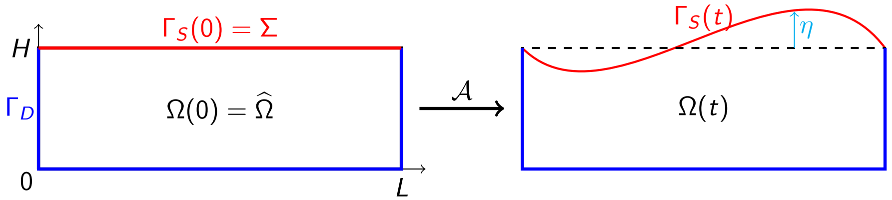
# Numerical solutions to fluid–structure interactions involving compressible fluids




$$\Omega(t) := \{(r, x_d) \in \Sigma \times \mathbb{R} \,:\, 0 < x_d < H + \eta(r)\}.$$

$$\mathcal{A} : \widehat{\Omega}(\widehat{x}) \to \Omega(x), \quad x = \mathcal{A}(t, \widehat{x}) = \left(\widehat{r}, \hat{x}_d \frac{H + \eta(t, r)}{H}\right).$$

$\widehat{\Omega} = \Omega_0 = \Sigma \times [0, H]$ $\qquad \partial\Omega = \Gamma_S(t) \cup \Gamma_D$

$\Gamma_D$: fixed walls $\quad \Gamma_S(t)$: top surface $\quad \eta$: displacement of the plate

**Hyperthesis**

The structure moves only in the vertical direction.

# Governing equations

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \text{ in } I \times \Omega, \tag{1a}$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div} \boldsymbol{\tau} + \varrho \mathbf{f}, \text{ in } I \times \Omega. \tag{1b}$$

$$\partial_t z + \alpha \Delta^2 \eta - \beta \Delta \eta = g + \mathbf{e}_d \cdot \mathbf{F}, \ z = \partial_t \eta, \quad \text{in} \quad I \times \Sigma, \tag{1c}$$

$\varrho = \varrho(t, x)$: is the fluid density $\quad$ $\mathbf{u} = \mathbf{u}(t, x)$: the velocity field

$g = g(t, r)$ given function $\quad$ $z$: the velocity of the structure

$$\mathbf{u}|_{\Gamma_S} \circ \mathcal{A} = z \mathbf{e}_d, \quad \mathbf{F} = -(\boldsymbol{\tau} \cdot \mathbf{n}) \circ \mathcal{A}\, \mathcal{J}, \quad \boldsymbol{\tau} = \mathbf{S} - p \mathbf{I},$$
$$\mathbf{S} = 2\mu \mathbf{D}(\mathbf{u}) + \lambda \mathrm{div} \mathbf{u}\, \mathbf{I}, \quad p = a \varrho^\gamma, \quad \mu > 0, \ \lambda \geq 0, \ a > 0, \ \gamma > 1.$$

$\mathcal{J}$: determinant of the Jacobian of the ALE mapping $\mathcal{A}$

$$\mathbf{u}|_{\Gamma_D} = 0,$$
$$\eta|_{\partial \Sigma} = 0, \quad \nabla \eta|_{\partial \Sigma} = 0,$$

Existence theory is recently done [Breit and Schwarzacher, ARMA, 2018].

No literature results on numerical analysis

Aim: a numerical approximation satisfies the density positivity, mass conservation, energy dissipation, the renormalized continuity equation, and consistent to the weak solution

## Backward Euler

$$\delta_t r^k(\widehat{x}) = \frac{r^k(\widehat{x}) - r^{k-1}(\widehat{x})}{\tau}, \quad \delta_t^2 r^k(\widehat{x}) = \frac{\delta_t r^k(\widehat{x}) - \delta_t r^{k-1}(\widehat{x})}{\tau}.$$

## Material derivative

$$D_t^{\mathcal{A}} r_\tau^k = \frac{r_\tau^k - r_\tau^{k-1} \circ X_k^{k-1}}{\tau}, \quad X_k^{k-1} = \mathcal{A}_{k-1} \circ \mathcal{A}_k^{-1}.$$

## "Conservative" derivative

$$D_t r^k = \frac{r^k - r^{k-1} \circ X_k^{k-1} \mathcal{J}_k^{k-1}}{\tau}.$$

## $\mathcal{P}^0$, $\mathcal{P}^1_{CR}$, $\mathcal{P}^2$

$$X_h = \left\{ \varphi \in L^1(\Omega) \;\middle|\; \varphi|_K = \mathrm{const} \in \mathbb{R} \text{ for any } K \in \mathcal{T}_h \right\},$$

$$\mathbf{Y}_h = \left\{ v \in L^2(\Omega) \;\middle|\; v|_K = \text{affine function on } K \in \mathcal{T}_h, \int_\sigma [\![ v ]\!]_\sigma \, \mathrm{d}Sx = 0 \text{ for } \sigma \in \mathcal{E}_{\mathrm{I}} \right\}$$

$$W_h = \left\{ q \in L^2(\Sigma) \;\middle|\; q|_\sigma \in \mathcal{P}^2(\sigma) \text{ for } \sigma \in \Sigma_h \right\}.$$

## Lemma 1 (Discrete Reynolds transport )

$$\delta_t \int_{\Omega^k} r^k \,\mathrm{d}x = \int_{\Omega^k} D_t r^k \,\mathrm{d}x = \int_{\Omega^k} \left( D_t^{\mathcal{A}} r^k + \mathrm{div}\mathbf{w}_\tau^k \; r^{k-1} \circ X_k^{k-1} \right) \mathrm{d}x.$$

Geometrical conservation law holds.

$$\widehat{\mathbf{w}}_\tau^k(\widehat{x}) = \frac{\mathcal{A}_k - \mathcal{A}_{k-1}}{\tau} \text{ domain velocity}$$

$$\boxed{\tau \,\mathrm{div}\mathbf{w}_\tau^k = 1 - \mathcal{J}_k^{k-1}.}$$

# Fully discrete scheme on the current domain

We seek a solution $(\varrho_\tau^k, \mathbf{u}_\tau^k, \eta_\tau^k) \in (X_h(\Omega^k), \mathbf{Y}_h(\Omega^k), W_h(\Sigma))$ for all $k \in \{1, \dots, N_t\}$ and for all $(\varphi, \boldsymbol{\phi}, \psi) \in (X_h(\Omega^k), \mathbf{Y}_h(\Omega^k), W_h(\Sigma))$ with $\Pi_{\mathcal{E}}[\boldsymbol{\phi} \circ \mathcal{A}] = \Pi_{\mathcal{E}}[\Pi_p[\psi]]\mathbf{e}_d$, such that the following hold:

$$\int_{\Omega^k} D_t \varrho_\tau^k \varphi \,\mathrm{d}x + \int_{\Omega^k} \mathrm{div}_\tau^{\mathrm{up}}(\varrho_\tau^k, \mathbf{v}_\tau^k)\varphi \,\mathrm{d}x = 0; \tag{2a}$$

$$\begin{aligned} &\int_{\Omega^k} D_t \left(\varrho_\tau^k \left\langle \mathbf{u}_\tau^k \right\rangle\right) \cdot \boldsymbol{\phi} + \mathrm{div}_\tau^{\mathrm{up}}(\varrho_\tau^k \left\langle \mathbf{u}_\tau^k \right\rangle, \mathbf{v}_\tau^k) \cdot \boldsymbol{\phi} \,\mathrm{d}x + \int_{\Omega^k} \mathbf{S}(\nabla \mathbf{u}_\tau^k) : \nabla \boldsymbol{\phi} \,\mathrm{d}x \\ &\quad + 2\mu \sum_{\sigma \in \mathcal{E}_{\mathrm{I}}^k} \int_\sigma \frac{1}{h} \left[\!\left[ \mathbf{u}_\tau^k \right]\!\right] \left[\!\left[ \boldsymbol{\phi} \right]\!\right] \mathrm{d}Sx - \int_{\Omega^k} p(\varrho_\tau^k) \mathrm{div} \boldsymbol{\phi} \,\mathrm{d}x + \int_\Sigma \delta_t z_\tau^k \psi \,\mathrm{d}Sx \\ &\quad + \int_\Sigma \left(\alpha \Delta \eta_\tau^k \Delta \psi + \beta \nabla \eta_\tau^k \nabla \psi\right) \mathrm{d}Sx = \int_{\Omega^k} \varrho_\tau^k \mathbf{f}_\tau^k \cdot \boldsymbol{\phi} \,\mathrm{d}x + \int_\Sigma g_\tau^k \psi \,\mathrm{d}Sx; \end{aligned} \tag{2b}$$

where $z_\tau^k = \delta_t \eta_\tau^k,\ \mathbf{v}_\tau^k = \mathbf{u}_\tau^k - \mathbf{w}_\tau^k, \varrho_h^0 = \Pi_{\mathcal{T}}[\varrho_0],\ \mathbf{u}_\tau^0 \in \Pi_{\mathcal{T}}[\mathbf{u}_0],\ \eta_\tau^0 = 0,\ z_\tau^0 = 0$ and the boundary conditions

$$\left\langle \mathbf{v}_\tau \right\rangle_\sigma = 0, \quad \left[\!\left[ \varrho_\tau^k \right]\!\right]_\sigma = 0,\ \forall\, \sigma \in \mathcal{E}_{\mathrm{E}}.$$

## Renormalized continuity equation

Let $(\varrho_\tau, \mathbf{u}_\tau, \eta_\tau)$ be a solution of the scheme (2). Let $B \in C^2(R)$.

$$\delta_t \left( \int_{\Omega^k} B(\varrho_\tau^k) \,\mathrm{d}x \right) + \int_{\Omega^k} \left( \varrho_\tau^k B'(\varrho_\tau^k) - B(\varrho_\tau^k) \right) \mathrm{div} \mathbf{u}_\tau^k \,\mathrm{d}x + D_0 = 0.$$

$$D_1 = \frac{1}{\tau} \int_{\Omega^k} \mathcal{J}_k^{k-1} \left( B(\varrho_\tau^{k-1} \circ X_k^{k-1}) - B(\varrho_\tau^k) - B'(\varrho_\tau^k) \big( \varrho_\tau^{k-1} \circ X_k^{k-1} - \varrho_\tau^k \big) \right) \mathrm{d}x$$

$$D_2 = \sum_{\sigma \in \mathcal{E}_{\mathrm{I}}^k} \int_\sigma B''(\zeta) \left[\!\left[ \varrho_\tau^k \right]\!\right]^2 \left( h^\varepsilon + \frac{1}{2} \left| \left\langle \mathbf{v}_\tau^k \cdot \mathbf{n} \right\rangle_\sigma \right| \right) \mathrm{d}Sx$$

$D_1, D_2 \geq 0$ for convex $B$.

## Mass conservation

$$\int_{\Omega^k} \varrho_\tau^k \,\mathrm{d}x = \int_{\Omega^{k-1}} \varrho_\tau^{k-1} \,\mathrm{d}x = \cdots = \int_{\widehat{\Omega}} \varrho_\tau^0 \,\mathrm{d}x =: M_0, \quad \text{for all } k = 1, \ldots, N_t.$$

Setting $\varphi = 1$ in (2a) implies $0 = \int_{\Omega^k} D_t \varrho_\tau^k \,\mathrm{d}x = \delta_t \left( \int_{\Omega^k} \varrho_\tau^k \,\mathrm{d}x \right)$

## Theorem 2 (Existence of a positive solution.)

*Let the assumption of no-touching hold. Let* $0 < \varrho_\tau^{k-1} \in X_h(\Omega^{k-1})$, $(\mathbf{u}_\tau^{k-1}, \eta_\tau^{k-1}, z_\tau^{k-1}) \in \mathbf{Y}_h(\Omega^{k-1}) \times W_h(\Sigma) \times W_h(\Sigma)$ *be given. Then there exists* $0 < \varrho_\tau^k \in X_h(\Omega)$ *and* $(\mathbf{u}_\tau^k, \eta_\tau^k, z_\tau^k := \frac{\eta_\tau^k - \eta_\tau^{k-1}}{\tau}) \in \mathbf{Y}_h(\Omega) \times W_h(\Sigma) \times W_h(\Sigma)$ *satisfying the discrete problem* (2).

## Internal energy balance

Let $(\varrho_\tau, \mathbf{u}_\tau, \eta_\tau)$ be a solution of the discrete problem (2).

$$\delta_t \left( \int_{\Omega^k} \mathcal{H}(\varrho_\tau^k) \,\mathrm{d}x \right) + \int_{\Omega^k} p(\varrho_\tau^k) \mathrm{div} \mathbf{u}_\tau^k \,\mathrm{d}x + D_1 + D_2 = 0.$$

$$\mathcal{H}(\varrho) = \varrho \int_1^\varrho \frac{p(s)}{s^2} ds$$

# Theorem 3 ( Energy stability of the fully-discrete scheme)

*Let* $\left(\varrho_\tau^k, \mathbf{u}_\tau^k, \eta_\tau^k\right)_{k=1}^{N_T}$ *be a family of numerical solutions obtained by the scheme* (2)*. Then for any* $N = 1, \ldots, N_T$ *the energy is stable in the following sense*

$$\int_{\Omega^N} E_f^N \,\mathrm{d}x + \int_\Sigma E_s^N \,\mathrm{d}Sx + \tau \sum_{k=1}^N \int_{\Omega^k} \left(2\mu |\mathbf{D}(\mathbf{u}_\tau^k)|^2 + \lambda |\mathrm{div}\mathbf{u}_\tau^k|^2\right) \mathrm{d}x$$

$$+ \frac{\tau^2}{2} \sum_{k=1}^N \int_\Sigma \left(|\delta_t z_\tau^k|^2 + \alpha \left|\Delta z_\tau^k\right|^2 + \beta \left|\nabla z_\tau^k\right|^2\right) \mathrm{d}Sx + \tau \sum_{k=1}^N \int_{\Omega^k} \frac{\tau}{2} \varrho_\tau^{k-1} \circ X^k \left| D_t \left\langle \mathbf{u}_\tau^k \right. \right.$$

$$+ \tau \sum_{k=1}^N (D_1 + D_2) + \tau \sum_{k=1}^N \sum_{\sigma \in \mathcal{E}_{\mathrm{I}}^k} \int_\sigma \left(\frac{1}{2} \varrho_\tau^{k,up} |\mathbf{v}_\tau^k \cdot \mathbf{n}| + h^\varepsilon \left\langle \varrho_\tau^k \right\rangle_\sigma\right) [\![\langle \mathbf{u}_\tau^k \rangle]\!]^2 \,\mathrm{d}Sx$$

$$= \int_{\widehat{\Omega}} E_f^0 \,\mathrm{d}x + \int_\Sigma E_s^0 \,\mathrm{d}Sx + \tau \sum_{k=1}^N \int_{\Omega^k} \varrho_\tau^k \mathbf{f}_\tau^k \cdot \mathbf{u}_\tau^k \,\mathrm{d}x + \tau \sum_{k=1}^N \int_\Sigma g_\tau^k z_\tau^k \,\mathrm{d}Sx$$

$$E_f^k = \frac{1}{2} \varrho_\tau^k \left|\langle \mathbf{u}_\tau^k \rangle\right|^2 + \mathcal{H}(\varrho_\tau^k), \quad E_s^k = \frac{1}{2}(|z_\tau^k|^2 + \alpha |\Delta \eta_\tau^k|^2 + \beta |\nabla \eta_\tau^k|^2).$$

# Theorem 4 (Consistency of the fully discrete scheme)

*Let* $(\varrho_\tau, \mathbf{u}_\tau, \eta_\tau)$ *be the numerical solution of the scheme* (2) *with* $h \approx \tau$, $\gamma > 6/5$. *Then for any* $\psi \in C_0^2(0,T;\Sigma)$, $\varphi \in C_0^2(0,T;\Omega)$, *and* $\boldsymbol{\phi} \in C_0^2(0,T;\Omega^d)$, *there exists* $\theta_1, \theta_2 > 0$ *such that*

$$-\int_\Omega \varrho_\tau^0 \varphi^0 \,\mathrm{d}x - \int_I \int_{\Omega(t)} \left(\varrho_\tau \partial_t \varphi + \varrho_\tau \mathbf{u}_\tau \cdot \nabla \varphi\right) \mathrm{d}x = \mathcal{O}(\tau^{\theta_1}), \tag{3}$$

*and*

$$\begin{aligned}
-\int_\Omega \varrho_\tau^0 \mathbf{u}_\tau^0 \cdot \boldsymbol{\phi}^0 \,\mathrm{d}x - \int_I \int_{\Omega(t)} \left(\varrho_\tau \mathbf{u}_\tau \cdot \partial_t \boldsymbol{\phi} + \varrho_\tau \mathbf{u}_\tau \otimes \mathbf{u}_\tau : \nabla \boldsymbol{\phi}\right) \mathrm{d}x \\
+ \int_I \int_{\Omega(t)} \mathbf{S}(\nabla \mathbf{u}_\tau^k) : \nabla \boldsymbol{\phi} \,\mathrm{d}x - \int_I \int_{\Omega(t)} p(\varrho_\tau) \mathrm{div} \boldsymbol{\phi} \,\mathrm{d}x \\
- \int_\Sigma \partial_t \eta(0) \psi^0 \,\mathrm{d}Sx - \int_I \int_\Sigma \delta_t \eta_\tau \partial_t \psi \,\mathrm{d}Sx \,\mathrm{d}t + \int_I \int_\Sigma K'(\eta_\tau) \psi \,\mathrm{d}Sx \,\mathrm{d}t \\
= \int_I \int_\Sigma g_\tau \psi \,\mathrm{d}Sx + \int_I \int_{\Omega(t)} \mathbf{f}_\tau \cdot \boldsymbol{\phi} \,\mathrm{d}x + \mathcal{O}(\tau^{\theta_2}).
\end{aligned} \tag{4}$$

# Future work

## FSI

1. Convergence to weak/strong solution; convergence rate
2. High order scheme
3. Full Navier–Stokes–Fourier system

## Fixed domain

1. High order scheme ($\varrho > 0$): convergence, convergence rate

Any questions/comments/possible collaboration is welcome