On solving linear algebraic systems arising from Shishkin mesh discretizations

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joint work with

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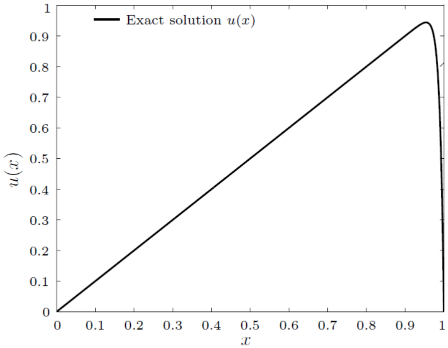
One-dimensional case

[Echeverría, Liesen, Tichý, Szyld, 2018]

Convection-diffusion boundary value problem

$$-\epsilon\, u'' + \alpha\, u' + \beta\, u = f \quad \text{in} \quad \Omega = (0,1), \quad 0 < \epsilon \ll \alpha, \quad \beta \geq 0.$$



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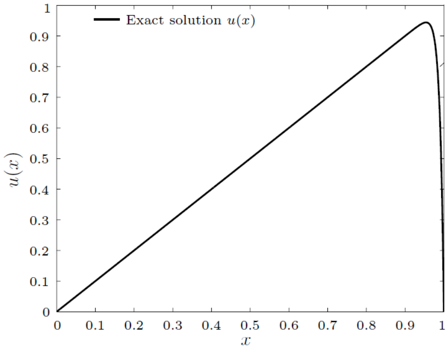
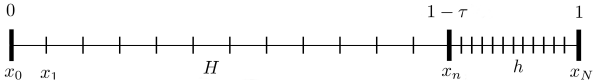


Shishkin mesh



The standard upwind difference scheme

$$\mathcal{A} = \left[\begin{array}{c|c|c} A_H & & \\ & b_H & \\ \hline c & a & b \\ \hline & c_h & \\ & & A_h \end{array}\right].$$

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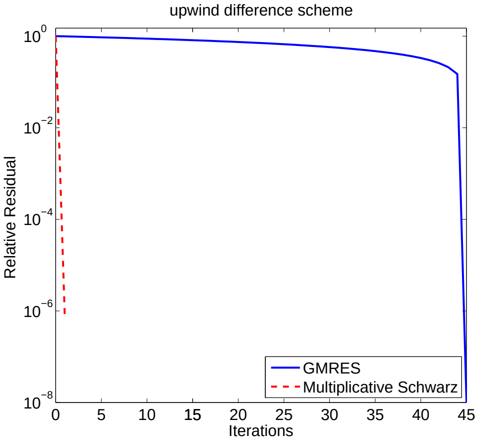
Use the **multiplicative Schwarz method** to solve $\mathcal{A}x = b$,

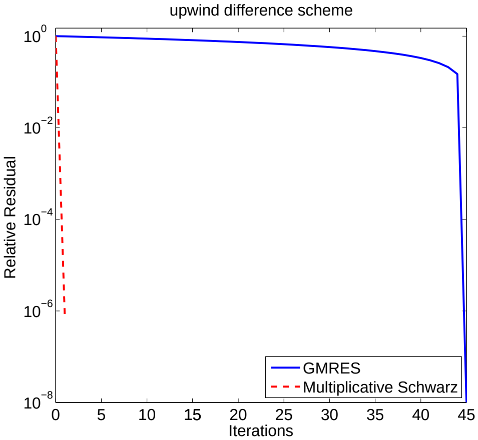
$$x^{(k)} \; = \; T\,x^{(k-1)} + v, \quad T = (I - P_2)(I - P_1),$$

$$\|x - x^{(k)}\| \; \leq \; \|T^k\|\,\|x - x_0\|.$$

Results [Echeverría, Liesen, Tichý, Szyld, 2018]



Results [Echeverría, Liesen, Tichý, Szyld, 2018]



$$\frac{\|e^{(k)}\|_\infty}{\|e^{(0)}\|_\infty} \leq \rho^k, \qquad \rho < \frac{\epsilon}{\epsilon + \frac{\alpha}{N}}$$

Schwarz method as a preconditioner

- Consistent scheme

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- **Preconditioned** system

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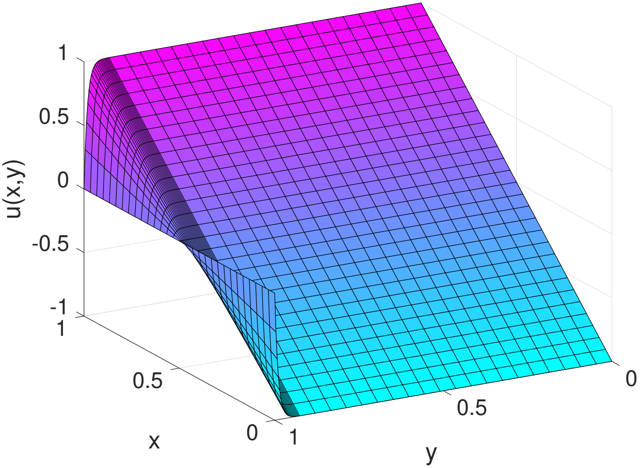
... a **motivation** for more dimensional cases.

Two-dimensional case

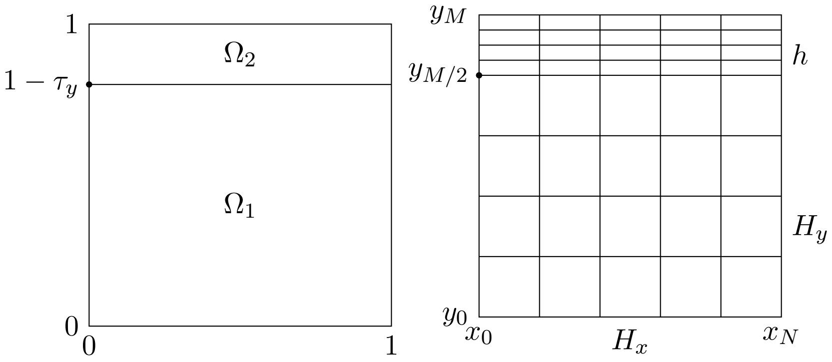
[Echeverría, Liesen, Tichý, 2020]

Problems with one boundary layer

$$-\epsilon\Delta u + \alpha u_y + \beta u = f \text{ in } \Omega = (0,1) \times (0,1)$$



Shishkin mesh



Use the standard upwind difference scheme.

A general algebraic problem

$$
\mathcal{A} = \left[\begin{array}{c|c|c}
\widehat{A}_H & B_H & \\
\hline
C & A & B \\
\hline
 & C_h & \widehat{A}_h
\end{array}\right]
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- Convergence of the **multiplicative Schwarz method**?
- Structure of T ?

$$x^{(k)} \;=\; T\,x^{(k-1)} + v,$$

$$\|x - x^{(k)}\| \;\leq\; \|T^k\|\,\|x - x_0\|.$$

Results for the two-dimension case

[Echeverría, Liesen, Tichý, 2020] - submitted

Using [Echeverría, Liesen, Nabben, 2018] we have shown

$$\frac{\|e^{(k)}\|_\infty}{\|e^{(0)}\|_\infty} \leq \rho^k \,, \qquad \rho < \frac{\epsilon}{\epsilon + \frac{\alpha}{M}} \,.$$

- Low-rank structure of T, $\text{rank}(T) = N$,
- Schwarz can be used as a preconditioner,
- Preconditioned GMRES $\rightarrow$ at most $N + 1$ iterations.

Open problems

Practical implementation of the Schwarz method

To use the iterative scheme

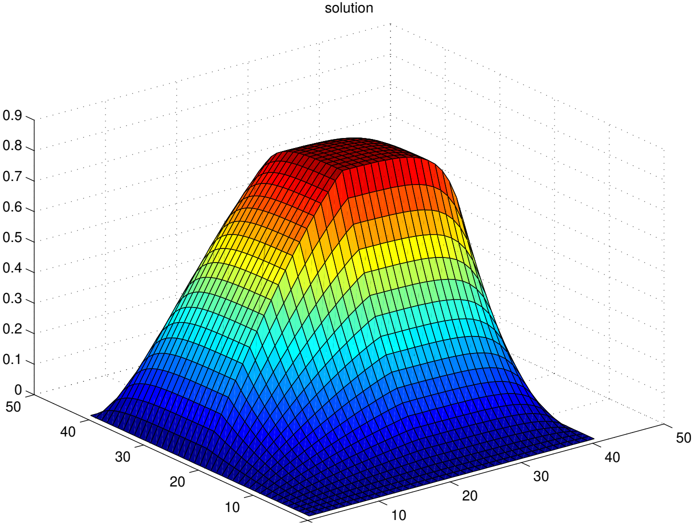
$$x^{(k)} = T\,x^{(k-1)} + v, \quad T = (I - P_2)(I - P_1),$$

we need to solve **linear systems with submatrices** of $\mathcal{A}$.

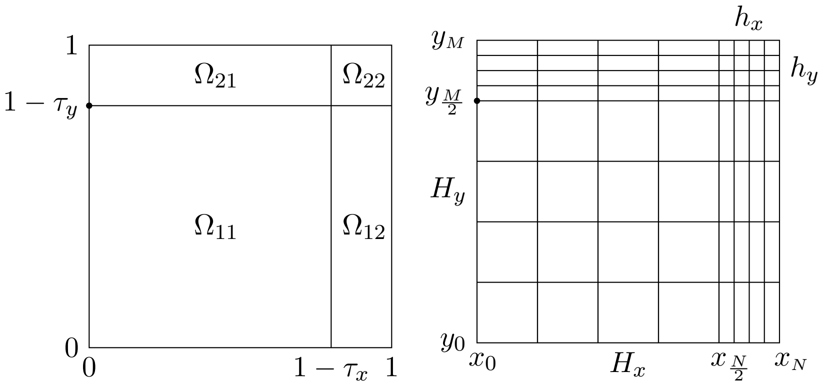
- Schur complement and fast Toeplitz solvers?
- Inexact solvers?

Two boundary layers

$$-\epsilon \Delta u + \alpha_1 u_x + \alpha_2 u_y + \beta u = f$$



Shishkin mesh



- How to define the multiplicative Schwarz method?
- Structure of $\mathcal{A}$?
- Is T low-rank?

Related papers

- C. Echeverría, J. Liesen, and P. Tichý, [Analysis of the multiplicative Schwarz method for matrices with a special block structure, submitted, 2020.]
- C. Echeverría, J. Liesen, and R. Nabben, [Block diagonal dominance of matrices revisited: bounds for the norms of inverses and eigenvalue inclusion sets, Linear Algebra Appl., 553, pp. 365–383, 2018.]
- C. Echeverría, J. Liesen, P. Tichý, and D. Szyld, [Convergence of the multiplicative Schwarz method for singularly perturbed convection-diffusion problems discretized on a Shishkin mesh, Electron. Trans. Numer. Anal., 48, pp. 40–62, 2018.]
- H-G. Roos, M. Stynes, L. Tobiska, [Robust Numerical Methods for Singularly Perturbed Differential Equations, second edition, Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2008, 604 pp.]
- M. Stynes, [Steady-state convection-diffusion problems, Acta Numerica, 14 (2005), pp. 445–508.]

Thank you for your attention!