Department of Numerical Mathematics

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Ondřej Bartoš, Vít Dolejší

Guaranteed goal-oriented error estimates

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Model problem

Continuous problem:
Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. Find $u : \overline{\Omega} \to \mathbb{R}$ satisfying

$$-\Delta u = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

and evaluate $J(u)$, e.g. $\int_\Gamma u \, \mathrm{d}S$ or $\int_{\Omega_J} u \, \mathrm{d}x$, where $\Gamma \subset \partial\Omega$, $\Omega_J \subset \Omega$.

Weak formulation: Find $u, z \in H_0^1(\Omega)$ such that

$$a_h(u, \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega),$$

$$a_h(\psi, z) = J(\psi) \quad \forall \psi \in H_0^1(\Omega).$$

Discrete SIPG solution: Find $u_h, z_h \in S_h^{\mathbf{p}}$ such that

$$a_h(u_h, \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in S_h^{\mathbf{p}},$$

$$a_h(\psi_h, z_h) = J(\psi_h) \quad \forall \psi_h \in S_h^{\mathbf{p}}.$$

Then $J(u) = a_h(u, z) = (f, z)$ and $J(u_h) = a_h(u_h, z_h) = (f, z_h)$.

$$a_h(u, v) = \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, \mathrm{d}x - \sum_{\gamma \in \mathcal{F}_h} \int_\gamma \langle \nabla u \rangle \cdot \mathbf{n}[v] + \langle \nabla v \rangle \cdot \mathbf{n}[u] \, \mathrm{d}S + \sum_{\gamma \in \mathcal{F}_h} \int_\gamma \sigma [u][v] \, \mathrm{d}S$$

Lifting operator $l_\gamma : L^2(\gamma) \to [\mathbb{P}_p(K_\gamma)]^2$ defined on neighbours of γ by

$$(l_\gamma([u_h]), \varphi)_{K_\gamma} = ([u_h], \langle\varphi\rangle \cdot \mathbf{n})_\gamma \quad \forall \varphi \in [\mathbb{P}_p(K_\gamma)]^2.$$

Global lifting operator $l : L^2(\mathcal{F}_h) \to [S_h^{\mathbf{p}}]^2$ is

$$l([u_h]) = \sum_{\gamma\in\mathcal{F}_h} l_\gamma([u_h]_\gamma).$$

It follows that $l([u_h])|_K = \sum_{\gamma\in\partial K} l_\gamma([u_h]_\gamma)|_K$, $l([u]) = l([z]) = 0$.

$$\sum_{\gamma\in\mathcal{F}_h} \int_\gamma \langle\nabla u_h\rangle \cdot \mathbf{n}[v_h]\,\mathrm{d}S = \sum_{\gamma\in\mathcal{F}_h} \int_\gamma l([v_h]) \cdot \nabla u_h\,\mathrm{d}S$$

Setting $G_h(u_h) = \nabla(u_h) - l([u_h])$, $G_h(z_h) = \nabla(z_h) - l([z_h])$, we obtain

$$a_h(u_h, v_h) = \sum_{K\in\mathcal{T}_h} \int_K G(u_h) \cdot G(v_h)\,\mathrm{d}x$$
$$- \sum_{K\in\mathcal{T}_h} \int_K l([u_h]) \cdot l([v_h])\,\mathrm{d}x + \sum_{\gamma\in\mathcal{F}_h} \int_\gamma \sigma[u_h][v_h]\,\mathrm{d}S$$

$$
\begin{aligned}
J(u-u_h) &= a_h(u-u_h, z) \\
&= a_h(u-u_h, z-z_h) \\
&= a_h(u-u_h, z_h^+ - z_h) + a_h(u-u_h, z-z_h^+) \\
&= r_h(u_h)(z_h^+ - z_h) + a_h(u-u_h, z-z_h^+) \\
&= a_h(u-u_h, z_h^+ - z_h) + a_h(u_h^+ - u_h, z-z_h^+) + a_h(u-u_h^+, z-z_h^+) \\
&= r_h(u_h)(z_h^+ - z_h) + r_h^*(z_h^+)(u_h^+ - u_h) + a_h(u-u_h^+, z-z_h^+)
\end{aligned}
$$

The last term is equal to

$$
\begin{aligned}
a_h(u-u_h^+, z-z_h^+) = \sum_{K\in\mathcal{T}_h}\int_K (\nabla(u-u_h^+) + l([u_h^+]))\cdot(\nabla(z-z_h^+) + l([z_h^+]))\,\mathrm{d}x \\
- \sum_{K\in\mathcal{T}_h}\int_K l([u_h^+])\cdot l([z_h^+])\,\mathrm{d}x + \sum_{\gamma\in\mathcal{F}_h}\int_\gamma \sigma[u_h^+][z_h^+]\,\mathrm{d}S.
\end{aligned}
$$

Setting $\mathcal{L}(u_h^+, z_h^+) = \sum_{K\in\mathcal{T}_h}\int_K l([u_h^+])\cdot l([z_h^+])\,\mathrm{d}x$,
$J_h^\sigma(u_h^+, z_h^+) = \sum_{\gamma\in\mathcal{F}_h}\int_\gamma \sigma[u_h^+][z_h^+]\,\mathrm{d}S$, we arrive at an estimate

$$
\begin{aligned}
|a_h(u-u_h^+, z-z_h^+)| \le \left\|\nabla u - G_h(u_h^+)\right\|_{L^2(\Omega)} \left\|\nabla z - G_h(z_h^+)\right\|_{L^2(\Omega)} \\
+ \left|-\mathcal{L}(u_h^+, z_h^+) + J_h^\sigma(u_h^+, z_h^+)\right|.
\end{aligned}
$$

Error estimate

Let $s_h \in H_0^1(\Omega) \cap C(\Omega)$, $\sigma_h \in H(\mathrm{div}, \Omega)$, $(\nabla \cdot \sigma_h, 1)_K = (f, 1)_K \; \forall K \in \mathcal{T}_h$. Then

$$\left\| \nabla u - G_h(u_h^+) \right\|_{L^2(\Omega)}^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K^2$$

with

$$\eta_K^2 = \left(\left\| G_h(u_h^+) + \sigma_h \right\|_K + \frac{h_K}{\pi} \left\| f - \nabla \cdot \sigma_h \right\|_K + (\eta_{\Gamma_N, K}) \right)^2 + \left(\left\| G_h(u_h^+) - \nabla s_h \right\|_K + (\eta_{\Gamma_D, K}) \right)^2.$$

Both $s_h \in S_h^{\mathbf{p}+1} \cap H_0^1(\Omega)$ and $\sigma_h \in RTN_{\mathbf{p}+1}$ can be reconstructed by solving local mixed problems on patches around vertices.
$\sigma_h^a \in RTN_{\mathbf{p}+1}^a$ s.t. $\nabla \cdot \sigma_h^a = \Pi_{\nabla \cdot RTN_{\mathbf{p}+1}^a} (\psi_a f - \nabla \psi_a \cdot G(u_h^+))$ minimizes $\left\| \psi_a G(u_h^+) + \sigma_h^a \right\|_{\omega_a}$.
s_h^a minimizes $\left\| \nabla (\psi_a u_h^+ - s_h^a) \right\|_{\omega_a}$ over a suitable space.
We then have $\sigma_h = \sum_{a \in \mathcal{V}_h} \sigma_h^a$ and $s_h = \sum_{a \in \mathcal{V}_h} s_h^a$.