Numerical solution of partial differential equations

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Course “Discontinuous Galerkin Method”
https://www2.karlin.mff.cuni.cz/~dolejsi/Vyuka/DGM.html

Partial differential equations

Why partial differential equations?

- many processes can be described (approximately) by PDEs
  - fluid dynamics, hydrology, heat and mass transfer, medicine, environmental protection, financial mathematics, etc.
  - these PDEs represent a mathematical description of physical, chemical, biological, etc. rules and/or laws
- some simplification usually necessary $\Longrightarrow$ model error
- these PDEs are usually too complicated for an exact solution

Numerical solution of PDEs

- we solve PDEs approximately (numerically)
- we define new simplified (finite dimensional, solvable) problem $\Longrightarrow$ discretization error

Exact and approximate problems

Abstract problem described by PDEs

- let V be a functional space, we seek $u \in V$ such that
  (EP) $$\mathcal{L}u = f$$
- $\mathcal{L}$ is a differential operator, f is a right-hand side,
- let solution of (EP) exists and is unique

Abstract numerical method

- let V_h be a space, $\dim(V_h) < \infty$, $V_h \subset V$ or $V_h \not\subset V$,
- we seek $u_h \in V_h$ such that
  (AP) $$\mathcal{L}_h u_h = f_h,$$
- $\mathcal{L}_h$ is a discrete operator, f_h is an approximation of f.
- problem (AP) has to be quickly solvable

Goals of the numerical solution of PDEs

Numerical analysis

- existence and uniqueness of u_h
- stability $\|u_h\| < \infty$
- convergence: $u_h \to u$ if dof $= \dim(V_h) \to \infty$
- estimate $\|u - u_h\|$ in terms of dof (a priori estimate)
- estimate $\|u - u_h\|$ based on u_h (a posteriori estimate)
- robustness: validity of previous items for large range of data

Numerical realization

- algorithm for fast evaluation of u_h (efficiency)
- stability of the method in the finite precision arithmetic
- adaptive strategies = adaptive changes of V_h

Numerical method

Numerical method in practise

- finite sequence of mathematical operations
- output is the approximate solution u_h

Construction of a numerical method for the given PDE

- discretization (space, time)
- setting of arising algebraic systems (numerical quadratures)
- (iterative) solution of nonlinear algebraic systems
- solution of linear algebraic systems

Type of discretizations

finite difference method, finite element method, finite volume method, spectral method, wavelets method, etc.

Choice of the numerical method

Which numerical method is the best one?
Depends on many aspects of the PDE considered

- physical background of the PDE
- expected regularity of the unknown exact solution
- presence of local phenomena
- outputs of interest
  - usual condition $\|u - u_h\| \leq TOL$ is not always practical
  - goal is the quantity of interest $J(u_h)$,
    $\Rightarrow$ error: $|J(u) - J(u_h)| \leq TOL$

Example

- conservation laws should be discretized by a conservative numerical method

Two basic physical processes

Diffusion

- $\frac{\partial u}{\partial t} - \nabla \cdot (a(u) \nabla u) = g$
- parabolic (elliptic) equation
- quantity is spread in all directions
- influence is decreasing for increasing distance of the source

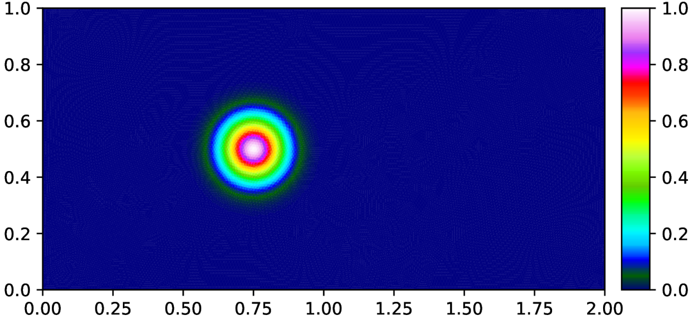
Convection

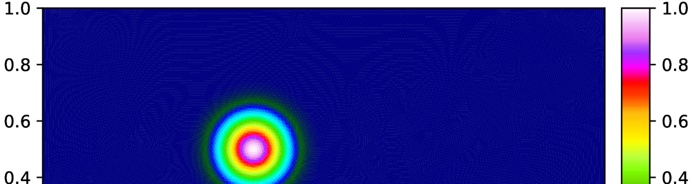
- $\frac{\partial u}{\partial t} - \nabla \cdot \vec{f}(u) = g$
- hyperbolic equation
- quantity is spread only in the direction of convection $\vec{f}(u)$
- influence is (almost) independent w.r.t. the distance of the source

Examples of physical features (1)

Only diffusion

$$\frac{\partial u}{\partial t} - \varepsilon \Delta u = 0$$

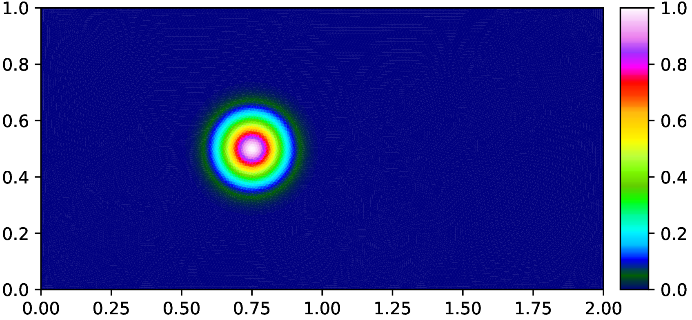


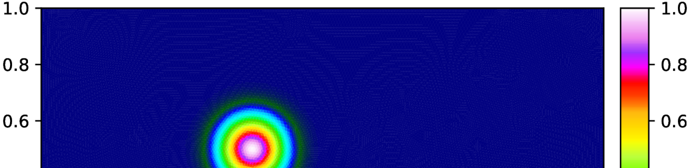


Examples of physical features (2)

Only convection:

$$\frac{\partial u}{\partial t} + \nabla \cdot (\vec{f}(u)) = 0, \quad \vec{f}(u) = (1,0)^T$$

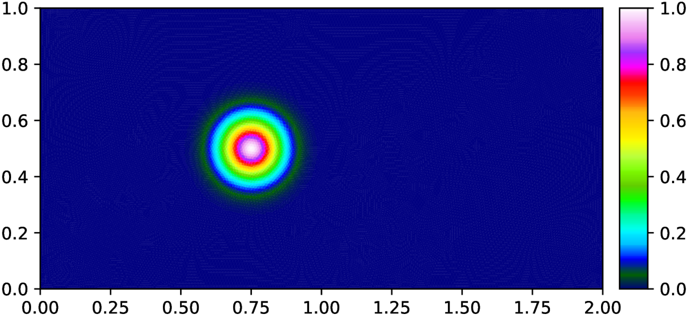


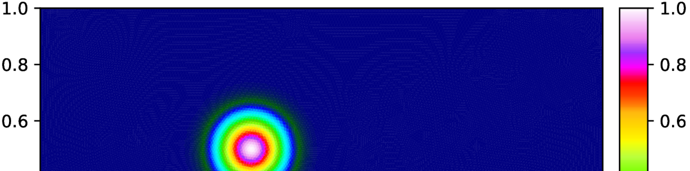


Examples of physical features (3)

Convection + small diffusion

$$\frac{\partial u}{\partial t} + \nabla \cdot (\vec{f}(u)) - \epsilon \Delta u = 0, \quad \vec{f}(u) = (1, 0)^T$$





Elliptic and parabolic PDE vs. hyperbolic PDE

Cauchy problem

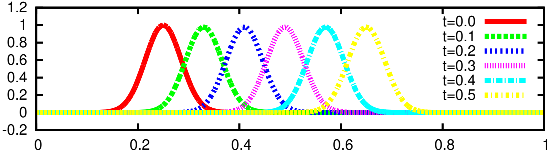
$u(x,t) : \mathbb{R} \times (0,T) \to \mathbb{R}$:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = \exp[(x - 1/4)^2]$$

$$\varepsilon = 0 \quad \Longrightarrow \quad u(x,t) = \exp[(x - 1/4 - t)^2]$$

$\varepsilon > 0 \quad \Longrightarrow \quad$ solution is smeared



exact solutions for $\varepsilon = 0$, $\varepsilon = 10^{-4}$, $\varepsilon = 10^{-3}$, $\varepsilon = 10^{-2}$

Importance of the character of PDE

Why is important to know the previous properties?

- numerical solution is a kind of approximation
- many sources of inaccuracies:
  - discretization errors (finite dimensional approximation)
  - iterative errors (approximate solution of algebraic systems)
  - rounding errors (finite precision arithmetic)
- these inaccuracies are propagated by PDEs

Linear convection problem (no diffusion)

- exact solution: a simple propagation of the initial solution
- numerical solution: initial solution is propagated but smeared
- numerical solution corresponds to convection+diffusion
- this effect is called numerical diffusion

Possible pitfalls

Effect of numerical diffusion

- zero diffusion does not exist in reality
- if numerical diffusion larger than physical one
  $\Longrightarrow$ numerical solution can be completely wrong
- e.g., numerical solution is steady whereas reality is unsteady

Effect of "finite h"

- we can prove that the proposed method is convergent
- approximate solution contains unphysical effects, e.g., spurious oscillations, negative temperature, etc.
- analysis is wrong?
- No, it converges for $h \to 0$, the solution is bad for finite h

1D convection-diffusion equation

$$u : (0,1) \to \mathbb{R} : \qquad -\varepsilon u'' + u' = f, \quad u(0) = u(1) = 0, \ \varepsilon > 0.$$

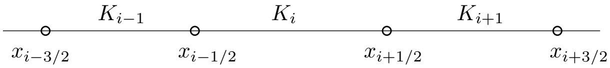
solution has a steep gradient near $x = 1$ (boundary layer)

Weak formulation

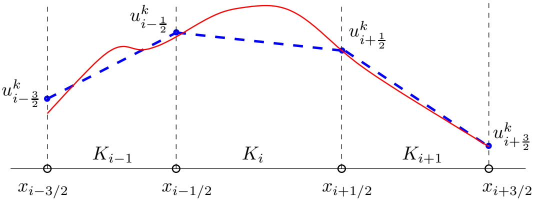
$u \in H_0^1((0,1))$: $\int_0^1 (\varepsilon u' v' + u' v)\,\mathrm{d}x = \int_0^1 f\, v\,\mathrm{d}x \quad \forall v \in H_0^1((0,1))$

Partition of domain

$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \dots x_{N+\frac{1}{2}} = 1, \quad K_i := [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}],\ i = 1, \dots, N$



Finite element method



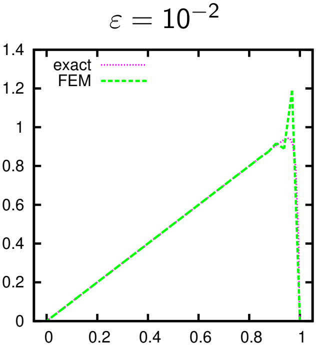
FEM solution

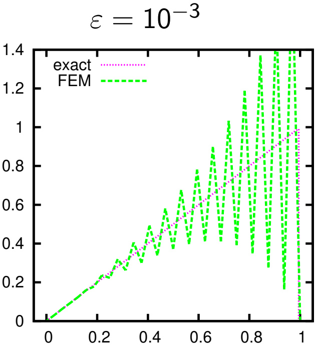
- $V_h = \{v_h \in C_0([0,1]);\ v_h|_{K_i} = P^1(K_i),\ i = 1,\dots,N\}$
- $u_h \in V_h$:

$$\int_0^1 (\varepsilon u_h' v_h' + u_h' v_h)\,\mathrm{d}x = \int_0^1 f\, v_h\,\mathrm{d}x \quad \forall v_h \in V_h$$

- reasonable discretization of diffusion $\Rightarrow$ we prove convergence
- discretization of convective term "does not respect physics"

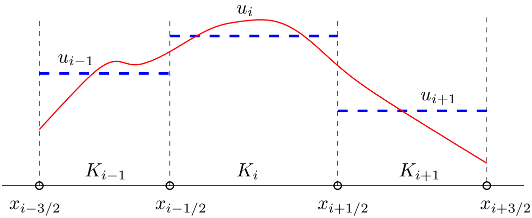
Finite element method





- Solution suffers from spurious oscillations for small ε
- A stabilization is a possible remedy

Finite volume method



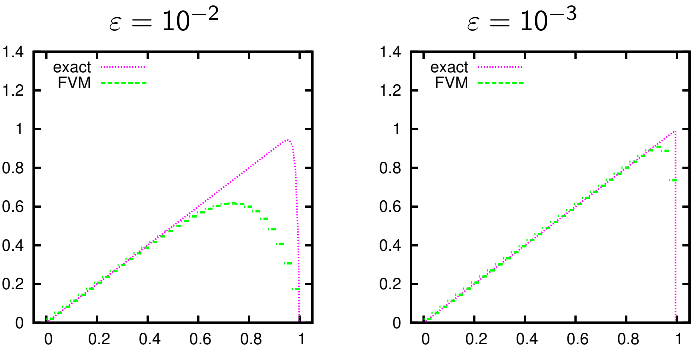
Piecewise constant approximation

- $V_h = \{v_h \in L^2([0,1]);\ v_h|_{K_i} = P^0(K_i),\ i = 1, \dots, N\}$
- we integrate $-\varepsilon u'' + au' = 1$ over K_i and use Gauss theorem

$$-\varepsilon[u'(\cdot)]_{x_{i-1/2}}^{x_{i+1/2}} + a[u(\cdot)]_{x_{i-1/2}}^{x_{i+1/2}} = |K_i|$$

- $u|_{x_{i+\frac{1}{2}}} = ??$ upwinding: $a > 0 \ \Rightarrow u|_{x_{i+\frac{1}{2}}} := u_i$

Finite volume method



- Oscillations free approximation
- Low accuracy for larger ε
- A higher order reconstruction is a possible remedy

Comparison of FEM and FVM

comparison of FEM and FVM for time-independent convective problem

Finite element method

- continuous approximation
- high order of accuracy
- many theoretical results
- fine for diffusive problems

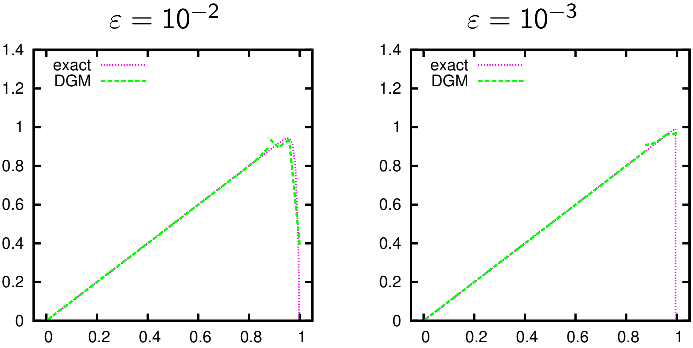
Finite volume method

- discontinuous approximation
- low order of accuracy
- lack of theory
- fine for convective problems

Discontinuous Galerkin method

- piecewise polynomial discontinuous approximation
- theoretical justification
- higher freedom (adaptation, parallelization, etc.)

Discontinuous Galerkin method



(P_4-approximation, same number of DoF)

- not ideal but works very well for both ε
- additional techniques (remedies) are possible

Overview of DGM (1)

Basic properties – positive

- efficient method for the numerical solution of various PDEs
- piecewise polynomial BUT discontinuous approximation
- suitable for very large range of problems
  - elliptic, parabolic, hyperbolic
  - linear, nonlinear, degenerate
- space-time DGMs are available
- flexibility in the mesh design
  - non-matching and non-uniform grids
  - anisotropic grids
  - varying polynomial approximation degrees
- (nice) block structure of arising algebraic systems
- easy paralelization

Overview of DGM (2)

Basic properties – theoretical

- formulation of the method is more complicated
- numerical analysis of the method is more complicated

Basic properties – practical

- more degrees of freedom $\Rightarrow$ larger algebraic systems
  - it can be compensated by mesh adaptation
- less of available "standard" libraries,
  - multi-level preconditioners
  - domain decomposition preconditioners

A lot of work to do!

Plan of the course

Outline

- Abstract error analysis
- DGM for the Laplace problem: complete error analysis
- numerical approximation based on upwinding
- DGM for the nonlinear convection-diffusion equation
- DGM for time dependent problems
- DGM for compressible flow problems and other applications

Organization issues

- standard lectures (lecture notes are available)
- 3 quizes during the semestr
- oral exam