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Complete systems of invariants for rank 1 curves in Lagrange Grassmannians

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Abstract. Curves in Lagrange Grassmannians naturally appear when one studies intrinsically "the Jacobi equations for extremals", associated with control systems and geometric structures. In this way one reduces the problem of construction of the curvature-type invariants for these objects to the much more concrete problem of finding of invariants of curves in Lagrange Grassmannians w.r.t. the action of the linear Symplectic group. In the present paper we develop a new approach to differential geometry of so-called rank 1 curves in Lagrange Grassmannian, i.e., the curves with velocities being rank one linear mappings (under the standard identification of the tangent space to a point of the Lagrange Grassmannian with an appropriate space of linear mappings). The curves of this class are associated with "the Jacobi equations for extremals", corresponding to control systems with scalar control and to rank 2 vector distributions. In particular, we construct the tuple of m principal invariants, where m is equal to half of dimension of the ambient linear symplectic space, such that for a given tuple of arbitrary m smooth functions there exists the unique, up to a symplectic transformation, rank 1 curve having this tuple, as the tuple of the principal invariants. This approach extends and essentially simplifies the results of [**4**], where only the uniqueness part was proved and in rather cumbersome way. It is based on the construction of the new canonical moving frame with the most simple structural equation.

1. Statement of the problem and the results

Let W be a $2m$-dimensional linear space provided with a symplectic form σ. Recall that an m-dimensional subspace Λ of W is called Lagrangian, if $\sigma|_\Lambda = 0$. Lagrange Grassmannian $L(W)$ of W is the set of all Lagrangian subspaces of W. The linear Symplectic group acts naturally on $L(W)$. Invariants of curves in Lagrange Grassmannian w.r.t. this action are called symplectic The motivation to study differential geometry of curves in Lagrange Grassmannians comes from optimal control problems: it turns out that to any extremal of rather general control systems one can assign a special curve in some Lagrange Grassmannian, called the Jacobi curve (see [**1**], [**2**], and Introduction to [**4**] for the details). Symplectic invariants of Jacobi curves produce curvature-type differential invariants for these control systems.

The natural differential-geometric problem is to construct a complete system of symplectic invariants for curves in Lagrange Grassmannians, i.e., some set of invariants such that there exists the unique, up to a symplectic transformation, curve in Lagrange Grassmannian with the prescribed invariants from this set. Some methods for construction and calculation of symplectic invariants of the curves in

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Lagrange Grassmannians (including invariants of unparametrized curves) were given in [**4**] and [**5**]. Also the problem of finding a complete system of symplectic invariants for the special class of the so-called rank 1 curves in Lagrange Grassmannians (see Definition 1 below) were partially solved there. In the present paper we solve this problem for the mentioned class of curves completely by developing another approach for the construction of symplectic invariants.

Now we will briefly describe some main constructions of [**4**] in order to specify in what sense our problem was partially solved and why it was not solved completely there. Note only that some results of the present paper (for example, Theorem 2) do not depend on these constructions and in our opinion they are interesting by themselves. The key tool, used in [**4**] for construction of symplectic invariants of curves in Lagrange Grassmannians, is an *infinitesimal cross-ratio* of two tangent vectors V_0, V_1 at two distinct points Λ_0, Λ_1 in $L(W)$. In order to define it recall first that the tangent space $T_\Lambda L(W)$ to the Lagrangian Grassmannian $L(W)$ at the point Λ can be naturally identified with the space $\mathrm{Quad}(\Lambda)$ of all quadratic forms on linear space $\Lambda \subset W$ or with the space $\mathrm{Symm}(\Lambda)$ of self-adjoint linear mappings from the space Λ to the dual space Λ^*. Namely, take a curve $\Lambda(t) \in L(W)$ with $\Lambda(0) = \Lambda$. Given some vector $l \in \Lambda$, take a curve $l(\cdot)$ in W such that $l(t) \in \Lambda(t)$ for all t and $l(0) = l$. Define the quadratic form

$$q_{\Lambda(\cdot)}(l) = \sigma(\frac{d}{dt}l(0), l). \tag{1}$$

Using the fact that the spaces $\Lambda(t)$ are Lagrangian, it is easy to see that the form $q_{\Lambda(\cdot)}(l)$ depends only on $\frac{d}{dt}\Lambda(0)$. One can consider also the self-adjoint linear mapping from Λ to Λ^*, corresponding to this quadratic form. So, we have the mappings from $T_\Lambda L(W)$ to the spaces $\mathrm{Quad}(\Lambda)$ and $\mathrm{Symm}(\Lambda)$. A simple counting of dimensions shows that these mappings are bijections and they define the required identifications. Below we use these identifications without a special mentioning. Besides, given two Lagrangian subspaces Λ_0 and Λ_1, which are transversal, i.e. $\Lambda_0 \cap \Lambda_1 = 0$, the map $v \mapsto \sigma(v, \cdot)$, $\quad v \in \Lambda_1$, defines the canonical isomorphism from Λ_1 to Λ_0^*, which will be denoted by $\mathcal{B}_{\Lambda_0,\Lambda_1}$.

Now we are ready to define the infinitesimal cross-ratio of two tangent vectors V_0, V_1 at two distinct points Λ_0, Λ_1 in $L(W)$, where $\Lambda_0 \cap \Lambda_1 = 0$. By above, V_0 is the self-adjoint linear mapping from Λ_0 to Λ_0^*, while V_1 is the self-adjoint linear mapping from Λ_1 to Λ_1^*. But $\Lambda_0^* \cong \Lambda_1$ via $\Big(\mathcal{B}_{\Lambda_0,\Lambda_1}\Big)^{-1}$, while $\Lambda_1^* \cong \Lambda_0$ via $\Big(\mathcal{B}_{\Lambda_1,\Lambda_0}\Big)^{-1}$. Therefore

$$[V_0, V_1]_{\Lambda_0,\Lambda_1} \stackrel{def}{=} \Big(\mathcal{B}_{\Lambda_1,\Lambda_0}\Big)^{-1} \circ V_1 \circ \Big(\mathcal{B}_{\Lambda_0,\Lambda_1}\Big)^{-1} \circ V_0 \tag{2}$$

is well defined linear operator from the space Λ_0 to itself. This operator will be called an *infinitesimal cross-ratio* of a pair $(V_0, V_1) \in T_{\Lambda_0}L(W) \times T_{\Lambda_1}L(W)$. Actually, this notion is an infinitesimal version of the *cross-ratio* of four points in $L(W)$, which in turn is the generalization of the classical cross-ratio of four points on the projective line (see [**4**] or [**3**] for the details).

By constructions, the infinitesimal cross-ratio is the symplectic invariant of two tangent vectors at two distinct points of $L(W)$. Now let us show how to use this notion for the construction of symplectic invariants of a smooth curve $t \mapsto \Lambda(t)$ in $L(W)$. Suppose that the curve $\Lambda(\cdot)$ satisfies at some point τ the following condition: There exists a natural number s such that for any representative $\Lambda_\tau^s(\cdot)$ of the s-jet of $\Lambda(\cdot)$ at τ, there exists t such that $\Lambda^s_\tau(t) \cap \Lambda(\tau) = 0$. In this case the curve $\Lambda(\cdot)$ is called *ample at* τ. The curve $\Lambda(\cdot)$ is called *ample*, if the last condition hold at any point τ of its segment of definition. To clarify this definition, let us give it in some coordinates: Let $W \cong \mathbb{R}^m \times \mathbb{R}^m$. The curve $t \mapsto \{(x, S_t x) : x \in \mathbb{R}^n\}$ is ample at τ if and only if the function $t \mapsto det(S_t - S_\tau)$ has zero of *finite order* $k(\tau)$ at τ. The number $k(\tau)$ is called *the weight* of an ample curve $\Lambda(\cdot)$ at τ. Obviously, $k(\tau)$ is an integer valued upper semicontinuous function of τ. Therefore it is locally constant on the open dense subset of the segment of definition of the curve. Note that any analytic monotone curve in $L(W)$, the image of which is not a point, is either ample or becomes ample in the Lagrange Grassmannian of another symplectic space, which obtained from W after an appropriate symplectic factorization by the common subspace of all subspaces $\Lambda(t)$ (see Lemma 2.1 of [**4**]). The term " monotone curve" means that its velocities are either nonnegative definite quadratic forms at any point or nonpositive definite quadratic forms at any point. Any curve $\Lambda(\cdot)$ in $L(W)$ such that $\dot{\Lambda}(t)$ is nondegenerated quadratic form on $\Lambda(t)$ (nondegenerated self-adjoint mapping from $\Lambda(t)$ to $\Lambda(t)^*$) has constant weight equal to m $(= \frac{1}{2}\dim W)$. We call such curves *regular*. Note that the set of all curves with constant weight in $L(W)$ is much wider than the set of all regular curves in $L(W)$. For example, any ample curve of rank 1 in $L(W)$ (see Definition 1 below) at a generic point has the weight equal to m^2.

The following proposition shows how to extract symplectic invariants from the infinitesimal cross-ratio:

Proposition 1. (see [**4**], Lemma 4.2) *If the curve $\Lambda : I \mapsto L(W)$ has the constant finite weight k on the segment of the definition I , then the following asymptotic holds*

$$\operatorname{trace}\Big[\dot{\Lambda}(t_0) \ \big|\ \dot{\Lambda}(t_1)\Big]_{\Lambda(t_0),\Lambda(t_1)} = -\frac{k}{(t_0 - t_1)^2} - g_{\Lambda}(t_0, t_1), \tag{3}$$

where $g_{\Lambda}(t_0, t_1)$ is a smooth function in the neighborhood of diagonal $\{(t, t) | t \in I\}$.

Let us give the coordinate expression for the function $g_{\Lambda}(t_0, t_1)$: If $W \cong \mathbb{R}^m \times \mathbb{R}^m$, and $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$ then

$$g_{\Lambda}(t_0, t_1) = \frac{\partial^2}{\partial t_0 \partial t_1} \ln\left(\frac{\det(S_{t_0} - S_{t_1})}{(t_0 - t_1)^k}\right) \tag{4}$$

(the proof of the last formula follows from [**4**], see relations (4.9),(4.11), and Lemma 4.2 there).

The function g_{Λ} is a "generating function" for the symplectic invariants in the following sense: suppose that it has the following expansion in the formal Taylor series at the point (t, t) :

$$g(t_0, t) \approx \sum_{i=0}^{\infty} \beta_i(t)(t_0 - t)^i, \tag{5}$$

then all coefficients $\beta_i(t)$ are symplectic invariants of the curve $\Lambda(\cdot)$. In particular, the first appearing in (5) coefficient $\beta_0(t)$ $(= g_{\Lambda}(t, t))$ produces the Ricci curvature, if one calculates it for Jacobi curves of Riemannian geodesics.

The natural questions are whether the function g_{Λ} contains all information about the curve $\Lambda(\cdot)$ and what tuple of coefficients in expansion (5) of the function g_{Λ} constitutes a complete system of invariants of Λ? These questions were investigated in [**4**], section 7, for so-called rank 1 curves.

DEFINITION 1. *We say that a curve $\Lambda(\cdot)$ in $L(W)$ has rank r at a point t, if its velocity $\dot\Lambda(t)$ is the linear self-adjoint mapping from $\Lambda(t)$ to $\Lambda(t)^*$ of rank r. A curve $\Lambda(\cdot)$ is called a rank r curve in $L(W)$, if it has rank r at any point t.*

The motivation to study curves in $L(W)$ of rank less than $\frac{1}{2}\dim W$ at any point (which at first glance looks as rather degenerated case) comes from the fact that Jacobi curves associated with extremals of control systems with r-dimensional control space and n-dimensional state space (where $r < n$) are curves of rank not greater than r in Lagrange Grassmannian of symplectic spaces of dimension equal usually to $2(n-1)$ or $2(n-2)$ (see Introduction to [**4**] for the details). In particular, rank 1 curves in Lagrange Grassmannians appear as Jacobi curves associated with extremals of control systems with scalar control and with so-called abnormal extremals of rank 2 vector distributions (subbundles of the tangent bundle, see [**6**] for the details). The fact that Jacobi curves are ample corresponds to some kind of controllability of the corresponding control system and to complete nonholonomicity (nonintegrability) of the corresponding distribution.

The main results in [**4**], concerning rank 1 curves in $L(W)$, is the following

THEOREM 1. (see [**4**], Theorem 2) *The tuple $\{\beta_{2i}(t)\}_{i=0}^{m-1}$, where $\beta_j(t)$ as in (5), determines the curve $\Lambda(t)$, of rank 1 and the constant finite weight uniquely, up to a symplectic transformation.*

For any $0 \leqslant i \leqslant m-1$ the function $\beta_{2i}(t)$ will be called the $(i+1)$*-th principal curvatures* of the rank 1 curve $\Lambda(t)$. Theorem 1 actually states the uniqueness of rank 1 curve with the prescribed principal curvatures. *But the result about existence of rank 1 curve with the prescribed principal curvatures was missing.* Now I will try to explain the reason for it. Recall that the basis $(E_1,\dots,E_m,F_1,\dots F_m)$ of W is called *Darboux*, if

$$\sigma(E_i,E_j)=\sigma(F_i,F_j)=0,\quad \sigma(F_i,E_j)=\delta_{ij},\quad 1\leqslant i,j\leqslant m, \tag{6}$$

where δ_{ij} is the Kronecker symbol. To prove Theorem 1 we have constructed in [**4**] a special canonical moving Darboux's frame for the given rank 1 curve in $L(W)$. To construct this frame we have used first the natural affine structure on the set $\Lambda(\tau)^{\pitchfork}$ of all Lagrangian subspaces transversal to $\Lambda(\tau)$, secondly the expansion of $t\mapsto\Lambda(t)$ (considered as the curve in this affine space $\Lambda(\tau)^{\pitchfork}$ with a singularity at $t=\tau$) into the Laurent series at $t=\tau$, and finally the fact that the free term of this Laurent series is the well-defined Lagrangian subspace transversal to $\Lambda(\tau)$, the *derivative subspace* (see Appendix below for the details). The first disadvantage of the canonical frame from [**4**] is that the number of nontrivial entries in the matrix of its structural equation is much greater than the number of the functional parameters in our equivalence problem (which is equal to dimension of $m\times m$ symmetric matrices of rank 1, i.e., to m). So, this matrix does not give automatically a complete system of invariants: we need to choose some of the entries and to prove that all other nontrivial entries can be expressed by the chosen ones. Besides, the entries in the matrix of the considered structural equation are expressed in some nontrivial way by the principal curvatures. So, in order to prove Theorem 1 we had to analyze these expressions, which was rather nontrivial task. Another disadvantage is that the canonical frame from [**4**] is not determined in the explicit way by the matrix of its structural equation: Even if some frame satisfies the structural equation with some prescribed functions substituted instead of the appropriate principal curvatures, it is not clear a priori whether this frame is canonical for the curve. Therefore it is not clear a priori whether the prescribed functions are exactly the corresponding principal curvatures of the curve. This is the reason why using this frame we did not succeed to prove the existence of the curve with the prescribed tuple of principal curvatures (in Remark 3 below we indicate the main technical difficulty that we met in this way).

In the present paper we solve positively the problem of existence of the rank 1 curves of the constant weight with the prescribed tuple of principal curvatures. For this we introduce a new very natural canonical moving Darboux's frame for a rank 1 curve in $L(W)$, which is uniquely defined by the matrix of its structural equation. It allows to obtain a new tuple of principal curvatures for which the uniqueness an existence results follow automatically. Namely, we have the following

THEOREM 2. *Let W be a $2m$-dimensional linear symplectic space. For the given rank 1 curve $\Lambda(\cdot)$ of the constant finite weight in $L(W)$ there exists the moving Darboux frame*

$$\big(E_1(t),\dots,E_m(t),F_1(t),\dots,F_m(t)\big)$$

such that the following two conditions hold:

(1) $\Lambda(t)=\operatorname{span}\big(E_1(t),\dots,E_m(t)\big)$*;*

(2) The moving frame $\big(E_1(t),\dots,E_m(t),F_1(t),\dots,F_m(t)\big)$ satisfies the following structural equation:

$$\begin{cases} E_i'(t)=E_{i+1}(t), \quad 1\leqslant i\leqslant m-1\\ E_m'(t)=\pm F_m(t)\\ F_1'(t)=\lambda_m(t)E_1(t)\\ F_i'(t)=\lambda_{m-i+1}(t)E_i(t)-F_{i-1}(t), \quad 2\leqslant i\leqslant m \end{cases} \tag{7}$$

(in the second equation of (7) the sign " $+$ " appears if the quadratic form $\dot\Lambda(t)$ is nonnegative definite, while the sign " $-$ " appears if the quadratic form $\dot\Lambda(t)$ is nonnegative definite).

In addition, if the moving frame $\big(E_1(t),\dots,E_m(t),F_1(t),\dots,F_m(t)\big)$ satisfies conditions 1-3, then the only frame, which is different from it and satisfies the same conditions, is $\big(-E_1(t),\dots,-E_m(t),-F_1(t),\dots,-F_m(t)\big)$.

The proof of Theorem 2 will be given in section 2. It consists basically of three steps: First the condition of rank 1 and the constant weight allows to construct the curve of complete flags in W, associated with our curve (see Lemmas 1, 2, and Remark 2 below), secondly the presence of the symplectic structure allows to normalize the one-dimensional subspaces of the flags, which in turn gives the canonical basis on each subspace $\Lambda(t)$ (see Lemma 3 and formula (24)), and finally we complete this basis to the moving Darboux frame with the "most simple" structural equation (having the maximal possible number of zero entries in the matrix corresponding to it).

From the uniqueness of the frame in Theorem 2 it follows immediately that each function $\lambda_i(t)$ in its structural equation is a symplectic invariant. It will be called the *ith modified principal curvature* of the curve $\Lambda(\cdot)$. Also, as a direct consequence of Theorem 2 we obtain the following

THEOREM 3. *For the given tuple of m smooth functions $\{\rho_i\}_{i=1}^m$ there exists the unique rank 1 curve in the Lagrange Grassmannian $L(W)$ such that its i-th modified principal curvature coincides with $\rho_i(t)$ for any $1\leqslant i\leqslant m$.*

In other words, the tuple of the modified principal curvatures, defined by the structural equation (7), constitutes the complete system of symplectic invariants for rank 1 curves of the constant rank. Besides, we have

PROPOSITION 2. *The following relations between the tuple of the principal curvatures $\{\beta_{2i}(t)\}_{i=0}^{m-1}$ from (5) and the tuple of the modified principal curvatures $\{\lambda_i(t)\}_{i=1}^{m}$ from (7) hold:*

$$\lambda_i(t) = C_i\beta_{2i-2} + \Phi_i(t), \quad 1 \leqslant i \leqslant m \tag{8}$$

where C_i are nonzero constants, any $\Phi_i(t)$ is some polynomial expression (over $\mathbb{R}$) without free term w.r.t. the functions $\beta_{2j}(t)$, $0 \leqslant j \leqslant i-2$ and their derivatives.

REMARK 1. Actually, the fact that the constants C_i, appearing in (8), are nonzero follows from Theorem 3: Assuming the converse, take the smallest $\bar{i}$ such that $C_{\bar{i}} = 0$. Then from all relations (8) with $1 \leqslant i < \bar{i}$ it follows that $\lambda_{\bar{i}}(t)$ is some polynomial expression w.r.t. the functions $\lambda_i(t)$, $1 \leqslant i < \bar{i}$ and their derivatives. But this contradicts the fact that the functions $\lambda_i(t)$, $1 \leqslant i \leqslant m$, are independent according to Theorem 3.

The proof of Proposition 2 will be given in section 3. As a direct consequence of Theorem 3 and the previous proposition we obtain the following extension of Theorem 1:

THEOREM 4. *For the given tuple of m smooth functions $\{\rho_i\}_{i=1}^{m}$ there exists the unique rank 1 curve in the Lagrange Grassmannian $L(W)$ such that its i-th principal curvature coincides with $\rho_i(t)$ for any $1 \leqslant i \leqslant m$.*

In other words, the tuple of the principal curvatures, defined by expansion (5), constitutes the complete system of symplectic invariants of rank 1 curves of the constant rank, the proof of which was the original goal of the present paper.

Note that a kind of the complete system of invariants for regular curves (i.e., with nondegenerated velocities $\dot{\Lambda}(t)$) was constructed in [**3**]. In the forthcoming paper we will use the ideology of the proof of Theorem 2 in order to construct a kind of complete system of symplectic invariants for generic curve $\Lambda(\cdot)$ of arbitrary rank. Let us briefly describe what objects can be obtained in this way. First one can construct a kind of a canonical parallel transform along the curve $\Lambda(\cdot)$ instead of the canonical basis $\big(E_1(t), \dots, E_m(t)\big)$ from Theorem 2 in the case of rank 1 curves. Namely, it turns out that any subspace $\Lambda(t)$ admits the canonical splitting $\Lambda(t) = \Lambda_1(t) \oplus \ldots \oplus \Lambda_s(t)$ such that on each subspace $\Lambda_i(t)$ the canonical Euclidean structure is defined $\big($for rank 1 curves $s = m$, $\Lambda_i(t) = \mathrm{span}\big(E_i(t)\big)\big)$; then for any t_0 an t_1 there exists the canonical linear mapping $P_{t_0,t_1} : \Lambda(t_0) \mapsto \Lambda(t_1)$ such that $P_{t_0,t_1}\big(\Lambda_i(t_0)\big) = \Lambda_i(t_1)$ and P_{t_0,t_1} sends the Euclidean structure of $\Lambda_i(t_0)$ to the Euclidean structure of $\Lambda_i(t_1)$ for all $1 \leqslant i \leqslant s$. Moreover, $P_{t_1,t_2} \circ P_{t_0,t_1} = P_{t_0,t_2}$ and $P_{t,t} = \mathrm{Id}$. Secondly one can define the canonical complement of $\Lambda(t)$ to W, i.e., the subspace $\Lambda^{\mathrm{comp}}(t) \in L(W)$ such that $W = \Lambda(t) \oplus \Lambda^{\mathrm{comp}}(t)$. The main idea, lying in all these constructions is that if we choose some orthonormal basis $\big(e_{1i}(t), \dots, e_{1m_i}(t)\big)$ on each subspace $\Lambda_i(t)$ (w.r.t. the canonical Euclidean structure on it), where $m_i = \dim \Lambda_i(t)$, and afterwards we take the basis on $\Lambda^{\mathrm{comp}}(t)$ dual (w.r.t. the symplectic form σ) to the basis $\big(\{e_{j1}(t)\}_{j=1}^{m_1}, \ \dots, \ \{e_{js}(t)\}_{j=1}^{m_s}\big)$ of $\Lambda(t)$, then the structural equation of the obtained moving Darboux frame in W (called *almost canonical moving frame*) has to be of the simplest possible form (with the maximal possible trivial blocks in the matrix, corresponding to this structural equation). All nontrivial blocks in this matrix correspond to some invariant operators associated with our curve, which constitute a kind of the complete system of symplectic invariants of the curve. Note that in general the subspace $\Lambda^{\mathrm{comp}}(t)$, obtained in this way, is different from the derivative subspace $\Lambda^0(t)$, the construction of which is described in Appendix: they coincide only for regular curves.

Finally let us describe a method for construction of invariants for curves in the Grassmannian $G_n(V)$ of n-dimensional subspace of $2n$-dimensional linear space V w.r.t. the action of General Linear group $GL(V)$. It turns out that this problem can be reduced to the previous problem for the curves in Lagrange Grassmannian by an appropriate symplectification. Indeed, the $4n$-dimensional linear space $V\times V^*$ can be provided with the natural symplectic structure $\sigma\big((x_i,y_1),(x_2,y_2)\big)=y_2(x_1)-y_1(x_2)$, where $x_1,x_2\in V$ and $y_1,y_2\in V^*$. To any curve $\Lambda(\cdot)$ in $G_n(V)$ one can assign canonically the curve in Lagrange Grassmannian $L(V\times V^*)$. For this let

$$\Lambda^{(*)}(t)=\mathrm{span}\{p\in V^*: p(v)=0\ \ \forall v\in\Lambda(t)\}.$$

Then the curve $\Lambda(\cdot)\times\Lambda^{(*)}(t)$ is the curve in $L(V\times V^*)$. Moreover, if the curve $\Lambda(\cdot)$ is ample in $G_n(V)$ [1], then the curve $\Lambda(\cdot)\times\Lambda^{(*)}(t)$ is ample. Any symplectic invariant of it is the invariant of the original curve. Besides, in this way one can construct the (almost) canonical moving frame also for the curves in $G_n(V)$ (in space $V\times V^*$). Of course, the curves obtained by the described symplectification are special curves in $L(V\times V^*)$, so some invariants from the structural equation of its (almost) canonical moving frame depend somehow one on another.

If we start with a curve $\Lambda(\cdot)$ in the Grassmannian $G_k(V)$ of k-dimensional subspaces in V ($\dim V=2n$), where $k\neq n$, then $\Lambda(\cdot)\times\Lambda^{(*)}(t)$ is also the curve in $L(V\times V^*)$, but it is never ample. So, we cannot apply directly the procedure of symplectification, described above. But in many cases one can build from the curves in $G_k(V)$ the curves in $G_n(V)$ in a canonical way, combining operations of extension and contraction, defined by relations (9) and (10) below, and then use the symplectification.

2. Proof of Theorem 2

First let us introduce some notations. Set $\mathcal{D}^{(0)}\Lambda(t)=\mathcal{D}_{(0)}\Lambda(t)=\Lambda(t)$ and define inductively the following subspaces $\mathcal{D}_{(i)}\Lambda(\tau)$ and $\mathcal{D}^{(i)}\Lambda(\tau)$ for any $i\in\mathbb{N}$:

$$\mathcal{D}_{(i)}\Lambda(\tau)\overset{def}{=}\left\{v\in W:\begin{array}{l}\exists\text{ a curve } l(\cdot)\text{ such that } l(t)\in\mathcal{D}_{(i-1)}\Lambda(t)\ \forall t,\\ l(\tau)=v\text{ and } l'(\tau)\in\mathcal{D}_{(i-1)}\Lambda(\tau)\end{array}\right\}\tag{9}$$

$$\mathcal{D}^{(i)}\Lambda(\tau)\overset{def}{=}\mathcal{D}^{(i-1)}\Lambda(\tau)+\left\{v\in W:\begin{array}{l}\exists\text{ a curve } l(\cdot)\text{ in } W\text{ such that}\\ l(t)\in\mathcal{D}^{(i-1)}\Lambda(t)\ \forall t\text{ and } v=l'(\tau)\end{array}\right\}\tag{10}$$

The subspaces $\mathcal{D}_{(i)}\Lambda(\tau)$ and $\mathcal{D}^{(i)}\Lambda(\tau)$ are called respectively *the ith contraction* and the ith extension of the curve $\Lambda(\cdot)$ at the point τ. In particular, directly from the definitions we have

$$\mathcal{D}_{(1)}\Lambda(\tau)=\ker\dot\Lambda(\tau),\tag{11}$$

$$\mathrm{rank}\,\dot\Lambda(\tau)=\dim\,\mathcal{D}^{(1)}\Lambda(\tau)-\dim\,\Lambda(\tau).\tag{12}$$

[1] The notion of ample curve is defined in $G_n(V)$ in the same way as in Lagrange Grassmannian.

Moreover, if for a given subspace $L \subset W$ we denote by $L^{\angle}$ its skew-symmetric complement, i.e. $L^{\angle} = \{v \in W : \sigma(v, l) = 0\ \forall l \in L\}$, then directly from the definitions it is not hard to show that the subspaces $\mathcal{D}_{(i)}\Lambda(\tau)$ and $\mathcal{D}^{(i)}\Lambda(\tau)$ are related in the following way:

$$\mathcal{D}_{(i)}\Lambda(\tau) = \left(\mathcal{D}^{(i)}\Lambda(\tau)\right)^{\angle}. \tag{13}$$

Also, from the definition the curve $\Lambda(\cdot)$ is ample at τ if and only if there exists $p \in \mathbb{N}$ such that

$$\mathcal{D}^{(p)}\Lambda(\tau) = W \text{ or, equivalently, } \mathcal{D}_{(p)}\Lambda(\tau) = 0. \tag{14}$$

Besides, if we suppose that the rank of $\dot{\Lambda}(t)$ is constant and equal to r for any t, then easily

$$\begin{array}{l} \dim\, \mathcal{D}^{(i)}\Lambda(t) - \dim\, \mathcal{D}^{(i-1)}\Lambda(t) \leqslant r, \\ \dim\, \mathcal{D}_{(i-1)}\Lambda(t) - \dim\, \mathcal{D}_{(i)}\Lambda(t) \leqslant r, \end{array}, \quad i \in \mathbb{N} \tag{15}$$

Now suppose that the curve $\Lambda(\cdot)$ has constant rank 1 on some segment I, i.e., $\mathrm{rank}\dot{\Lambda}(t) = 1$ for any $t \in I$. Our goal is to give more convenient characterization of the property of the rank 1 curve to be of the constant finite weight. This characterization is given in the following two lemmas, which was actually proved in [**4**]. Here we reformulate them in our new notations. We also give a proof of the first of them, because it is short, while for the proof of the second one we refer to the corresponding statements from [**4**].

Lemma 1. (compare with Proposition 3 in [**4**]) *Assume that* $\dim W = 2m$. *If an ample curve* $\Lambda : I \mapsto L(W)$ *has rank 1 in the segment* I*, then out of some discrete subset* $\mathcal{C} \in I$*, one has*

$$\dim \mathcal{D}^{(i)}\Lambda(t) = m + i, \quad 1 \leqslant i \leqslant m, \tag{16}$$

or, equivalently,

$$\dim \mathcal{D}_{(i)}\Lambda(t) = m - i, \quad 1 \leqslant i \leqslant m, \tag{17}$$

Proof. Actually we have to prove that the set $\mathcal{C}$ of points, where the condition (16) fails, has no accumulation point. Otherwise, if $\bar{t}$ is an accumulation point of $\mathcal{C}$, then immediately from (15) it follows that there are a natural number i_0, $1 \leqslant i_0 \leqslant m-1$, and a sequence of points $\{t_i\}_{k=1}^{\infty}$, converging to $\bar{t}$, such that $\mathcal{D}^{(i_0)}\Lambda(t_k) = \mathcal{D}^{(i_0+1)}\Lambda(t_k)$ for all $k \in \mathbb{N}$. It implies that $\mathcal{D}^{(j)}\Lambda(\bar{t}) = \mathcal{D}^{(i_0)}\Lambda(\bar{t})$ for any $j \geqslant i_0$. But by (15) again $\dim\, \mathcal{D}^{(i_0)}\Lambda(\bar{t}) \leqslant m + i_0 < 2m$. Hence $\mathcal{D}^{(j)}\Lambda(\bar{t}) \neq W$ for any $j \in \mathbb{N}$. So, by (14) the curve $\Lambda(\cdot)$ is not ample at $\bar{t}$, which contradicts our assumptions. □

Lemma 2. (see Corollary 1 and item 1 of Corollary 2 in [**4**]) *An ample curve* $\Lambda : I \mapsto L(W)$ *has the constant finite weight in the segment* I *if and only if the relations (16) or, equivalently, (17) hold at any point of* I*. In this case the weight is equal to* m^2.

REMARK 2. Actually, from the last two lemmas it follows that with any rank 1 curve of the constant weight one can associate the following curve of complete flags in W:

$$t \mapsto \Big(\mathcal{D}_{(m-1)}\Lambda(t) \subset \ldots \subset \mathcal{D}_{(1)}\Lambda(t) \subset \Lambda(t) \subset \mathcal{D}^{(1)}\Lambda(t) \subset \ldots \subset \mathcal{D}^{(m-1)}\Lambda(t)\Big).$$

Now let us start to prove Theorem 2. From now one $\Lambda(\cdot)$ is a rank 1 curve of the constant weight in $L(W)$. Without loss of generality it can be assumed that the velocities $\dot\Lambda(t)$ are nonnegative definite quadratic forms. In this case the curve $\Lambda(\cdot)$ is called *monotone increasing*. By the previous lemma $\dim \mathcal{D}_{(m-1)}\Lambda(t) = 1$. For any t choose a vector $\epsilon(t)$ such that

$$\mathcal{D}_{(m-1)}\Lambda(t) = \operatorname{span}\Big(\epsilon(t)\Big) \tag{18}$$

and the curve $t \mapsto \epsilon(t)$ is smooth. From (9) it follows easily that a smooth curve $\epsilon(\cdot)$ satisfies (18) for any t if and only if the following relations hold

$$\begin{cases} \Lambda(t) = \operatorname{span}\big(\epsilon(t), \epsilon'(t), \ldots, \epsilon^{(m-1)}(t)\big), \\ \epsilon^{(m)}(t) \notin \Lambda(t) \end{cases} \tag{19}$$

The following lemma gives the canonical normalization of the vector function $\epsilon(t)$:

LEMMA 3. *There exists the unique, up to the reflection $v \mapsto -v$, smooth curve $\epsilon(\cdot)$ of vectors in W, satisfying (18), such that $\sigma\big(\epsilon^{(m)}(t), \epsilon^{(m-1)}(t)\big) = 1$.*

PROOF. Let $\epsilon(\cdot)$ and $\tilde\epsilon(\cdot)$ be two smooth curves of vectors in W, satisfying (18). Then there exists a smooth scalar function $\alpha(t)$ such that $\tilde\epsilon(t) = \alpha(t)\epsilon(t)$. The last equation implies that

$$\tilde\epsilon^{(i)}(t) \equiv \alpha(t)\epsilon^{(i)}(t) \mod\Big(\operatorname{span}\big(\epsilon(t), \ldots \epsilon^{(i-1)}(t)\big)\Big). \tag{20}$$

Note that from the first relation of (19) it follows that

$$\sigma(\epsilon^{(m)}(t), \epsilon^{(i)}(t)) = 0 \quad 0 \leqslant i \leqslant m-2. \tag{21}$$

Indeed, from the fact that all subspaces $\Lambda(t)$ are Lagrangian and the first relation of (19) it follows that

$$\sigma(\epsilon^{(m-1)}(t), \epsilon^{(i)}(t)) \equiv 0, \quad \sigma(\epsilon^{(m-1)}(t), \epsilon^{(i+1)}(t)) \equiv 0 \quad 0 \leqslant i \leqslant m-2. \tag{22}$$

Differentiating the first identity of (22) and using the second one, we obtain (21). Therefore (20) yields that

$$\sigma\big(\tilde\epsilon^{(m)}(t), \tilde\epsilon^{(m-1)}(t)\big) = \alpha^2(t)\sigma\big(\epsilon^{(m)}(t), \epsilon^{(m-1)}(t)\big). \tag{23}$$

Since by assumption $\Lambda(\cdot)$ is monotone increasing, one has that

$$\sigma\big(\epsilon^{(m)}(t), \epsilon^{(m-1)}(t)\big) = \dot\Lambda(t)\big(\epsilon^{(m-1)}(t)\big) \geqslant 0$$

(we use the identification of $\dot\Lambda(t)$ with the quadratic form, see (1); here $\dot\Lambda(t)(v)$ is the value of the quadratic form $\dot\Lambda(t)$ at a vector v). On the other hand,

$$\sigma\big(\epsilon^{(m)}(t), \epsilon^{(m-1)}(t)\big) \neq 0$$

Assuming the converse and taking into account (21), we obtain that $\epsilon^{(m)}(t) \in \Lambda(t)$, which contradicts the second relation in (19). So, $\sigma\big(\epsilon^{(m)}(t), \epsilon^{(m-1)}(t)\big) > 0$. Setting $\alpha(t) = \pm\Big(\sigma\big(\epsilon^{(m)}(t), \epsilon^{(m-1)}(t)\big)\Big)^{-1/2}$, we obtain from (23) that

$$\sigma\big(\tilde{\epsilon}^{(m)}(t), \tilde{\epsilon}^{(m-1)}(t)\big) = 1.$$

It remains only to notice that by our constructions $\tilde{\epsilon}(t)$, satisfying the last relation, is defined up to the sign. □

Now suppose that $\epsilon(t)$ is as in the previous lemma. We set

$$E_i(t) = \epsilon^{(i-1)}(t), \quad 1 \leqslant i \leqslant m; \quad F_m(t) = \epsilon^{(m)}(t) \tag{24}$$

By the first relation of (19) the tuple $\big(E_1(t), \dots, E_m(t)\big)$ satisfies the first condition of Theorem 2, while together with $F_m(t)$ it satisfies the first two equations of (7). Besides, by (21) the choice of the vectors, defined in (24), does not contradict the relations (6).

To finish the proof of the theorem it remains to complete the tuple

$$\big(E_1(t), \dots, E_m(t), F_m(t)\big)$$

to the moving Darboux frame in W, which satisfies the last two equations of (7), and to show that such complement is unique (the freedom in the sign, mentioned in the last sentence of Theorem 2 will follow then from the freedom up to the sign in the choice of $\epsilon(t)$ in Lemma 3).

For this we analyze the structural equations of all possible moving Darboux's frames, and choose among them one, which has the maximal possible number of zero entries in the matrix of its structural equation. First take some tuple $\{\overline{F}_i(t)\}_{i=1}^{m-1}$ such that

$$\big(E_1(t), \dots, E_m(t), \overline{F}_1(t), \dots, \overline{F}_{m-1}(t), F_m(t)\big) \tag{25}$$

is a moving Darboux's frame in W. Then from the definition of Darboux's basis (see (6)), and the first two equations of (7) it follows that there exist functions $\bar{\xi}_{ij}(t)$, $1 \leqslant i, j \leqslant m-1$ such that

$$\overline{F}_i'(t) = \sum_{j=1} \bar{\xi}_{ij}(t) E_j(t) - (1 - \delta_{1i})\overline{F}_{i-1}(t), \quad \bar{\xi}_{ij}(t) = \bar{\xi}_{ji}(t), \quad 1 \leqslant i, j \leqslant m-1, \tag{26}$$

where δ_{kl} is the Kronecker symbol. Further, by the definition of Darboux's basis for the given tuple $\{\widehat{F}_i(t)\}_{i=1}^{m-1}$ of curves of vector in W the frame

$$\big(E_1(t), \dots, E_m(t), \widehat{F}_1(t), \dots, \widehat{F}_{m-1}(t), F_m(t)\big) \tag{27}$$

is a moving Darboux's frame in W if and only if there exist functions $b_{ij}(t)$, $1 \leqslant i, j \leqslant m-1$ such that

$$\begin{cases} \widehat{F}_i(t) = \overline{F}_i(t) + \displaystyle\sum_{j=1}^{m-1} b_{ij}(t) E_j(t), \quad 1 \leqslant i \leqslant m-1 \\ b_{ij}(t) = b_{ji}(t), \quad 1 \leqslant i, j \leqslant m-1. \end{cases} \tag{28}$$

Besides, similarly to (26), for the tuple $\{\widehat{F}_i(t)\}_{i=1}^{m-1}$ there exist functions $\hat{\xi}_{ij}(t)$, $1 \leqslant i, j \leqslant m-1$ such that

$$\widehat{F}_i'(t) = \sum_{j=1} \hat{\xi}_{ij}(t) E_j(t) - (1 - \delta_{1i})\widehat{F}_{i-1}(t), \quad \hat{\xi}_{ij}(t) = \hat{\xi}_{ji}(t), \quad 1 \leqslant i, j \leqslant m-1. \tag{29}$$

Now we are ready to find the transformation rule from the coefficients $\bar{\xi}_{ij}$ of the structural equation for the original frame (25) to the coefficients $\hat{\xi}_{ij}$ of the structural equation for the frame (27): Set

$$b_{im}(t) = b_{mi}(t) = 0, \quad 1 \leqslant i \leqslant m. \tag{30}$$

Then, substituting (28) into (29), and using (26) one can easily obtain

$$\hat{\xi}_{ij}(t) = \bar{\xi}_{ij}(t) + b'_{ij}(t) + (1-\delta_{1i})b_{i-1,j}(t) + (1-\delta_{1j})b_{i,j-1}(t). \tag{31}$$

From transformation rule (31) it follows immediately that Theorem 2 will follow from the following

LEMMA 4. *For the given smooth curve of $m \times m$ symmetric matrices $\overline{\Omega}(t) = \big(\bar{\xi}_{ij}(t)\big)_{ij=1}^{m}$ there exists the unique smooth curve of symmetric $m \times m$ matrices $B(t) = \big(b_{ij}(t)\big)_{ij=1}^{m}$, satisfying (30), such that*

$$\bar{\xi}_{ij}(t) + b'_{ij}(t) + (1-\delta_{1i})b_{i-1,j}(t) + (1-\delta_{1j})b_{i,j-1}(t) = 0, \quad i \neq j, \tag{32}$$

where δ_{kl} is the Kronecker symbol.

PROOF. For $m = 1$ there is nothing to prove. Suppose that $m > 1$. Note that by the symmetricity of equations (32) w.r.t. permutation $(ij) \mapsto (ji)$ it is enough to prove existence and uniqueness of $b_{ij}(t)$ with $i \geqslant j$. We will "fill" step by step the lower triangle (including the diagonal) of the matrix $B(t)$ starting from the $(m-1)$th row (the last row is given by (30)). Taking into account (30), from equation (32) for $i = m$ it follows that

$$b_{m-1,j}(t) = -\bar{\xi}_{mj}(t), \quad 1 \leqslant j \leqslant m-1.$$

In this way and using symmetricity we have filled $(m-1)$th row of $B(t)$.

Now suppose by induction that for some $2 < i \leqslant m-1$ we have filled $\bar{i}$th rows of the matrix $B(t)$ for all $\bar{i} \geqslant i$ (note that if $i = 2$, we are already done from symmetricity). We would like to determine the $(i-1)$th row. From equation (32) for $j = 1$ it follows that

$$b_{i-1,1}(t) = -\bar{\xi}_{i,1}(t) - b'_{i,1}(t). \tag{33}$$

Since the righthand side of (33) is determined by the induction hypothesis, $b_{i-1,1}(t)$ is determined. Other elements $b_{i-1,j}(t)$ with $2 \leqslant j \leqslant i$ are determined from the following recursive formula

$$b_{i-1,j}(t) = -\bar{\xi}_{ij}(t) - b'_{ij}(t) - b_{i-1,j-1}(t),$$

which follows from (32). In this way and using symmetricity we have filled $(i-1)$th row of $B(t)$. The proof by induction is completed. □

Let $B(t) = \big(b_{ij}(t)\big)_{ij=1}^{m}$ be as in the previous lemma. Then, setting $F_i(t) = \widehat{F}_i(t)$, $1 \leqslant i \leqslant m-1$, where $\widehat{F}_i(t)$ is defined by the first equation of (28), we obtain the moving Darboux frame, required in Theorem 2. Note that from (31) with $j = i$ it follows that the functions λ_i from (7) satisfy

$$\lambda_{m-i+1}(t) = \bar{\xi}_{ii}(t) + b'_{ii}(t) + 2(1-\delta_{1i})b_{i,i-1}(t), \quad 1 \leqslant i \leqslant m.$$

The proof of Theorem 2 is completed.

3. Proof of Proposition 2

Throughout this section we will use the following notations: For a given tuple $\{\psi_i(t)\}_{i=1}^N$ of smooth functions we denote by $\mathrm{Pol}\big(\{\psi_i(t)\}_{i=1}^N\big)$ any function, which can be expressed as a polynomial (over $\mathbb{R}$) without free term w.r.t. the functions $\psi_i(t)$, $1 \leqslant i \leqslant N$, and their derivatives. Also, we denote by $\mathrm{Lin}\big(\{\psi_i(t)\}_{i=1}^N\big)$ any function, which can be expressed as a linear combination (over $\mathbb{R}$) of the functions $\psi_i(t)$, $1 \leqslant i \leqslant N$, and their derivatives.

One can try to prove Proposition 2, expressing $\beta_{2i}(t)$, $0 \leqslant i \leqslant m-1$, by $\lambda_j(t)$ with the help of the structural equation (7) and formula (4): Let $W \cong \mathbb{R}^m \times \mathbb{R}^m, \Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$, $E_i(t) = \big(\pi_{1i}(t), \dots, \pi_{2m,i}(t)\big)$. Denote also by $\Pi_1(t)$ and $\Pi_2(t)$ the following $m \times m$-matrices:

$$\Pi_1(t) = (\pi_{ji}(t))_{1\leqslant j\leqslant m, 1\leqslant i\leqslant m}, \quad \Pi_2(t) = (\pi_{ji}(t))_{m+1\leqslant j\leqslant 2m, 1\leqslant i\leqslant m}$$

Then $S_t = \Pi_2(t)\Pi_1(t)^{-1}$. Using the structural equation (7) one can find derivatives of S_t of any order. So, using (4), one can compute in general the Taylor formula for g_Λ up to the required order. But in this way one meets rather cumbersome computations even in the case $m = 2$.

Our proof of Proposition 2 basically consists of the following two steps:

Step 1. From (7) one can express $E_1^{(2m)}(t)$ as a linear combination of $E_1^{(k)}(t)$, where $k = 0, \dots, 2m-1$: it turns out that if $E_1^{(2m)}(t) = \sum_{k=0}^{2m-1} \Gamma_k(t) E_1^{(k)}(t)$, then

$$\Gamma_{2i}(t) = (-1)^i \lambda_{m-i}(t) + \mathrm{Lin}\big(\{\lambda_j(t)\}_{j=1}^i\big), \quad 0 \leqslant i \leqslant m-1 \tag{34}$$

Indeed, using the first, second, and forth relations of (7), it is easy to show by induction that for all $1 \leqslant s \leqslant m-1$

(35)

$$F_{m-s}(t) = (-1)^s E_1^{(m+s)}(t) + \sum_{k=1}^{s-1}\Big((-1)^{s-k}\lambda_k(t) + \mathrm{Lin}\big(\{\lambda_l(t)\}_{l=1}^{k-1}\big)\Big) E_1^{(m+s-2k)}(t) +$$
$$\sum_{k=1}^{s-1}\Big((-1)^{s-k}(s-k)\lambda_k'(t) + \mathrm{Lin}\big(\{\lambda_l(t)\}_{l=1}^{k-1}\big)\Big) E_1^{(m+s-2k-1)}(t) + \lambda_s(t) E_1^{(m-s)}(t).$$

Then substituting (35) for $s = m-1$ in the third relation of (7) one gets (34).

Step 2. First let us give a sketch of what we are going to do. Let

$$\big(e_1(t), \dots, e_m(t), f_1(t), \dots f_m(t)\big) \tag{36}$$

be the canonical basis of the curve $\Lambda(\cdot)$, constructed in [**4**] (for the sake of completeness we will describe this construction in Appendix). Collecting some information about its structural equation from [**4**] and [**5**], we will express $e_1^{(2m)}(t)$ as a linear combination of $e_1(t), \dots e_1^{(2m-1)}(t)$. Namely, if $e_1^{(2m)}(t) = \sum_{k=0}^{2m-1} \gamma_k(t) e_1^{(k)}(t)$, then

$$\gamma_{2i}(t) = \overline{C}_i \beta_{2(m-1-i)}(t) + \mathrm{Pol}\big(\{\beta_{2j}(t)\}_{j=0}^{i-1}\big), \quad 0 \leqslant i \leqslant m-1. \tag{37}$$

On the other hand, from the structural equation again it will follow that

$$e_1(t) = \mathrm{const}\, E_1(t), \tag{38}$$

which implies that $\gamma_k(t) = \operatorname{const}\Gamma_k(t)$. Comparing (34) with (37) in view of the last relation, one gets (8), which together with Remark 1 implies Proposition 2.

Now we start to prove formulas (37) and (38). All information from [**4**] and [**5**] that we need about the frame (36) can be summarized in the following

LEMMA 5. *The frame (36), constructed in* [**4**], *satisfies the following equation*

$$\begin{cases} e_i'(t) = \sum_{j=1}^{m} \alpha_{ij}(t)e_j(t) + m^2\delta_{mi}f_m(t) \\ f_i'(t) = \sum_{j=1} \sigma_{ij}(t)e_j(t) - \sum_{j=1}^{m} \alpha_{ji}(t)f_j(t), \end{cases} \tag{39}$$

where $\sigma_{ij}(t) = \sigma_{ji}(t)$*;* δ_{mi} *is the Kronecker symbol;*

$$\alpha_{ij}(t) \equiv 0 \text{ for } i < j-1; \quad \alpha_{i-1,i}(t) \equiv \frac{(i-1)(2m-i+1)}{m-i+1}; \tag{40}$$

$$\alpha_{ii} = 0; \tag{41}$$

$$\alpha_{ij}(t) = \begin{cases} \nu_{ij}\beta_{i-j-1}(t) + \operatorname{Pol}\Big(\{\beta_{2s}(t)\}_{s=0}^{\frac{i-j-3}{2}}\Big), & i-j \text{ is positive odd}, \\ \operatorname{Pol}\Big(\{\beta_{2s}(t)\}_{s=0}^{\frac{i-j-2}{2}}\Big), & i-j \text{ is positive even}; \end{cases} \tag{42}$$

$$\sigma_{ij}(t) = \begin{cases} c_{ij}\beta_{2m-i-j}(t) + \operatorname{Pol}\Big(\{\beta_{2s}(t)\}_{s=0}^{m-\frac{i+j+2}{2}}\Big), & i+j \text{ is even}, \\ \operatorname{Pol}\Big(\{\beta_{2s}(t)\}_{s=0}^{m-\frac{i+j+1}{2}}\Big), & i+j \text{ is odd}; \end{cases} \tag{43}$$

ν_{ij} *and* c_{ij} *are some constants.*

Relations (40) are exactly items 1 and 2 of Lemma 7.3 from [**4**]. Relation (41) for $1 \leqslant i \leqslant m-1$ is exactly relation (1.79) in [**5**], while for $i = m$ it can be obtained easily from formula (7.52) of [**4**], taking into account identity (1.74) from [**5**]. Further, relation (42) for $1 \leqslant i \leqslant m-1$ is more specified version of item 3 of Lemma 7.3 from [**4**], which follows from the proof of this lemma, while for $i = m$ it can be obtained without difficulties from formula (7.52) of [**4**]. Finally, relation (43) in the case of even $i+j$ is exactly Lemma 1.6 of [**5**], while in the case of odd $i+j$ it follows from the proof of this lemma (see, for example, formula (1.67) there).

One can reformulate Lemma 5 in more convenient form if one denotes

$$\forall\, 1 \leqslant i \leqslant m: \quad e_{m+i}(t) \stackrel{def}{=} f_{m-i+1}(t); \quad \mathcal{E}(t) \stackrel{def}{=} \begin{pmatrix} e_1(t) \\ \vdots \\ e_{2m}(t) \end{pmatrix}. \tag{44}$$

If $\mathcal{M}(t)$ is $2m \times 2m$ matrix, $\mathcal{M}(t) = \{\mu_{ij}(t)\}_{i,j=1}^{2m}$, such that

$$\mathcal{E}'(t) = \mathcal{M}(t)\mathcal{E}(t), \tag{45}$$

then from (39)-(43) it follows easily that

$$\mu_{ij}(t)=\begin{cases} 0 & j>i+1 \text{ and } j=i \\ \chi_{ij} & j=i+1 \\ \chi_{ij}\beta_{i-j-1}(t)+\mathrm{Pol}\Big(\{\beta_{2s}(t)\}_{s=0}^{\frac{i-j-3}{2}}\Big), & i-j \text{ is positive odd,} \\ \mathrm{Pol}\Big(\{\beta_{2s}(t)\}_{s=0}^{\frac{i-j-2}{2}}\Big), & i-j \text{ is positive even,} \end{cases} \tag{46}$$

where χ_{ij} are constants. Set also $\chi_{2m,2m+1}=1$. Then combining (45) with (46), one can obtain without difficulties by induction that
(47)

$$(1-\delta_{i,2m})e_{i+1}(t)=\Big(\prod_{k=1}^{i}\chi_{k,k+1}\Big)^{-1}\Big(\sum_{j=0}^{i-2}\kappa_{ij}(t)e_1^{(j)}(t)+e_1^{(i)}(t)\Big),\quad 0\leqslant i\leqslant 2m-1,$$

where $\delta_{i,2m}$ is the Kronecker symbol,

$$\kappa_{ij}(t)=\begin{cases} \rho_{ij}\beta_{i-j-2}(t)+\mathrm{Pol}\Big(\{\beta_{2s}(t)\}_{s=0}^{\frac{i-j-4}{2}}\Big) & i-j \text{ is positive even} \\ \mathrm{Pol}\Big(\{\beta_{2s}(t)\}_{s=0}^{\frac{i-j-3}{2}}\Big) & i-j \text{ is positive odd} \end{cases}, \tag{48}$$

$$\rho_{ij}=\sum_{s=1}^{j+1}\left(\chi_{i-j-1+s,s}\prod_{l=s}^{i-j-2+s}\chi_{l,l+1}\right),\quad i-j \text{ is positive even.} \tag{49}$$

Relation (47), used for $i=2m$, implies (37). Actually, $\overline{C}_i$ in (37) can be taken as

$$\overline{C}_i=-\rho_{2m,i}\Big(\prod_{k=1}^{i}\chi_{k,k+1}\Big)^{-1}. \tag{50}$$

Further, relation (47), used for $i\leqslant m$, implies that $e_1(t)$, taken as $\epsilon(t)$, satisfies (19). Therefore $e_1(t)=\alpha(t)E_1(t)$. Moreover,

$$\sigma\big(e_1^{(m)},e_1^{(m-1)}\big)=\chi_{m,m+1}\prod_{l=1}^{m-1}\chi_{l,l+1}^2,$$

which implies that $\alpha(t)$ is constant (equal to $(\chi_{m,m+1})^{1/2}\prod_{l=1}^{m-1}\chi_{l,l+1}$). This proves formula (38), which completes the proof of Proposition 2.

Remark 3. Actually, the existence part of Theorem 4 will follow from (37) only (without using the modified principal curvatures $\lambda_j(t)$ and Proposition 2), if one shows that

$$\overline{C}_i\neq 0,\quad 0\leqslant i\leqslant m-1. \tag{51}$$

But even after finding explicit expressions for χ_{ij} we did not succeed to verify (51), using formula (50). Instead, the presence of the complete system of invariants $\{\lambda_i(t)\}_{i=1}^m$ and identity (8) imply (51) automatically, as was mentioned already in Remark 1 above.

4. Appendix

In this Appendix we briefly describe the construction of the canonical moving Darboux frame for a rank 1 curve of the constant weight, introduced in [**4**] and used in the previous section. The construction is based on the fact that the set $\Lambda^{\pitchfork}$ of all Lagrangian subspaces transversal to a subspace $\Lambda \in L(W)$ can be naturally endowed with the structure of an affine space over the linear space $\mathrm{Quad}\,(\Lambda^*)$ of quadratic forms on Λ^* or, equivalently, over the linear space $\mathrm{Symm}\,(\Lambda^*)$ of self-adjoint linear mappings from Λ^* to Λ. Indeed, as in Introduction, for a given $\Gamma \in \Lambda^{\pitchfork}$ denote by $\mathcal{B}_{\Lambda,\Gamma}$ the following linear mapping from Γ to Λ^* : $v \mapsto \sigma(v,\cdot), \quad v \in \Lambda_1,\, v \in \Gamma$. Then the operation of subtraction on $\Lambda^{\pitchfork}$ with values in $\mathrm{Quad}(\Lambda^*)$ can be defined as follows :

$$(\Gamma - \Delta)(l) = \sigma\big((\mathcal{B}_{\Lambda,\Gamma})^{-1}l, (\mathcal{B}_{\Lambda,\Delta})^{-1}l\big), \quad \Gamma, \Delta \in \Lambda^{\pitchfork},\ l \in \Lambda^*. \tag{52}$$

It is not difficult to show that $\Lambda^{\pitchfork}$ endowed with this operation of subtraction satisfies the axioms of affine space.

Consider now some curve $\Lambda(\cdot)$ in $L(W)$. Fix some parameter τ. Note that if the curve $\Lambda(\cdot)$ is ample at τ, then $\Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ for all t from a punctured neighborhood of τ. Then we obtain the curve $t \mapsto \Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ in the affine space $\Lambda(\tau)^{\pitchfork}$. Denote by $\Lambda_\tau(t)$ the identical embedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$. The velocity $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is an element of the underlying linear space, i.e., it is well defined self-adjoint mappings from Λ^* to Λ. Now let $\Lambda(\cdot)$ be a rank 1 curve in $L(W)$. For definiteness suppose that it is monotone increasing. Then $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is a nonpositive definite rank 1 self-adjoint linear mapping from Λ^* to Λ and for $t \neq \tau$ there exists a unique, up to the sign, vector $w(t,\tau) \in \Lambda(\tau)$ such that $\langle v, \frac{\partial}{\partial t}\Lambda_\tau(t)v\rangle = -\langle v, w(t,\tau)\rangle^2$ for any $v \in \Lambda(\tau)^*$. The properties of the vector function $t \mapsto w(t,\tau)$ can be summarized as follows (see [**4**], section 7, Proposition 4 and Corollary 2):

Proposition 3. *If $\Lambda(\cdot)$ is a rank 1 curve of the constant weight in $L(W)$, then for any τ the function $t \mapsto w(t,\tau)$ has a pole of order m at $t = \tau$. Moreover, if we write down the expansion of $t \mapsto w(t,\tau)$ in Laurent series at $t = \tau$,*

$$w(t,\tau) = \sum_{i=1}^{m} e_i(\tau)(t-\tau)^{i-1-l} + O(1), \tag{53}$$

then the vector coefficients $e_1(\tau), \dots, e_m(\tau)$ constitute a basis of the subspace $\Lambda(t)$.

So, formula (53) defines the canonical basis $e_1(\tau), \dots, e_m(\tau)$ on each subspace $\Lambda(\tau)$ of the rank 1 curve $\Lambda(\cdot)$ of constant weight. In order to complete this basis to some canonical moving Darboux's frame in W, one can exploit the affine structure again: Fixing an "origin" Δ in $\Lambda(\tau)^{\pitchfork}$ we obtain a vector function $t \mapsto \big(\Lambda_\tau(t) - \Delta\big)$ with values in $\mathrm{Quad}\,(\Lambda^*)$ (or $\mathrm{Symm}\,(\Lambda^*)$). Actually, the fact that the curve $\Lambda(\cdot)$ is ample at the point τ is equivalent to the fact that the vector function $t \mapsto \big(\Lambda_\tau(t) - \Delta\big)$ has the pole at $t = \tau$. Using only the axioms of affine space, one can prove easily that there exists a unique subspace $\Lambda^0(\tau) \in \Lambda^{\pitchfork}$ such that the free term in the expansion of the vector function $t \mapsto \big(\Lambda_\tau(t) - \Lambda^0(\tau)\big)$ to the Laurent series at τ is equal to zero. The curve $\tau \mapsto \Lambda^0(\tau)$ is called the *derivative curve of the ample curve* $\Lambda(\cdot)$. Now let $f_1(\tau), \dots, f_m(\tau)$ be a basis of $\Lambda^0(\tau)$ dual to the canonical basis of $\Lambda(\tau)$, i.e. $\sigma(f_i(\tau), e_j(\tau)) = \delta_{ij}$. The tuple $\big(e_1(\tau), \dots, e_m(\tau), f_1(\tau), \dots, f_m(\tau)\big)$ is exactly the canonical moving Darboux's frame in W, properties of which we used in section 3.

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