of interest. The symmetries of $\mathrm{P}V$, namely the diffeomorphisms of $\mathrm{P}V$ mapping lines to lines, are elements of the projective general linear group $\mathrm{PGL}(V)$. Thus every symmetry of $\mathrm{P}V$ is an equivalence class of symmetries of V; the infinitesimal version of this remark is that the infinitesimal projective symmetries of $\mathrm{P}V$ are obtained by projecting the linear vector fields from V onto $\mathrm{P}V$, where two linear vector fields give rise to the same projection if they differ by a constant multiple of the radial vector field. There are, of course, other diffeomorphisms of V mapping (affine) lines to lines, namely the translations, and these do not project to $\mathrm{P}V$: surprisingly, though, we shall see that the infinitesimal translations on V do have a close relationship to the structure of projective space.

4. The relationship between the theories

In [**3, 4**] we show that associated with every manifold M there are two principal bundles. In each case the group of the bundle is a subgroup of $\mathrm{PGL}(m + 1)$. In summary, the constructions are as follows.

In the affine case, the *Cartan bundle* $\mathcal{C}M \to M$ has group $\mathrm{H}_{m+1} \subset \mathrm{PGL}(m+1)$; if $h \in \mathrm{H}_{m+1}$ then

$$h = \varepsilon_m \begin{pmatrix} h_0^0 & h_1^0 & \dots & h_m^0 \\ 0 & h_1^1 & \dots & h_m^1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & h_1^m & \dots & h_m^m \end{pmatrix} \qquad \begin{cases} m \text{ even:} & \varepsilon_m = 1, \quad \det h = 1 \\ m \text{ odd:} & \varepsilon_m = \pm 1, \quad \det h = \pm 1 \end{cases}$$

A projective equivalence class of symmetric affine connections on M gives rise to a family of torsion-free Cartan connections on $\mathcal{C}M \to M$ with the same geodesics; one of these is the distinguished normal Cartan connection ω.

In the general case, the *projective Cartan bundle* $\mathcal{PC}M \to \mathrm{P}TM$ has group $\mathrm{K}_{m+1} \subset \mathrm{PGL}(m+1)$; if $k \in \mathrm{K}_{m+1}$ then

$$k = \varepsilon_m \begin{pmatrix} k_0^0 & k_1^0 & k_2^0 & \dots & k_m^0 \\ 0 & k_1^1 & k_2^1 & \dots & k_m^1 \\ 0 & 0 & k_2^2 & \dots & k_m^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & k_2^m & \dots & k_m^m \end{pmatrix} \qquad \begin{cases} m \text{ even:} & \varepsilon_m = 1, \quad \det k = 1 \\ m \text{ odd:} & \varepsilon_m = \pm 1, \quad \det k = \pm 1 \end{cases}$$

A projective equivalence class of reversible sprays on $T^\circ M$ gives rise to a family of Cartan connections on $\mathcal{PC}M \to \mathrm{P}TM$ with the same geodesics; again there is a distinguished normal Cartan connection ω.

Thus, in the affine case, we may construct two bundles

$$\mathcal{C}M \to M, \qquad \mathcal{PC}M \to \mathrm{P}TM\,;$$

it turns out, however, that the total spaces may be identified:

$$\mathcal{C}M \cong \mathcal{PC}M\,.$$

Similarly, from a symmetric affine connection ∇ on M and its geodesic spray Γ restricted to $T^\circ M$, we may construct two normal Cartan connection forms

$$\omega\colon T(\mathcal{C}M) \to \mathfrak{sl}(m+1), \qquad \omega\colon T(\mathcal{PC}M) \to \mathfrak{sl}(m+1)\,,$$

but after identifying $\mathcal{C}M$ and $\mathcal{PC}M$ we find, as the notation suggests, that the two forms are identical.

The remainder of this paper gives an outline of the methods used to obtain these results.

5. The construction in the affine case

The affine construction is modelled on the canonical flat structure of projective space itself, $V_\circ \to \mathrm{P}V$, described earlier: it involves a manifold with an additional dimension.

An extra dimension was used by T. Y. Thomas, and by J. H. C. Whitehead; also, more recently, by T. N. Bailey, M. G. Eastwood & A. R. Gover [**1**] and by C. Roberts [**7**]. Our approach uses, in particular, the (unoriented) *volume bundle*

$$\mathcal{V}M = \{\pm\theta \colon \theta \in \textstyle\bigwedge^m M, \theta \neq 0\}\,.$$

This is a principal $\mathbf{R}_+$-bundle with action $\mu_s\colon \mathcal{V}M \to \mathcal{V}M$ given by

$$[\pm\theta] \quad \mapsto \quad s^{1/(m+1)}[\pm\theta]$$

(note that this is similar to, but slightly different from, the formulation given by Roberts). We take coordinates x^α (with $\alpha = 0, 1, \ldots, m$) on $\mathcal{V}M$, where

$$[\pm\theta] = \left(x^0[\pm\theta]\right)^{m+1} [\pm dx^1 \wedge \ldots \wedge dx^m] \,,$$

so that x^0 is the $(m{+}1)$-th root of (the absolute value of) the usual volume coordinate on $\bigwedge^m M$. The volume bundle has as a fundamental vector field

$$\Upsilon = x^0 \frac{\partial}{\partial x^0} \,;$$

it also has a canonical odd scalar density

$$|d\Theta| = \pm(m+1)(x^0)^m dx^0 \wedge dx^1 \wedge \ldots \wedge dx^m$$

which is derived from the tautological m-form Θ on $\bigwedge^m M$.

The action μ_s of $\mathbf{R}_+$ on $\mathcal{V}M$ gives rise to the derived action $\mu_{s*} \colon T\mathcal{V}M \to T\mathcal{V}M$. We introduce the *Cartan algebroid* as the quotient manifold

$$(\mathcal{W}M = T\mathcal{V}M/\mu_{s*}) \to M$$

with coordinates

$$x^a, \quad w = (x^0)^{-1}\dot{x}^0, \quad \dot{x}^a \,;$$

the vector bundle $\mathcal{W}M \to M$ is in fact the Lie algebroid naturally associated with the principal bundle $\mathcal{V}M \to M$. The Cartan algebroid has a distinguished global section e_0, given in coordinates by $w = 1$, $\dot{x}^a = 0$, which is the projection of the fundamental vector field Υ on $\mathcal{V}M$. The projectivized Cartan algebroid

$$\mathrm{P}\mathcal{W}M = \mathcal{W}^\circ M/(\lambda \neq 0)$$

has m-dimensional projective spaces as fibres, each with a distinguished point $[e_0]$; these are the projective spaces used by Cartan. The analogy between $\mathrm{P}\mathcal{W}M \to M$ and the canonical flat model $\mathrm{P}\mathcal{W}V \to \mathrm{P}V$ described earlier should be clear.

To obtain the principal bundle on which the Cartan connection will be defined, we introduce the *simplex bundle* of the Cartan algebroid, $\mathcal{S}_\mathcal{W}M \to M$, defined by

$$\mathcal{S}_\mathcal{W}M = \{[\xi_0, \xi_1, \ldots, \xi_m]\} \quad \text{where } \{\xi_0, \xi_1, \ldots, \xi_m\} \text{ is a basis for } \mathcal{W}M.$$

Each element of $\mathcal{S}_\mathcal{W}M$ is therefore a reference simplex for the corresponding fibre of $\mathrm{P}\mathcal{W}M \to M$. There is a standard right action of $\mathrm{PGL}(m+1)$ on the fibres of the simplex bundle $\mathcal{S}_\mathcal{W}M \to M$, given as usual by matrix multiplication. The *Cartan bundle* $\mathcal{C}M \to M$, with $\mathcal{C}M \subset \mathcal{S}_\mathcal{W}M$, is defined by

$$\mathcal{C}M = \{[\xi_0, \xi_1, \ldots, \xi_m]\} \quad \text{where} \quad \begin{cases} \{\xi_0, \xi_1, \ldots, \xi_m\} \text{ is a basis for } \mathcal{W}_pM \\ \xi_0 = \lambda e_0 \end{cases}$$

Each element of the Cartan bundle is therefore a reference simplex for the corresponding fibre of $\mathrm{P}\mathcal{W}M \to M$ with the additional property that its first point $[\xi_0]$ is the distinguished point $[e_0]$. The standard right action of $\mathrm{PGL}(m+1)$ on the fibres of the simplex bundle restricts to a right action of H_{m+1} on the fibres of the Cartan bundle $\mathcal{C}M \to M$.

Roberts [**7**] defined a *TW* ('Thomas-Whitehead') connection $\widetilde{\nabla}$ to be a symmetric affine connection on $\mathcal{V}M$ with the additional properties

$$\widetilde{\nabla}\Upsilon = \mathrm{id}, \qquad [\Upsilon, \widetilde{\nabla}_X Y] = \widetilde{\nabla}_{[\Upsilon,X]}Y + \widetilde{\nabla}_X[\Upsilon, Y]$$

(the exact definition given in [**7**] involves a factor of $-1/(m+1)$ as a different definition of the volume bundle is used in that reference). Given this, each Ehresmann connection ϑ on $\nu\colon \mathcal{V}M \to M$ maps a symmetric TW-connection $\widetilde{\nabla}$ on $\mathcal{V}M$ to a symmetric connection ∇^{ϑ} on M by

$$\nabla^{\vartheta}_X Y = \nu_*\left(\widetilde{\nabla}_{X^{\mathrm{h}}} Y^{\mathrm{h}}\right)$$

(where X^{h}, Y^{h} are the horizontal lifts of X, Y by ϑ). Then different connections ϑ, ϑ' map $\widetilde{\nabla}$ to projectively equivalent affine connections ∇^{ϑ}, $\nabla^{\vartheta'}$ on M, and every affine connection in the projective equivalence class may be obtained in this way. We may then see that there is a unique TW-connection $\widetilde{\nabla}$ mapping to the projective equivalence class of ∇ satsifying the additional conditions

$$\operatorname{tr}(\widetilde{\nabla}) = 0, \qquad \operatorname{Ric}(\widetilde{\nabla}) = 0\,,$$

where the trace is taken with respect to the odd scalar density $|d\Theta|$; this is the *normal* TW-connection.

The normal TW-connection $\widetilde{\nabla}$ is, of course, a particular symmetric affine connection on $\mathcal{V}M$; in the usual way, it has an Ehresmann connection form $\widetilde{\omega}$ on the frame bundle $\mathcal{F}(\mathcal{V}M) \to \mathcal{V}M$,

$$\widetilde{\omega}\colon T(\mathcal{F}(\mathcal{V}M)) \to \mathfrak{gl}(m+1)\,.$$

This connection form $\widetilde{\omega}$ then gives rise to another Ehresmann connection form $\widehat{\omega}$ on the simplex bundle $\mathcal{S}_{\mathcal{W}}M \to M$,

$$\widehat{\omega}\colon T(\mathcal{S}_{\mathcal{W}}M) \to \mathfrak{sl}(m+1)\,,$$

by

$$\widehat{\omega}_{\xi}(v) = p_*(\widetilde{\omega}_{\zeta}(w))$$

where

$$\begin{aligned}
&\zeta \in \mathcal{F}(\mathcal{V}M) \text{ maps to } \xi \in \mathcal{S}_{\mathcal{W}}M\\
&w \in T_{\zeta}\mathcal{F}(\mathcal{V}M) \text{ maps to } v \in T_{\xi}\mathcal{S}_{\mathcal{W}}M\\
&p\colon \mathrm{GL}(m+1) \to \mathrm{PGL}(m+1)\\
&p_*\colon \mathfrak{gl}(m+1) \to \mathfrak{sl}(m+1)\,;
\end{aligned}$$

the conditions on the TW-connection ensure that $\widehat{\omega}$ is well-defined. We also find that $\ker\widehat{\omega} \cap T(\mathcal{C}M) \subset T(\mathcal{S}_{\mathcal{W}}M)$ contains only zero vectors, and so the restriction

$$\omega = \widehat{\omega}|_{T(\mathcal{C}M)}$$

is a Cartan connection (Sharpe [**9**]); this is the *normal Cartan projective connection* corresponding to the projective equivalence class of sprays. The unparametrized geodesics of this connection (in the sense of Cartan) are precisely the geodesics, in the usual sense, of the projective equivalence class of symmetric affine connections.

6. The construction in the general case

A general spray on the slit tangent bundle $T^{\circ}M$ is associated with a Berwald connection ∇ on the pull-back bundle $\tau^{\circ}_M TM$. This suggests consideration of the pull-back bundles

$$\pi_M^*(\mathcal{W}M) \to \mathrm{P}TM, \qquad \pi_M^*(\mathrm{P}\mathcal{W}M) \to \mathrm{P}TM$$

where $\pi_M\colon \mathrm{P}TM \to M$. Elements of $\pi_M^*(\mathrm{P}\mathcal{W}M)$ are pairs $([v],[\xi])$ where $v \in T_pM$ and $\xi \in \mathcal{W}_pM$. Then each fibre of $\pi_M^*(\mathrm{P}\mathcal{W}M) \to \mathrm{P}TM$ has a distinguished point

$$([v],[e_0])$$

as before, but now it also has a distinguished line

$$\{([v],[\xi])\}, \qquad \rho(\xi)=\mu v$$

where $\rho\colon \mathcal{W}M\to TM$. Note that $\rho(e_0)=0$, so that the distinguished point lies on the distingushed line. This line is the inverse image of the diagonal of $\pi_M^*(\mathrm{P}TM)$, and is therefore a geometric representation of the total time derivative operator

$$d/dt\colon TM\to\tau_M^*(TM), \qquad v\mapsto(v,v)\,.$$

Following the same philosophy as before, we consider the pull-back of the simplex bundle $\pi_M^*(\mathcal{S}_{\mathcal{W}}M)\to\mathrm{P}TM$. This has elements

$$([v],[\xi_0,\xi_1,\ldots,\xi_m]) \qquad \text{where } v\in T_pM \text{ and } \{\xi_0,\xi_1,\ldots,\xi_m\} \text{ is a basis for } \mathcal{W}_pM,$$

and as before there is a right action of $\mathrm{PGL}(m+1)$ on its fibres, given by matrix multiplication. The projective Cartan bundle $\mathcal{PC}M\to\mathrm{P}TM$ is the sub-bundle $\mathcal{PC}M\subset\pi_M^*(\mathcal{S}_{\mathcal{W}}M)$ given by

$$\mathcal{PC}M=\{([v],[\xi_0,\xi_1,\ldots,\xi_m])\}$$

where

$$\begin{cases} v\in T_pM \\ \{\xi_0,\xi_1,\ldots,\xi_m\} \text{ is a basis for } \mathcal{W}_pM \\ \xi_0=\lambda e_0 \\ \rho(\xi_1)=\mu v \end{cases}$$

Here, each element is a reference simplex for a fibre of $\pi_M^*(\mathrm{P}\mathcal{W}M)\to\mathrm{P}TM$, where the first point $[\xi_0]$ is the distinguished point $[e_0]$, and where the second point $[\xi_1]$ is some other point in the distinguished line. The standard right action of $\mathrm{PGL}(m+1)$ on the fibres of $\pi_M^*(\mathcal{S}_{\mathcal{W}}M)\to\mathrm{P}TM$ restricts to a right action of K_{m+1} on the fibres of the projective Cartan bundle $\mathcal{PC}M\to\mathrm{P}TM$.

A spray $\widetilde{\Gamma}$ on $T^\circ\mathcal{V}M$ defines a Berwald connection $\widetilde{\nabla}$ on the pull-back vector bundle

$$\tau_{\mathcal{V}M}^{\circ *}(T\mathcal{V}M)\to T^\circ\mathcal{V}M$$

in the usual way. We now say that $\widetilde{\nabla}$ is a BTW-connection if

$$\widetilde{\nabla}(\Upsilon\circ\tau_{\mathcal{V}M}^\circ)=\tau_{\mathcal{V}M*}^\circ$$

and if, in addition,

$$\begin{aligned} [\Upsilon^{\mathrm{c}},\widetilde{\nabla}_X Z] &= \widetilde{\nabla}_{[\Upsilon^{\mathrm{c}},X]}Z+\widetilde{\nabla}_X[\Upsilon^{\mathrm{c}},Z] \\ \text{and}\quad [\Upsilon^{\mathrm{v}},\widetilde{\nabla}_X Z] &= \widetilde{\nabla}_{[\Upsilon^{\mathrm{v}},X]}Z+\widetilde{\nabla}_X[\Upsilon^{\mathrm{v}},Z]\,. \end{aligned}$$

Here,

$$X\in\mathfrak{X}(T^\circ\mathcal{V}M), \quad Z\in\mathfrak{X}(\tau_{\mathcal{V}M}^\circ)\,,$$

$$\Upsilon^{\mathrm{c}},\Upsilon^{\mathrm{v}} \text{ are the complete and vertical lifts of } \Upsilon\,,$$

and the brackets $[\Upsilon^{\mathrm{c}},Z]$, $[\Upsilon^{\mathrm{v}},Z]$ are defined as commutators, using the fact that Υ^{c} and Υ^{v} are projectable to $\mathcal{V}M$. The spray $\widetilde{\Gamma}$ is said to be a BTW-spray if its Berwald connection $\widetilde{\nabla}$ is a BTW-connection; the coordinate representation of the spray is

$$\widetilde{\Gamma}=\dot{x}^\alpha\frac{\partial}{\partial x^\alpha}-\left(G^a+(x^0)^{-1}\dot{x}^0\dot{x}^a\right)\frac{\partial}{\partial\dot{x}^a}-x^0G^0\frac{\partial}{\partial\dot{x}^0}$$

where G^a, G^0 are functions pulled back from $T^\circ M$.

The relationship between TW-connections and equivalence classes of symmetric affine connections may be generalized in the following way, using the maps

$$T^\circ\mathcal{V}M \to \mathcal{W}^\circ M \xrightarrow{\rho} T^\circ M\,.$$

First, any BTW-spray $\widetilde{\Gamma}$ on $T^\circ\mathcal{V}M$ is projectable to a vector field $\bar{\Gamma}$ on $\mathcal{W}^\circ M$, represented in coordinates by

$$\bar{\Gamma} = \dot{x}^a\frac{\partial}{\partial x^a} - (G^a + w\dot{x}^a)\frac{\partial}{\partial \dot{x}^a} - (G^0 + w^2)\frac{\partial}{\partial w}\,.$$

Next, we consider sections $\sigma\colon T^\circ M \to \mathcal{W}^\circ M$ of ρ. Such a section is homogeneous if

$$\sigma(\lambda v) = \lambda\sigma(v)\,;$$

an example of such a homogeneous section would be the linear section defined by a connection ϑ on $\mathcal{V}M \to M$. Then any homogeneous section σ maps a BTW-spray $\widetilde{\Gamma}$ to a spray Γ^σ on $T^\circ M$ by

$$\Gamma^\sigma = \rho_*\big(\bar{\Gamma}\big|_\sigma\big)\,.$$

Different sections σ, σ' map $\widetilde{\Gamma}$ to projectively equivalent sprays Γ^σ, $\Gamma^{\sigma'}$, and there is a unique BTW-spray $\widetilde{\Gamma}$ mapping to the projective equivalence class of Γ whose Berwald connection $\widetilde{\nabla}$ satsifies

$$\operatorname{tr}(\widetilde{\nabla}) = 0, \qquad \operatorname{Ric}(\widetilde{\nabla}) = 0$$

(where the trace is taken with respect to the odd scalar density $|d\Theta|$); this is the *normal* BTW-connection, and the spray $\widetilde{\Gamma}$ is the normal BTW-spray.

As in the affine case, the normal BTW-connection $\widetilde{\nabla}$ gives rise to an Ehresmann connection form $\widetilde{\omega}$ on the pull-back of the frame bundle $\tau^{\circ *}_{\mathcal{V}M}(\mathcal{F}(\mathcal{V}M)) \to T^\circ\mathcal{V}M$,

$$\widetilde{\omega}\colon T(\tau^{\circ *}_{\mathcal{V}M}(\mathcal{F}(\mathcal{V}M))) \to \mathfrak{gl}(m+1)\,.$$

This connection form $\widetilde{\omega}$ then maps to an Ehresmann connection form $\widehat{\omega}$ on the pull-back of the simplex bundle $\pi^*_M(\mathcal{S}_\mathcal{W}M) \to \mathrm{P}TM$,

$$\widehat{\omega}\colon T(\pi^*_M(\mathcal{S}_\mathcal{W}M)) \to \mathfrak{sl}(m+1)\,,$$

by

$$\widehat{\omega}_\xi(v) = p_*(\widetilde{\omega}_\zeta(w))\,,$$

where

$$\begin{aligned}
&\zeta \in \tau^{\circ *}_{\mathcal{V}M}(\mathcal{F}(\mathcal{V}M)) \text{ maps to } \xi \in \pi^*_M(\mathcal{S}_\mathcal{W}M)\\
&w \in T_\zeta\tau^{\circ *}_{\mathcal{V}M}(\mathcal{F}(\mathcal{V}M)) \text{ maps to } v \in T_\xi\pi^*_M(\mathcal{S}_\mathcal{W}M)\\
&p\colon \mathrm{GL}(m+1) \to \mathrm{PGL}(m+1)\\
&p_*\colon \mathfrak{gl}(m+1) \to \mathfrak{sl}(m+1)\,;
\end{aligned}$$

the conditions on the BTW-connection ensure that $\widehat{\omega}$ is well-defined. We also find that $\ker\widehat{\omega} \cap T(\mathcal{PC}M) \subset T(\pi^*_M(\mathcal{S}_\mathcal{W}M))$ contains only zero vectors, and so the restriction

$$\omega = \widehat{\omega}|_{T(\mathcal{PC}M)}$$

is a Cartan connection (Sharpe **[9]**). Once again, the geodesics of this Cartan connection are the same as the geodesics of the projective equivalence class of sprays on $T^\circ M$.

We now recall that $\mathcal{C}M = \{[\xi_0, \xi_1, \ldots, \xi_m]\}$ where

$$\begin{cases} \{\xi_0, \xi_1, \ldots, \xi_m\} \text{ is a basis for } \mathcal{W}_p M \\ \xi_0 = \lambda e_0 \,, \end{cases}$$

whereas $\mathcal{PC}M = \{([v], [\xi_0, \xi_1, \ldots, \xi_m])\}$ where

$$\begin{cases} v \in T_p M \\ \{\xi_0, \xi_1, \ldots, \xi_m\} \text{ is a basis for } \mathcal{W}_p M \\ \xi_0 = \lambda e_0 \\ \rho(\xi_1) = \mu v \,, \end{cases}$$

so that we may identify $\mathcal{C}M \leftrightarrow \mathcal{PC}M$ by

$$[\xi_0, \xi_1, \ldots, \xi_m] \leftrightarrow ([\rho(\xi_1)], [\xi_0, \xi_1, \ldots, \xi_m]) \,.$$

It is instructive to write down a formula for the Cartan connection in coordinates. For a general equivalence class of sprays on $T^\circ M$, we use the projective Cartan bundle $\mathcal{PC}M \to \mathrm{P}TM$. A gauge is local section of this bundle: with a suitable choice, the pull-back of ω by the gauge section is represented (in coordinates x^a, y^i on $\mathrm{P}TM$ defined by the gauge, with $i = 2, \ldots, m$ and $y^i = (\dot{x}^1)^{-1}\dot{x}^i$) as the matrix

$$\begin{pmatrix} 0 & \left(\alpha_{1c} + y^k \alpha_{kc}\right) dx^c & \alpha_{jc} dx^c \\ dx^1 & \left(\Pi^1_{1c} + y^k \Pi^1_{kc}\right) dx^c & \Pi^1_{jc} dx^c \\ dx^i - y^i dx^1 & \omega^i_1 & \left(\Pi^i_{jc} - y^i \Pi^1_{jc}\right) dx^c \end{pmatrix}$$

where $\omega^i_1 = \left(\Pi^i_{1c} - y^i \Pi^1_{1c} + y^k(\Pi^i_{kc} - y^i \Pi^1_{kc})\right) dx^c + dy^i$, and α_{bc} is related to the curvature of the corresponding Berwald connection. If Γ is an affine spray (in other words, if the projective class of sprays contains an affine representative) then we may use instead the coarser fibration $\mathcal{C}M \to M$: with a suitable choice of gauge, ω is now represented as the matrix

$$\begin{pmatrix} 0 & \alpha_{bc} dx^c \\ dx^a & \Pi^a_{bc} dx^c \end{pmatrix}$$

These connections in gauged form are essentially the same as those given by Cartan [**2**].

7. Symmetries

Given any manifold M with reversible spray Γ on $T^\circ M$, an *infinitesimal affine symmetry* of Γ is a vector field X on M such that $[X^{\mathrm{c}}, \Gamma] = 0$; an *infinitesimal projective symmetry* satisfies, instead, $[X^{\mathrm{c}}, \Gamma] = f\Delta$ for some function f homogeneous of degree 1. As $[X^{\mathrm{c}}, \Delta] = 0$, we see that an infinitesimal projective symmetry of Γ is also such a symmetry of any projectively related spray. So an infinitesimal affine symmetry thus preserves the integral curves of the given spray, and hence the geodesics of the spray up to affine reparametrization, whereas an infinitesimal projective symmetry preserves only the projective class of sprays and the geometric geodesics.

In the case of projective space itself, the infinitesimal projective symmetries are the vector fields preserving straight lines; we have observed that these are in bijective correspondence with equivalence classes of linear vector fields on the generating vector space, modulo constant multiples of the radial vector field. The linear vector fields are themselves infinitesimal affine symmetries of the canonical flat connection on the vector space. We find that a similar situation arises in the more general case of a manifold M with a reversible spray Γ.

Let $\widetilde{\Gamma}$ be the BTW-spray of Γ, so that $\widetilde{\Gamma}$ is a spray on $T^\circ\mathcal{V}M$, and let X be an infinitesimal affine symmetry of $\widetilde{\Gamma}$. If, in coordinates, $X = X^\alpha\,\partial/\partial x^a$ we find from the BTW-spray conditions that

$$X^0 = A^0 x^0 + B^0, \qquad X^a = A^a + (x^0)^{-1}B^a$$

where the functions A^α, B^α are pulled back from M. Indeed, the Lie algebra $\mathfrak{A}$ of infinitesimal affine symmetries of $\widetilde{\Gamma}$ may be written as a (vector space) direct sum $\mathfrak{a}\oplus\mathfrak{b}$, where the elements $A\in\mathfrak{a}$ satisfy $[\Upsilon, A] = 0$ and the elements $B\in\mathfrak{b}$ satisfy $[\Upsilon, B] = -B$; $\mathfrak{a}$ is a subalgebra of $\mathfrak{A}$, whereas $\mathfrak{b}$ is an Abelian ideal. In the case of projective space, $\mathfrak{a}$ contains the linear vector fields, and $\mathfrak{b}$ the translations. Further coordinate calculations now show that $\Upsilon\in\mathfrak{a}$ is always an infinitesimal affine symmetry, and that any other element of $\mathfrak{a}$ projects to an infinitesimal projective symmetry of (the projective class of) Γ. In this way, we find a bijection between equivalence classes of elements of $\mathfrak{a}$, modulo constant multiples of Υ, and infinitesimal projective symmetries of Γ, just as in the case of projective space.

The rôle of the 'translations' $\mathfrak{b}$ is also quite surprising. These, of course, do not project to M. Nevertheless, we find as another consequence of the BTW-spray conditions that $\widetilde{\nabla}(B\circ\tau^\circ_{\mathcal{V}M}) = 0$, so that each element $B\in\mathfrak{b}$ gives rise to an integral section of the horizontal distribution determined by $\widetilde{\nabla}$. If $\dim\mathfrak{b} = m+1$, the horizontal distribution is therefore Frobenius integrable, and so the curvature of $\widetilde{\nabla}$ must vanish; thus the original spray Γ must be projectively flat, and in these circumstances the projective structure of M must locally be that of projective space.

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References

[1] Bailey, T.N., M.G. Eastwood and A.R. GoverThomas's structure bundle for conformal, projective and related structures Rocky Mountain J. Math. 2419941191–1217

[2] Cartan, E, Sur les variétés à connexion projective Bull. Soc. Math. France521924205–241

[3] Crampin, M. and D. J. Saunders, On projective connections: the affine case Ghent University 2004 (available at `http://maphyast.ugent.be`)

[4] Crampin, M. and D. J. Saunders, On projective connections: the general case Ghent University 2004 (available at `http:/maphyast.ugent.be`)

[5] Crampin, M. and D. J. Saunders, Affine and projective transformations of Berwald connections Ghent University 2004 (available at `http://maphyast.ugent.be`)

[6] Douglas, J, The general geometry of paths Ann. Math. 291928143–168

[7] Roberts, C, The projective connections of T.Y. Thomas and J.H.C. Whitehead applied to invariant connections Diff. Geom. Appl. 51995237–255

[8] Saunders, D. J, Sprays and Cartan projective connections Proceedings of the International Workshop on Global Ananlysis, Ankara, 2004 AIP Conference Proceedings Series – to appear

[9] Sharpe, R.W, Differential Geometry: Cartan's Generalization of Klein's Erlangen Program; Springer 1997

[10] Thomas, T.Y On the projective and equi-projective geometries of paths Proc. Nat. Acad. Sci. 11 1925 199–203

[11] Thomas, T.Y, A projective theory of affinely connected manifolds Math. Zeit. 25 1926 723–733

[12] Weyl, H, Zur Infinitesimalgeometry; Einordnung der projektiven und der konformen Auffassung Gött. Nach. 1921 99-112

[13] Whitehead, J.H.C The representation of projective spaces Ann. Math. 32 1931 327–360

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