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On warped product manifolds

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Abstract. Let (M,g) be a connected n-dimensional, $n \geqslant 3$, semi-Riemannian manifold of class C^∞. The manifold (M,g) is said to be pseudosymmetric if its curvature tensor R satisfies at every point of M the condition $R \cdot R = L_R Q(g,R)$. In this paper, we present a review which is related to the results by Ryszard Deszcz and his collaborators on Warped Product Manifolds. Firstly, we are going to give precise definitions of the symbols used, and give the necessary background material concerning the concepts we use, and derive for later use some formulas for warped products. Next, we enlarge to study of curvature conditions to warped products with a base manifold and a semi-Riemannian fiber realizing the same set of pseudosymmetric curvature conditions. Finally, we will present warped product spacetimes.

1. Preliminaries

Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞. Let (M,g) be an n-dimensional, $n \geqslant 3$, semi-Riemannian manifold. We denote by ∇, R, C, S and κ the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M,g), respectively. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X,Y) = S(X,Y)$, where $X,Y \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M. Further, we define the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X,Y)$ and $\mathcal{C}(X,Y)$ of $\Xi(M)$ by

$$\begin{aligned}
(X \wedge_A Y)Z &= A(Y,Z)X - A(X,Z)Y\,, \\
\mathcal{R}(X,Y)Z &= [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z\,, \\
\mathcal{C}(X,Y)Z &= \mathcal{R}(X,Y)Z - \frac{1}{n-2}(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y)Z\,,
\end{aligned}$$

respectively, where $X,Y,Z \in \Xi(M)$ and A is a symmetric $(0,2)$-tensor. Now the Riemann-Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the $(0,4)$-tensor G of (M,g) are defined by

$$\begin{aligned}
R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4)\,, \\
C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4)\,, \\
G(X_1,X_2,X_3,X_4) &= g((X_1 \wedge_g X_2)X_3,X_4)\,,
\end{aligned}$$

respectively, where $X,Y,Z,X_1,X_2,\ldots \in \Xi(M)$.

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Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be a $(0,4)$-tensor associated with $\mathcal{B}(X,Y)$ by

$$B(X_1,X_2,X_3,X_4) \;=\; g(\mathcal{B}(X_1,X_2)X_3,X_4)\,.$$

The tensor B is said to be a generalized curvature tensor if the following conditions are fulfilled

$$\begin{aligned} B(X_1,X_2,X_3,X_4)+B(X_2,X_3,X_1,X_4)+B(X_3,X_1,X_2,X_4) &= 0\,,\\ B(X_1,X_2,X_3,X_4) \;=\; B(X_3,X_4,X_1,X_2) &= 0\,. \end{aligned}$$

Clearly, the the tensors R, C and G are generalized curvature tensors. We extend the endomorphism $\mathcal{B}(X,Y)$ to derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X,Y)\cdot f = 0$ for any smooth function on M. Now for a $(0,k)$-tensor field T, $k \geqslant 1$, we can define the $(0,k+2)$-tensor $B\cdot T$ by

$$\begin{aligned} &(B\cdot T)(X_1,\ldots,X_k;X,Y) \;=\; (\mathcal{B}(X,Y)\cdot T)(X_1,\ldots,X_k;X,Y)\\ =\;& -T(\mathcal{B}(X,Y)X_1,X_2,\ldots,X_k)-\cdots-T(X_1,\ldots,X_{k-1},\mathcal{B}(X,Y)X_k)\,. \end{aligned}$$

In addition, if A is a symmetric $(0,2)$-tensor field then we define the $(0,k+2)$-tensor $Q(A,T)$ by

$$\begin{aligned} &Q(A,T)(X_1,\ldots,X_k;X,Y) \;=\; (X\wedge_A Y\cdot T)(X_1,\ldots,X_k;X,Y)\\ =\;& -T((X\wedge_A Y)X_1,X_2,\ldots,X_k)-\cdots-T(X_1,\ldots,X_{k-1},(X\wedge_A Y)X_k)\,. \end{aligned}$$

In particular, in this way, we obtain the $(0,6)$-tensors $B\cdot B$ and $Q(A,B)$.

Setting in the above formulas $\mathcal{B}=\mathcal{R}$ or $\mathcal{B}=\mathcal{C}$, $T=R$ or $T=C$ or $T=S$, $A=g$ or $A=S$, we get the tensors $R\cdot R$, $R\cdot C$, $C\cdot R$, $R\cdot S$, $Q(g,R)$, $Q(S,R)$, $Q(g,C)$ and $Q(g,S)$. In the same manner we obtain the other tensors of this kind.

Further, for symmetric $(0,2)$-tensors E and F we define their Kulkarni-Nomizu tensor $E\wedge F$ by

$$\begin{aligned} (E\wedge F)(X_1,X_2,X_3,X_4) \;=\;& E(X_1,X_4)F(X_2,X_3)+E(X_2,X_3)F(X_1,X_4)\\ &-E(X_1,X_3)F(X_2,X_4)-E(X_2,X_4)F(X_1,X_3)\,. \end{aligned}$$

It is easy to see that $E\wedge F$ is also a generalized curvature tensor. We extend this definition to the case when F is a $(0,k)$-tensor, $k\geqslant 2$, by

$$\begin{aligned} &(E\wedge F)(X_1,X_2,X_3,X_4;Y_3,\ldots,Y_k)\\ =\;& E(X_1,X_4)F(X_2,X_3,Y_3,\ldots,Y_k)+E(X_2,X_3)F(X_1,X_4,Y_3,\ldots,Y_k)\\ &-E(X_1,X_3)F(X_2,X_4,Y_3,\ldots,Y_k)-E(X_2,X_4)F(X_1,X_3,Y_3,\ldots,Y_k)\,.F \end{aligned}$$

The following three conditions are called *curvature conditions of pseudosymmetric type.*

A semi-Riemannian manifold is said to be pseudosymmetric if the condition

$$R\cdot R \;=\; L_R Q(g,R) \tag{1}$$

is satisfied on the set $\mathcal{U}_R=\{x\in M : R-\frac{\kappa}{n(n-1)}G\neq 0 \text{ at } x\}$, where L_R is some function on $\mathcal{U}_R$. It is clear that any semisymmetric manifold is pseuosymmetric.

A semi-Riemannian manifold is said to be Ricci-pseudosymmetric if the condition

$$R\cdot S \;=\; L_S Q(g,S) \tag{2}$$

holds on the set $\mathcal{U}_S=\{x\in M : S-\frac{\kappa}{n}g\neq 0 \text{ at } x\}$, where L_S is some function on $\mathcal{U}_S$. Evidently, any Einstein manifold is Ricci-pseuosymmetric.

A semi-Riemannian manifold is said to be Weyl-pseudosymmetric if the condition

$$R \cdot C \;=\; L_C Q(g, C) \tag{3}$$

is fulfilled on the set $\mathcal{U}_C = \{x \in M : C \neq 0 \text{ at } x\}$, where L_C is some function on $\mathcal{U}_C$.

2. Warped Product Manifolds

Let $(M, \bar{g})$ and $(N, \tilde{g})$, dim $M = p$, dim $N = n - p$, $1 \leqslant p < n$, be semi-Riemannian manifolds covered by systems of charts $\{\mathcal{U}; x^a\}$ and $\{\mathcal{V}; y^\alpha\}$, respectively. Let F be a positive smooth function on M. The *warped product* $M \times_F N$ of $(M, \bar{g})$ and $(N, \tilde{g})$ is the product manifold $M \times N$ with the metric $g = \bar{g} \times_F \tilde{g}$ defined by

$$g = \bar{g} \times_F \tilde{g} = \pi_1^* \bar{g} + (F o \pi_1) \pi_2^* \tilde{g},$$

where $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are the natural projections on M and N, respectively. Let $\{\mathcal{U} \times \mathcal{V} : x^1, ..., x^p, x^{p+1} = y^1, ..., x^n = y^{n-p}\}$ be a product chart for $M \times N$. The local components of the metric $g = \bar{g} \times_F \tilde{g}$ with respect to this chart are

$$g_{rs} = \begin{cases} \bar{g}_{ab} & \text{if} \quad r = a, s = b, \\ F\tilde{g}_{\alpha\beta} & \text{if} \quad r = \alpha, s = \beta, \\ 0 & \quad \text{otherwise,} \end{cases}$$

where $a, b, c, ... \in \{1, ..., p\}$, $\alpha, \beta, \gamma, ... \in \{p+1, ..., n\}$ and $r, s, t, ... \in \{1, ..., n\}$. We will denote by bars (resp., by tildes) the tensors formed from $\bar{g}$ (resp., $\tilde{g}$).

The local components Γ^r_{st} of the Levi-Civita connection ∇ of $M \times_F N$ are the following

$$\begin{aligned} \Gamma^a_{bc} &= \bar{\Gamma}^a_{bc}, \Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma}, \Gamma^a_{\alpha\beta} = -\frac{1}{2}\bar{g}^{ab} F_b \tilde{g}_{\alpha\beta}, \Gamma^\alpha_{a\beta} = \frac{1}{2F} F_a \delta^\alpha_\beta\,, \\ \Gamma^a_{\alpha b} &= \Gamma^\alpha_{ab} = 0, F_a = \partial_a F = \frac{\partial F}{\partial x^a}. \end{aligned}$$

The local components R_{rstu} of the Riemann-Christoffel curvature tensor R and the local components S_{st} of the Ricci tensor S of the warped product $M \times_F N$ which may not vanish identically are the following:

$$\begin{aligned} R_{abcd} &= \bar{R}_{abcd}\,, \\ R_{\alpha ab\beta} &= -\frac{1}{2} T_{ab} \tilde{g}_{\alpha\beta}\,, \\ R_{\alpha\beta\gamma\delta} &= F\tilde{R}_{\alpha\beta\gamma\delta} - \frac{1}{4}\Delta F \tilde{G}_{\alpha\beta\gamma\delta}\,, \\ S_{ab} &= \bar{S}_{ab} - \frac{n-p}{2}\frac{1}{F} T_{ab}\,, \\ S_{\alpha\beta} &= \tilde{S}_{\alpha\beta} - \frac{1}{2}(tr(T) + \frac{n-p-1}{2F}\Delta F)\tilde{g}_{\alpha\beta}\,, \end{aligned}$$

where $T_{ab} = \Delta_b F_a - \frac{1}{2F} F_a F_b$, $tr(T) = \bar{g}^{ab} T_{ab}$, $\Delta F = \Delta_{\bar{g}} F = \bar{g}^{ab} F_a F_b$ and T is the (0,2)-tensor with the local components T_{ab}. The scalar curvature κ of $M \times_F N$ satisfies the following relation

$$\kappa = \bar{\kappa} + \frac{1}{F}\tilde{\kappa} - \frac{n-p}{F}(tr(T) + \frac{n-p-1}{4F}\Delta F).$$

The class of warped product manifolds, shortly warped products, is an extension of the class of products of semi-Riemannian manifolds.

3. Conformally Flat Warped Products with n=4

LEMMA 1. *([4]) The only local components of the Weyl conformal curvature tensor C of a $4-$dimensional warped product $M \times_F N$ which are not identically zero are*

$$\begin{aligned} C_{\alpha 11\beta} &= -\frac{1}{2}\tilde{g}_{11}(\tilde{S}_{\alpha\beta} - \frac{\tilde{\kappa}}{3}\tilde{g}_{\alpha\beta})\,, \\ C_{\alpha\beta\gamma\delta} &= \frac{F}{2}(\tilde{g}_{\alpha\delta}\tilde{S}_{\beta\gamma} - \tilde{g}_{\alpha\gamma}\tilde{S}_{\beta\delta} + \tilde{g}_{\beta\gamma}\tilde{S}_{\alpha\delta} - \tilde{g}_{\beta\delta}\tilde{S}_{\alpha\gamma}) - \frac{F\tilde{\kappa}}{3}\tilde{G}_{\alpha\beta\gamma\delta}\,, \end{aligned}$$

provided that dim $M = 1$;

$$\begin{aligned} C_{abcd} &= \frac{P}{F}G_{abcd}, \quad P = \frac{1}{6}(F\bar{\kappa} + \tilde{\kappa} + tr(T) - \frac{\Delta F}{2F})\,, \\ C_{\alpha\beta\gamma\delta} &= \frac{P}{F}G_{\alpha\beta\gamma\delta}\,, \\ C_{\alpha ab\beta} &= -\frac{P}{2F}G_{\alpha ab\beta}\,, \end{aligned}$$

provided that dim $M = 2$;

$$\begin{aligned} C_{4ab4} &= -\frac{1}{2}\tilde{g}_{44}W_{ab}, \quad W_{ab} = V_{ab} - \frac{tr(V)}{3}\bar{g}_{ab}, \quad V_{ab} = F\bar{S}_{ab} + \frac{1}{2}T_{ab}\,, \\ C_{abcd} &= \frac{1}{2F}(\bar{g}_{ad}W_{bc} + \bar{g}_{bc}W_{ad} - \bar{g}_{ac}W_{bd} - \bar{g}_{bd}W_{ac})\,, \end{aligned}$$

provided that dim $M = 3$.

From the above Lemma the following Theorem follows immediately:

THEOREM 1. *([4]) Suppose dim $M \times_F N = 4$.*
(i) If dim $M = 1$, then $M \times_F N$ is conformally flat if and only if

$$\tilde{S} = \frac{\tilde{\kappa}}{3}\tilde{g}.$$

(ii)If dim $M = 2$, then $M \times_F N$ is conformally flat if and only if

$$tr(T) = -\frac{F\kappa}{3}.$$

(iii) If dim $M = 3$, then $M \times_F N$ is conformally flat if and only if

$$F\bar{S} + \frac{T}{2} = \frac{1}{3}(F\bar{\kappa} + \frac{tr(T)}{2})\bar{g}.$$

REMARK 1. *The assertion (iii) of Theorem 4.1 can be easily generalized as follows:*

The manifold $M \times_F N$, dim $M = n-1$, $n \geqslant 4$, is conformally flat if and only if $\bar{C} = 0$ and $F\bar{S} + \frac{(n-3)T}{2} = \frac{1}{n-1}(F\bar{\kappa} + \frac{(n-3)tr(T)}{2})\bar{g}$ on M.

An example of a conformally flat warped product $M \times_F N$, dim $N \geqslant 3$, is described in the following:

EXAMPLE. *([3]) Let M be an open connected domain of $\mathbb{R}^3[x^1, x^2, x^3]$. Define functions $G = G(x^1, x^2, x^3; t) = G(x^2; t)$ and $H = H(x^1, x^2, x^3; t) = H(x^2; t)$ depending on a point $x \in M$ and a parameter t as follows:*

$$\begin{aligned} G &= 2\frac{k}{l}x^2, \;\; H = 2(1-\frac{k}{l})x^2\,, \\ k &= \frac{4}{h}t^2 - 2t, \;\; l = \frac{4}{h}t^2, \;\; h = \frac{1}{(n-3)(n-4)}\tilde{\kappa}, \;\; t \in \mathbb{R} - \{0, \frac{1}{2}h\}\,. \end{aligned}$$

Define on M a family of metrics depending on the parameter t by the formulas

$$\bar{g}_{ab} = \begin{cases} exp(G+H) & \text{if} \quad a = b = 2, \\ exp \quad G & \text{if} \quad a = 1, b = 3 \quad \text{or} \quad a = 3, b = 1, \\ 0 & \text{otherwise}, \end{cases}$$

and a warping function F on M (depending also on t) as $F = l\; exp\,(G+H)$. Finally, let $(N, \tilde{g})$, dim $N \geqslant 3$, be a manifold with non-zero constant curvature. Then the manifold $M \times_F N$ is conformally flat and not pseudosymmetric.

Another consequence of Lemma 4.1 is

THEOREM 2. *Suppose dim $M \times_F N = 4$ and dim $M = 2$. Then $C \cdot C = -\frac{P}{2F}Q(g, C)$ on $M \times_F N$.*

4. Not Conformally Flat Warped Products with n=4 satisfying $R \cdot C = L_C Q(g, C)$

PROPOSITION 1. *Suppose the condition (3) is satisfied on $M \times_F N$, where dim $M = 1$ and dim $N = 3$. If C is non-zero at $x \in M \times_F N$ then (1) holds at x.*

PROPOSITION 2. *Suppose the condition (3) is satisfied on $M \times_F N$, where dim $M = 2$ and dim $N = 2$. If C is non-zero at $x \in M \times_F N$ then (1) holds at x.*

PROPOSITION 3. *Suppose the condition (3) is satisfied on $M \times_F N$, where dim $M = 3$ and dim $N = 1$. If C is non-zero at $x \in M \times_F N$ then (1) holds at x.*

If $n \geqslant 5$, then the relations (1) and (3) are equivalent for all points at which C is non-zero. In particular, the conditions $R \cdot C = 0$ and $R \cdot R = 0$ are equivalent at each point of (M, g) at which $C \neq 0$. On $4-$dimensional manifolds, this is not always the case.

The following Theorem shows that the relations (1) and (3) are equivalent at every point of a $4-$dimensional warped product manifold at which C does not vanish.

Combining Propositions $5.1, 5.2, 5.3$ we get

THEOREM 3. *([4]) Suppose dim $M \times_F N = 4$. If at a point of $M \times_F N$ the tensor C is non-zero, then the relations (1) and (3) are equivalent at this point.*

5. Non-Einstein $4-$dimensional Warped Products satisfying $R \cdot S = L_S Q(g, S)$

PROPOSITION 4. *Suppose the condition (2) is satisfied on $M \times_F N$, where dim $M = 1$ and dim $N = 3$. If $S - \frac{\kappa}{4}g$ is non-zero at $x \in M \times_F N$ then (1) holds at x.*

PROPOSITION 5. *Suppose the condition (2) is satisfied on $M \times_F N$, where dim $M =$dim $N = 2$. If $S - \frac{\kappa}{4}g$ is non-zero at $x \in M \times_F N$ then (1) holds at x.*

Proposition 6. *Suppose the condition (2) is satisfied on $M \times_F N$, where dim $M = 3$ and dim $N = 1$. If $S - \frac{\kappa}{4}g$ is non-zero at $x \in M \times_F N$ then (1) holds at x.*

Any pseudosymmetric manifold is Ricci-pseudosymmetric. However, the converse fails in general.

The relations (1) and (2) are equivalent on manifolds with vanishing Weyl Conformal Curvature tensor C. Namely, we have

Lemma 2. *([3]) If C vanishes at $x \in M$ and $n \geqslant 3$, then at x the following three identities are equivalent to each other:*

$$((n-2)\alpha + \frac{\kappa}{n-1})(S - \frac{\kappa}{n}) = s^2 - \frac{tr(S^2)}{n}g,$$

$$R \cdot S = \alpha Q(g,S),\ R \cdot R = \alpha Q(g,R),$$

where $\alpha \in R$.

In non-Einstein case the following Theorem shows that the relations (1) and (2) are equivalent at every point of a 4−dimensional warped product manifold at which the tensor $S - \frac{\kappa}{n}g$ does not vanish.

Combining Propositions 6.1, 6.2, 6.3 and Lemma 6.1, we get

Theorem 4. *([4]) Suppose dim $M \times_F N = 4$. If $S - \frac{\kappa}{4}g$ is non-zero at a point $x \in M \times_F N$, then the relations (1) and (2) are equivalent at this point.*

6. Warped Products realizing $C \cdot C = L_C Q(g, C)$

Now, we will consider a condition of pseudosymmetry type imposed on the Weyl tensor C.

There are warped products $M \times_F N$, dim M = dim $N = 2$ realizing $C \cdot C = L_C Q(g, C)$ which are not pseudosymmetric manifolds; where L_c is defined by (3). But there exist warped products $M \times_F N$, dim $M \geqslant 2$, dim $N \geqslant 2$, satisfying $C \cdot C = L_C Q(g, C)$.

We will consider warped products $M \times_F N$, dim $M = 1$, dim $N \geqslant 3$ or dim $M \geqslant 3$, dim $N = 1$, satisfying $C \cdot C = L_C Q(g, C)$. In addition, we will show that in these sets of warped products there are manifolds realizing $C \cdot C = L_C Q(g, C)$.

(i) Warped products $M \times_F N$, dim $M = 1$, dim $N \geqslant 3$

We will prove that in this class of warped products there exist manifolds realizing $C \cdot C = L_C Q(g, C)$.

Theorem 5. *([7]) Let $(M, \bar{g})$ be a 1−dimensional manifold and let $(N, \tilde{g})$ be a 3−dimensional manifold. Then the warped product $M \times_F N$ fulfils $C \cdot C = L_C Q(g, C)$ if and only if $(N, \tilde{g})$ is quasi-Einstein.*

Theorem 6. *([7]) Let $(M, \bar{g})$ be a 1−dimensional manifold and $(N, \tilde{g})$, dim $N \geqslant 4$, be an $(n-1)$−dimensional conformally flat manifold. Then the warped product $M \times_F N$ fulfils $C \cdot C = L_C Q(g, C)$ if and only if $(N, \tilde{g})$ is quasi-Einstein.*

(ii) Warped products $M \times_F N$, dim $M \geqslant 3$, dim $N = 1$

Let A be a symmetric $(0, 2)$-tensor field on M. We denote by $\tilde{A}$ the $(1, 1)$- tensor field related with A by $A(X, Y) = g(\tilde{A}X, Y)$. Now, we define the tensor field A^2 by $A^2(X, Y) = A(\tilde{A}X, Y)$.

THEOREM 7. *([7]) Let $(N,\tilde{g})$ be a $1-$dimensional manifold and $(M,\bar{g})$ be a $3-$dimensional manifold. Then the warped product $M\times_F N$ fulfils $C\cdot C = L_C Q(g,C)$ if and only if the tensor*

$$X_{ad} = \bar{S}_{ad} + \frac{n-3}{2}\frac{1}{F}T_{ad} - \frac{1}{n-1}(\bar{\kappa} + \frac{n-3}{2}\frac{tr(T)}{F})\bar{g}_{ad}$$

satisfies

$$X^2_{bc} = (n-2)(n-3)L_C X_{bc} + \lambda\bar{g}_{bc},$$

where λ is some function.

We give now an example of a $3-$dimensional manifold satisfying the above equation.

EXAMPLE. *([7]) Let M be a nonempty, connected open subset of $\mathbb{R}^{n-1}$, $n \geqslant 4$, equipped with its standard metric $\bar{g}$. Further, we define on M the function F by*

$$F(x^1,...,x^{n-2}) = \frac{1}{4}k^2,\ \ k = (x^1)^2 + ... + (x^{n-2})^2 + c,$$

where c is a positive constant. The tensor X fulfils on M the condition

$$X^2 = -\frac{n-3}{2(n-1)}\frac{k}{F}X + \frac{1}{2}\frac{(n-3)^2}{(n-1)^2}\frac{k^2}{F^2}\bar{g}.$$

7. Warped Product Spacetimes

Warped products play an important role in Riemannian Geometry as well as in general relativity [1]. Many well-known spacetimes of general relativity are warped products; e.g. the Schwarzschild spacetimes, the Kottler spacetimes, the Reissner-Nordström spacetimes as well as Robertson-Walker spacetimes.

By a spacetime, we mean a $4-$dimensional manifold endowed with a semi-Riemannian metric of Lorentz signature $(-,+,+,+)$.

(i) Warped product spacetimes with $1-$dimensional base

A warped product $M\times_F N$ of a $1-$dimensional manifold $(M,\bar{g})$, $\bar{g}_{11} = -1$, and a $3-$dimensional Riemannian space $(N,\tilde{g})$ of constant curvature, with a warping function F, is said to be a *Robertson-Walker Spacetime* [8]. More generally, one also considers warped products $M\times_F N$ of $(M,\bar{g})$, $\dim M = 1$, $\bar{g}_{11} = -1$, with a warping function F and an $(n-1)-$dimensional Riemannian manifold $(N,\tilde{g})$, $n \geqslant 4$. Such warped products are called generalized Robertson-Walker Spacetimes [1].

THEOREM 8. *([5]) The Robertson-Walker Spacetimes are pseudosymmetric.*

$4-$dimensional Robertson-Walker Spacetimes realize 3 independent curvature conditions of pseudosymmetric type:

(i) $R\cdot R = L_R Q(g,R)$, where L_R is some function on the set $\mathcal{U}_R = \{x \in M : R - \frac{\kappa}{n(n-1)}G \neq 0 \text{ at } x\}$

(ii) $R\cdot R - Q(S,R) = L_C Q(g,C)$, where L_C is some function on the set $\mathcal{U}_C = \{x \in M : C \neq 0 \text{ at } x\}$.

(iii) $R\cdot C = LQ(S,C)$, where L is some function on the set $\mathcal{U} = \{x \in M : Q(S,C) \neq 0 \text{ at } x\}$.

THEOREM 9. *([2]) Let $M \times_F N$ be a warped product of a $1-$dimensional manifold $(M, \bar{g})$, $\bar{g}_{11} = \pm 1$, the warping function F, $F(x^1) = a\,exp(bx^1)$, $a = const. > 0$, $b = const. \neq 0$ and a conformally flat semi-Riemannian manifold $(N, \tilde{g})$, dim $N = n-1 \geqslant 3$, such that its scalar curvature $\tilde{\kappa}$ vanishes and rank $\tilde{S} = 1$. Then the following relations are satisfied on $M \times_F N$:*

$$\begin{aligned} R \cdot R &= \frac{\kappa}{n(n-1)} Q(g, R)\,, \\ R \cdot R - Q(S, R) &= -\frac{\kappa(n-2)}{n(n-1)} Q(g, C)\,, \\ R \cdot C &= \frac{1}{n-1} Q(S, C)\,. \end{aligned}$$

THEOREM 10. *([6]) $4-$dimensional Robertson-Walker Spacetimes are quasi-Einstein manifolds.*

(ii) Warped product spacetimes with $2-$dimensional base

EXAMPLE. *([6]) Let $M = \{(t, r) \in \mathbb{R}^2, r > 0\}$ be a $2-$dimensional manifold with the metric tensor $\bar{g}$ defined by*

$$\bar{g}_{ab} = \begin{cases} -AB & \text{if} \quad a = b = 1, \\ \frac{1}{B} & \text{if} \quad a = b = 2, \\ 0 & \quad \text{otherwise,} \end{cases}$$

where $A = A(t)$ and $B = B(r)$ are positive smooth functions on M, $F = F(r) = r^2$, and $(N, \tilde{g})$, dim $N = n-2 \geqslant 2$, is a manifold of constant curvature. The manifold $M \times_F N$ satisfies the condition (1), with $L = -\frac{1}{2r}\frac{\partial B}{\partial r}$.

In the special case, when the function B is given by $B = 1 - \frac{2m}{r}$ (resp., $B = 1 - (2m/r) - (1/3)lr^2$, $B = 1 - (2m/r) - (k^2/r^2)$, $A \equiv 1$, and $(N, \tilde{g})$ is the unit $2-$dimensional sphere, the manifold $M \times_F N$ is said to be *Schwarzschild spacetime* (resp., *a Kottler- and a Reissner-Nordström spacetime*)[8], where m, l and k are some non-zero constants. Thus we have

THEOREM 11. *([6]) The Schwarzschild, Kottler- and a Reissner-Nordström spacetimes are all pseudosymmetric.*

(iii) Warped product spacetimes with higher dimensional base

Let $(M, \bar{g})$, dim $M = 3$, be a Riemannian manifold with positive definite metric $\bar{g}$, I an open interval of $\mathbb{R}$ with the standard metric $\tilde{g}$, $\tilde{g}_{44} = -1$ and F a positive smooth fuction on M. Then the warped product $M \times_F I$ is called a *standard static spacetime [8].*

EXAMPLE. *([6]) Assume that M is an open subset of $\mathbb{R}^p - \{(0, ..., 0)\}$, $p \geqslant 3$, $\bar{g}_{ab} = \delta_{ab}$, $a, b \in \{1, ..., p\}$, $F = F(x^1, ..., x^p) = \frac{1}{4}(\Sigma_{i=1}(x^i)^2)^2$ and $N = I$ an open interval of $\mathbb{R}$ with the standard metric $\tilde{g}$. Then the warped product $M \times_F I$ is a conformally flat and pseudosymmetric manifold with $L = -\frac{tr(T)}{2pF}$.*

By modifying the above example in the following way, we obtain an example of a non conformally flat pseudosymmetric manifold.

EXAMPLE. *([6]) Let M is an open subset of $\mathbb{R}^p$, $p \geqslant 3$, $\bar{g}_{ab} = \delta_{ab}$, $a, b \in \{1, ..., p\}$, $F = F(x^1, ..., x^p) = \frac{1}{4}((x^1)^2 + ... + (x^p)^2 + k)^2$, where k is a positive constant and I*

an open interval of $\mathbb{R}$ *with the standard metric* $\tilde{g}$. *Then the warped product* $M \times_F I$ *is a non conformally flat, pseudosymmetric manifold with* $L = -\frac{tr(T)}{2pF}$.

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