First Progress Meeting - Traffic flow

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GACR Project 20-01074S
Task 2.3: Coupled nonlinear hyperbolic conservation laws
26^{th} November 2020

Traffic flow

- Described by traffic density $\rho(x, t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}$

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$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x} Q_e\left(\rho(x,t)\right) = 0$$

Junctions

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- Rankine-Hugoniot condition

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- Traffic distribution

$$Q_e(\rho_j(a_j^+, t)) = \sum_{i=1}^{n} \alpha_{j,i} Q_e(\rho_i(b_i^-, t)), \quad \forall j = n+1, \dots, n+m$$

Discontinuous Galerkin method

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$$\int_\Omega (\rho_h)_t \varphi \,\mathrm{d}x + \sum_i H\left((\rho_h)_i^{(L)}, (\rho_h)_i^{(R)}\right) [\varphi]_i - \sum_K \int_K Q_e(\rho_h)\varphi' \,\mathrm{d}x = 0$$

Numerical flux

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- Godunov numerical flux:

$$H\left(\rho_h^{(L)}, \rho_h^{(R)}\right) = \begin{cases} \min_{\rho_h^{(L)} \leq \rho \leq \rho_h^{(R)}} Q_e(\rho), & \text{if } \rho_h^{(L)} < \rho_h^{(R)} \\ \max_{\rho_h^{(R)} \leq \rho \leq \rho_h^{(L)}} Q_e(\rho), & \text{if } \rho_h^{(L)} \geq \rho_h^{(R)} \end{cases}$$

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$$H_i(t) := \sum_{j=n+1}^{n+m} \alpha_{j,i} H\left(\rho_{hi}^{(L)}(b_i,t), \rho_{hj}^{(R)}(a_j,t)\right)$$

$$H_j(t) := \sum_{i=1}^{n} \alpha_{j,i} H\left(\rho_{hi}^{(L)}(b_i,t), \rho_{hj}^{(R)}(a_j,t)\right)$$

Lemma (Discrete Rankine–Hugoniot condition)

The numerical fluxes H_i and H_j satisfy the discrete version of the Rankine–Hugoniot condition:

$$\sum_{i=1}^{n} H_i(t) = \sum_{j=n+1}^{n+m} H_j(t).$$

Definition (DG formulation on a simple network)

Incoming roads: For all $i = 1\dots, n$ and all $\varphi_i \in S_{h_i}$

$$\int_{a_i}^{b_i} (\rho_{hi})_t \varphi_i \,\mathrm{d}x - \sum_{K\in\mathcal{T}_{hi}} \int_K Q_e(\rho_{hi})\varphi_i' \,\mathrm{d}x + \sum_{x\in\mathcal{F}_{hi}^I} H(\rho_{hi}^{(L)}, \rho_{hi}^{(R)})\,[\varphi_i]$$
$$+H_i\varphi_i^{(L)}(b_i) - H(\rho_{Di}, \rho_{hi}^{(R)}(a_i))\varphi_i^{(R)}(a_i) = 0.$$

Outgoing roads: For all $j = n+1, \dots, n+m$ and all $\varphi_j \in S_{h_j}$

$$\int_{a_j}^{b_j} (\rho_{hj})_t \varphi_j \,\mathrm{d}x - \sum_{K\in\mathcal{T}_{hj}} \int_K Q_e(\rho_{hj})\varphi_j' \,\mathrm{d}x + \sum_{x\in\mathcal{F}_{hj}^I} H(\rho_{hj}^{(L)}, \rho_{hj}^{(R)})\,[\varphi_j]$$
$$+Q_e(\rho_{hj}^{(L)}(b_j))\varphi_j^{(L)}(b_j) - H_j\varphi_j^{(R)}(a_j) = 0.$$

Theorem (Conservation property of the DG scheme)

The DG scheme from Definition conserves the total number of vehicles in the network in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{n+m}\int_{a_k}^{b_k}\rho_{hk}\,\mathrm{d}x=\sum_{i=1}^{n}H\big(\rho_{Di},\rho_{hi}^{(R)}(a_i)\big)-\sum_{j=n+1}^{n+m}Q_e\big(\rho_{hj}^{(L)}(b_j)\big).$$

Theorem (Traffic distribution error)

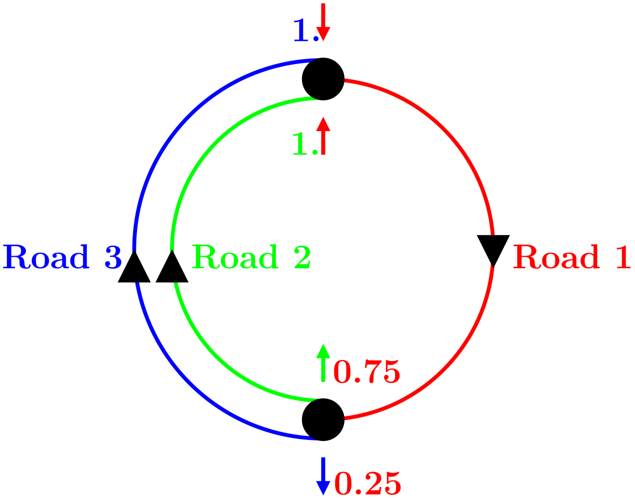
The numerical fluxes H_i and H_j satisfy

$$H_j(t) = \sum_{i=1}^{n} \alpha_{j,i} H_i(t) + E_j(t)$$

for all $j = n+1, \ldots, n+m$, where the error term is

$$E_j(t) = \sum_{i=1}^{n} \sum_{\substack{l=n+1 \\ l \neq j}}^{n+m} \alpha_{j,i}\alpha_{l,i}\big(H_{i,j}(t) - H_{i,l}(t)\big).$$

Network – Numerical flux vs. Maximum possible flux



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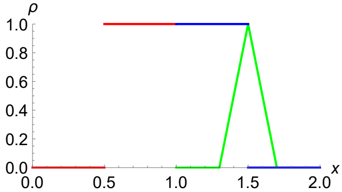

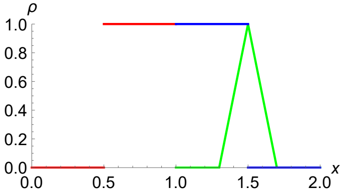

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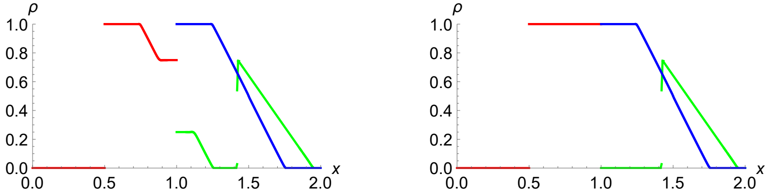

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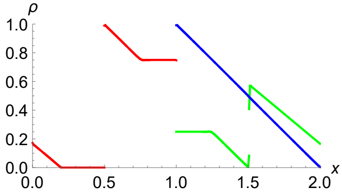

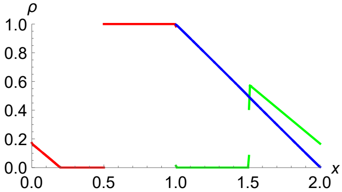

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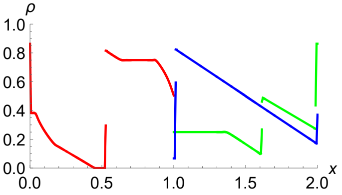

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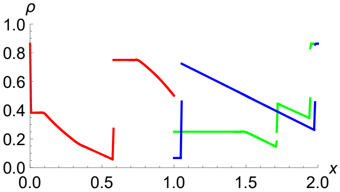

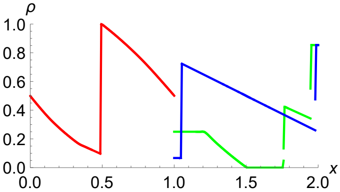

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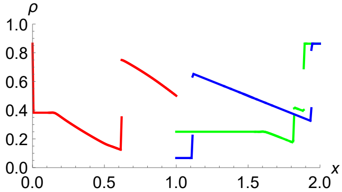

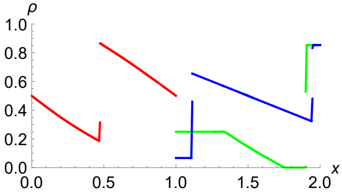

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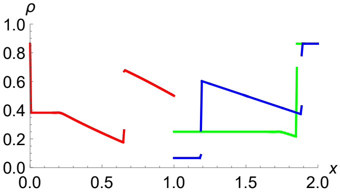

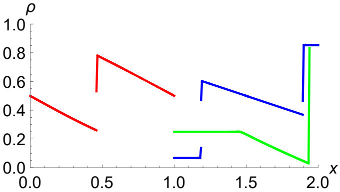

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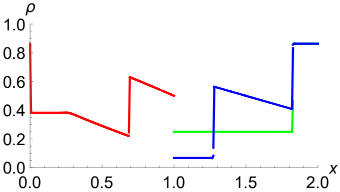

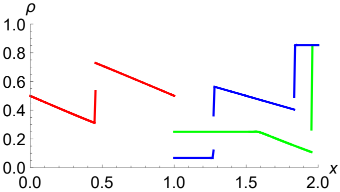

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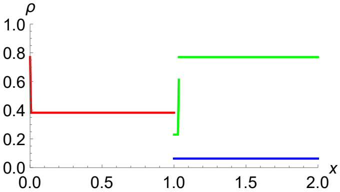

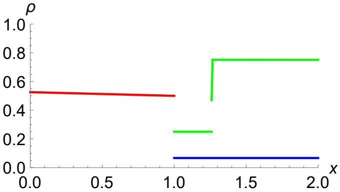

Godunov's distributive numerical flux

$$H_i(t) := \sum_{j=n+1}^{n+m} H_{i,j}\big(\rho_{hi}^{(L)}(b_i,t), \rho_{hj}^{(R)}(a_j,t), \alpha_{j,i}\big)$$

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$$H_{i,j}\left(\rho_i^{(L)}, \rho_j^{(R)}, \alpha_{j,i}\right) = \min\{\alpha_{j,i} f_{in}\left(\rho_i^{(L)}\right), f_{out}\left(\rho_j^{(R)}\right)\}$$

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$$f_{in}\left(\rho_i^{(L)}\right) = \begin{cases} Q_e\left(\rho_i^{(L)}\right), & \text{if } \rho_i^{(L)} < \rho_* \\ Q_e\left(\rho_*\right), & \text{if } \rho_i^{(L)} \geq \rho_* \end{cases} \quad f_{out}\left(\rho_j^{(R)}\right) = \begin{cases} Q_e\left(\rho_*\right), & \text{if } \rho_j^{(R)} \leq \rho_* \\ Q_e\left(\rho_j^{(R)}\right), & \text{if } \rho_j^{(R)} > \rho_* \end{cases}$$

Right of way

Example: 2 incoming, 2 outgoing roads. Road 1 is main road, Road 2 is side road. Road 3 and Road 4 are outgoing roads.

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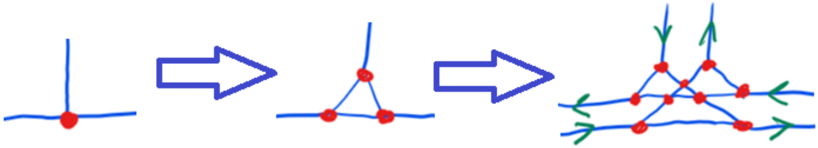
$$H_{1,3} = \min\{\alpha_{3,1} f_{in}(\rho_1^{(L)}), f_{out}(\rho_3^{(R)})\}$$
$$H_{1,4} = \min\{\alpha_{4,1} f_{in}(\rho_1^{(L)}), f_{out}(\rho_4^{(R)})\}$$
$$H_{2,3} = \min\{\alpha_{3,2} f_{in}(\rho_2^{(L)}), \max\{0, f_{out}(\rho_3^{(R)}) - \alpha_{3,1} f_{in}(\rho_1^{(L)})\}\}$$
$$H_{2,4} = \min\{\alpha_{4,2} f_{in}(\rho_2^{(L)}), \max\{0, f_{out}(\rho_4^{(R)}) - \alpha_{4,1} f_{in}(\rho_1^{(L)})\}\}$$

Pseudo 2D junction



Publications

L. VACEK, V. KUČERA: *Discontinuous Galerkin method for macroscopic traffic flow models on networks,* Communications on Applied Mathematics and Computation (submitted)

Thank you