Goal-oriented error estimates

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Introduction

Problem setting

- linear PDE: $\mathscr{L} u = f$
- FE discretization $a_h(u_h, \varphi_h) = \ell_h(\varphi_h)\ \forall \varphi_h \in V_h$
  $\iff \quad \mathbb{A}\boldsymbol{x} = \boldsymbol{b}$
- iterative solver gives approximations $\boldsymbol{x}_k \leftrightarrow u_h^k$,
- how to solve the system efficiently?
- how many iterations k is optimal?

Answer

- algebraic error $\ll$ discretization error

Goal-oriented error estimates

- we are interested in a linear target functional J,
- error $= J(u) - J(u_h^k) = J(u - u_h^k)$

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Abstract error estimate including algebraic errors

- $J(u - u_h^k) = r_h(u_h^k)(z - w_h) + r_h(u_h^k)(w_h) =: e_{\mathrm{S},k} + e_{\mathrm{A},k}$

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Higher-order reconstruction

- $z \approx z_h^+ := \mathscr{R}(z_h^k)$, $\quad z_h^k$ solves approximately $\mathbb{A}^\mathsf{T} \boldsymbol{y} = \boldsymbol{c}$

Computable error estimates

- $J(u - u_h^k) \approx r_h(u_h^k)(z_h^+ - z_h^k) + r_h(u_h^k)(z_h^k) =: \eta_{\mathrm{S},k} + \eta_{\mathrm{A},k}$

Numerical computations

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- dual problem: $\mathbb{A}^\mathsf{T}\boldsymbol{y} = \boldsymbol{c}$, approximations $\boldsymbol{y}_k \leftrightarrow z_h^k$
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- dual problem: $\mathbb{A}^\mathsf{T}\boldsymbol{y} = \boldsymbol{c}$, approximations $\boldsymbol{y}_k \leftrightarrow z_h^k$
- advantageous to solve both systems simultaneously

Goal-oriented error estimates (2)

Abstract error estimate

$$J(u - u_h^k) = r_h(u_h^k)(z - w_h) + r_h(u_h^k)(w_h)$$

Higher-order reconstruction

- $z \approx z_h^+ := \mathscr{R}(z_h^k)$, $\quad z_h^k$ solves approximately $\mathbb{A}^\mathsf{T}\boldsymbol{y} = \boldsymbol{c}$

Computable error estimates

- $J(u - u_h^k) \approx r_h(u_h^k)(z_h^+ - z_h^k) + r_h(u_h^k)(z_h^k) =: \eta_{\mathrm{S},k} + \eta_{\mathrm{A},k}$

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Stopping criterion

- $\min(\eta_{\mathrm{A},k}, \eta_{\mathrm{A},k}^*) \leq \omega \quad \& \quad \min(\zeta_{\mathrm{A},k}, \zeta_{\mathrm{A},k}^*) \leq \omega$
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Plan within project: Task 1.3 (with Filip Roskovec)

Extension to nonlinear problem

- primal problem: $a_h(u_h, \varphi_h) = 0 \quad \forall \varphi_h \in V_h$
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Solution of primal problem

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- $\hat{\eta}_\mathrm{S} = r_h(u_h^+)(z - z_h^+) \leq \left\| u - u_h^+ \right\|_E \left\| z - z_h^+ \right\|_E$
- standard (energetic) estimates of $\left\| u - u_h^+ \right\|_E$ and $\left\| z - z_h^+ \right\|_E$
- flux reconstructions (Vohralík, Papež,...)
- these estimates don't need to be accurate

Plan within project: Task 1.2 (with Ondra Bartoš)

Error estimates are not guaranteed

- $J(u - u_h^k) \approx r_h(u_h^k)(z_h^+ - z_h^k) + r_h(u_h^k)(z_h^k) = \eta_{\mathrm{S},k} + \eta_{\mathrm{A},k}$

Guaranteed error estimates – idea

- primal & dual solutions: $u_h, z_h \in V_h$
- primal & dual solutions: $u_h^+, z_h^+ \in V_h^+$ (globally $p_K + 1$)
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