Estimating the algebraic error using flux reconstruction

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February 11, 2019

Setting

Poisson problem: $-\operatorname{div}(\nabla u) = f$ in Ω, $\quad u = 0$ on $\partial\Omega$,

Weak solution $u \in V \equiv H_0^1(\Omega)$,

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in V\,,$$

flux $\boldsymbol{\sigma} \equiv -\nabla u \in \mathbf{H}(\mathrm{div}, \Omega)$, $\operatorname{div} \boldsymbol{\sigma} = f$.

FEM discrete approximation $u_h \in V_h \subset V$,

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h\,.$$

Algebraic problem, using the basis $\Phi = \{\phi_1, \ldots, \phi_N\}$ of V_h,

$$\mathbf{A}\mathrm{U} = \mathrm{F}, \qquad (\mathbf{A})_{j\ell} = (\nabla\phi_\ell, \nabla\phi_j), \quad \mathrm{F}_j = (f, \phi_j), \quad u_h = \Phi\mathrm{U}.$$

Inexact iterative **solution** $\mathrm{U}^i \approx \mathrm{U}$, $u_h^i = \Phi\mathrm{U}^i$,
residual $\mathrm{R}^i = \mathrm{F} - \mathbf{A}\mathrm{U}^i$.

Upper bound on the total error

Energy norm of the total error

$$\|\nabla(u - u_h^i)\| = \sup_{v \in V, \|\nabla v\| = 1} (\nabla(u - u_h^i), \nabla v) \,.$$

For any $\mathbf{d} \in \mathbf{H}(\mathrm{div}, \Omega)$,

$$\begin{aligned}
(\nabla(u - u_h^i), \nabla v) &= (f, v) - (\nabla u_h^i, \nabla v) \\
&= (f, v) + (\mathbf{d}, \nabla v) - (\mathbf{d}, \nabla v) - (\nabla u_h^i, \nabla v) \\
&= (f - \operatorname{div} \mathbf{d}, v) - (\nabla u_h^i + \mathbf{d}, \nabla v) \ .
\end{aligned}$$

Quasi-equilibrated flux reconstruction

We construct *representation* $r_h^i \in L^2(\Omega)$ of the algebraic residual R^i and the *approximate flux* $\mathbf{d}_h^i \in \mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ such that

$$\operatorname{div} \mathbf{d}_h^i = f_h - r_h^i \,.$$

Then

$$\begin{aligned}(\nabla(u - u_h^i), \nabla v) &= (f - \operatorname{div} \mathbf{d}_h^i, v) - (\nabla u_h^i + \mathbf{d}_h^i, \nabla v) \\ &= (f - f_h, v) + (r_h^i, v) - (\nabla u_h^i + \mathbf{d}_h^i, \nabla v)\,.\end{aligned}$$

Quasi-equilibrated flux reconstruction

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Then

$$\|\nabla(u - u_h^i)\| \leq \eta_{\mathrm{osc}} + \sup_{v \in V, \|\nabla v\| = 1} (r_h^i, v) + \|\nabla u_h^i + \mathbf{d}_h^i\| \,.$$

η_{osc}	data oscillation
$\sup_{v \in V, \|\nabla v\| = 1} (r_h^i, v)$	$\to$ algebraic error bound
$\|\nabla u_h^i + \mathbf{d}_h^i\|$	discretization error indicator

Residual representation and algebraic error

Constructing r_h^i such that

$$(r_h^i, \phi_j) = \mathrm{R}_j^i\,, \quad j = 1, \ldots, N\,,$$

where ϕ_j is a basis function of V_h and R_j^i is the associated element of R^i, we have for $v_h \in V_h$,

$$(\nabla(u_h - u_h^i), \nabla v_h) = (f, v_h) - (\nabla u_h^i, \nabla v_h) = (r_h^i, v_h)\,.$$

Recall that

$$\|\nabla(u_h - u_h^i)\| = \sup_{v_h \in V_h, \|\nabla v_h\| = 1} (\nabla(u_h - u_h^i), \nabla v_h)$$

and

$$\|\nabla(u_h - u_h^i)\| = \sup_{v_h \in V_h, \|\nabla v_h\| = 1} (r_h^i, v_h) \leq \sup_{v \in V, \|\nabla v\| = 1} (r_h^i, v)\,.$$

Algebraic residual representation

Construction of $r_h^i = \Phi \mathrm{C}^i \in V_h$ requires solution of

$$\mathbf{G}\mathrm{C}^i = \mathrm{R}^i\,, \qquad (\mathbf{G})_{j\ell} \equiv (\phi_\ell, \phi_j)\,.$$

In order to avoid solution of the system with the mass matrix $\mathbf{G}$, we construct the algebraic residual representation $r_h^i \notin V_h$, piecewise discontinuous polynomial of degree of u_h, *locally* on each element.

Algebraic residual representation

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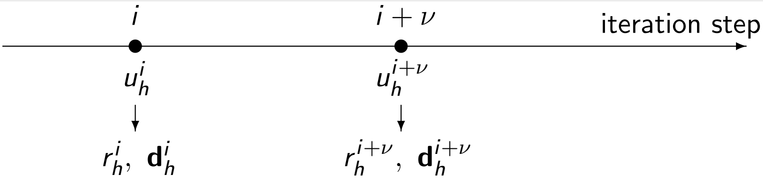
Using Cauchy–Schwarz and Friedrichs inequalities

$$(r_h^i, v) \leq \|r_h^i\| \cdot \|v\| \leq \|r_h^i\| \cdot C_F h_\Omega \|\nabla v\| \,,$$

which gives the first (worst-case) bound

$$\|\nabla(u_h - u_h^i)\| \leq C_F h_\Omega \|r_h^i\| \,.$$

Additional iteration steps



Flux reconstruction in i-th iteration, r_h^i is the representation of R^i,

$$\operatorname{div} \mathbf{d}_h^i = f_h - r_h^i \,,$$

in $(i+\nu)$-th iteration, $r_h^{i+\nu}$ is the representation of $\mathrm{R}^{i+\nu}$,

$$\operatorname{div} \mathbf{d}_h^{i+\nu} = f_h - r_h^{i+\nu} \,.$$

Then

$$r_h^i = -\operatorname{div} \mathbf{d}_h^i + \operatorname{div} \mathbf{d}_h^{i+\nu} + r_h^{i+\nu} \,.$$

Proposed upper bounds I

Then

$$(\nabla(u - u_h^i), \nabla v) = (f - f_h, v) + (r_h^i, v) - (\nabla u_h^i + \mathbf{d}_h^i, \nabla v)$$

$$= (f - f_h, v) + (\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}, \nabla v) + (r_h^{i+\nu}, v) - (\nabla u_h^i + \mathbf{d}_h^i, \nabla v),$$

and

$$(\nabla(u_h - u_h^i), \nabla v_h) = (\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}, \nabla v_h) + (r_h^{i+\nu}, v_h).$$

Upper bounds: [Papež, Strakoš, Vohralík (2016)]

$$\|\nabla(u - u_h^i)\| \leq \eta_{\mathrm{osc}} + \|\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}\| + C_F h_\Omega \|r_h^{i+\nu}\| + \|\nabla u_h^i + \mathbf{d}_h^i\|$$

$$\|\nabla(u_h - u_h^i)\| \leq \|\mathbf{d}_h^i - \mathbf{d}_h^{i+\nu}\| + C_F h_\Omega \|r_h^{i+\nu}\|$$

Proposed upper bounds II

In [Papež, Rüde, Vohralík, Wohlmuth (2019)] and [Papež, Vohralík (2019)] we propose a *hierarchical* construction of algebraic flux reconstruction $\mathbf{a}_h^i \in \mathbf{H}(\mathrm{div}, \Omega)$ such that

$$\operatorname{div} \mathbf{a}_h^i = r_h^i.$$

Then

$$(r_h^i, v) = (\operatorname{div} \mathbf{a}_h^i, v) = (\mathbf{a}_h^i, \nabla v) \leq \|\mathbf{a}_h^i\| \cdot \|\nabla v\|$$

and the bounds look like:

$$\|\nabla(u - u_h^i)\| \leq \eta_{\mathrm{osc}} + \|\mathbf{a}_h^i\| + \|\nabla u_h^i + \mathbf{d}_h^i\|$$

$$\|\nabla(u_h - u_h^i)\| \leq \|\mathbf{a}_h^i\|$$