TEST DGM #1

Solve the following tasks and send the solution to me by e-mail (`dolejsi@karlin.mff.cuni.cz`). It is sufficient to write the solution by hand on the paper, scan or make a snap by hand-phone. The references correspond to those in Lecture Notes
`http://msekce.karlin.mff.cuni.cz/~dolejsi/Vyuka/LectureNotes_DGM.pdf`

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1. Corollary 1.33 gives the boundedness of A_h by (1.122) in the form
$$|A_h(u,v)| \leq 2\|u\|_{1,\sigma}\|v\|_{1,\sigma} \quad \forall\, u,v \in H^2(\Omega,\mathcal{T}_h). \tag{1}$$
Why this inequality does not hold for $u,v \in H^1(\Omega,\mathcal{T}_h)$?

2. Lemma 1.39 gives the relation
$$A_h^{\mathrm{s},\sigma}(v_h,v_h) \geq \frac{1}{2}|\!|\!|v_h|\!|\!|^2 \quad \forall\, v_h \in S_{hp}\ \forall\, h \in (0,\bar{h}). \tag{2}$$
Why this estimate is not valid for $v_h \in H^2(\Omega,\mathcal{T}_h)$, namely which step in the proof is violated for $v_h \in H^2(\Omega,\mathcal{T}_h)$?

3. Let us assume that we have found the value C_W such that the form $A_h^{\mathrm{i},\sigma}(u,v) = a_h^{\mathrm{i}}(u,v) + J_h^{\sigma}(u,v)$ corresponding to the IIPG variant is coercive for the choice of the penalty weight σ according to (1.104), i.e.
$$\sigma|_{\Gamma} = \sigma_{\Gamma} = \frac{C_W}{h_{\Gamma}}, \quad \Gamma \in \mathcal{F}_h^{ID}. \tag{3}$$
   (a) Does this value C_W give the coercivity of the form $A_h^{\mathrm{s},\sigma}(u,v) = a_h^{\mathrm{s}}(u,v) + J_h^{\sigma}(u,v)$ also for the SIPG variant?
   (b) If not, which value of C_W has to be chosen for the SIPG variant?
   (c) What does happen if we solve the Laplace problem by DGM where the constant C_W is too small (i.e., the coercivity is not guaranteed)?

4. Why the optimal L^2-error estimate (1.162)
$$\|e_h\|_{L^2(\Omega)} \leq C_3 h^{\mu}|u|_{H^{\mu}(\Omega)}, \tag{4}$$
is not true for the NIPG and IIPG techniques? Namely, which step in the proof of Theorem 1.49 is not valid for the NIPG and IIPG variants?

5. In Section 1.7.2, the $L^2(\Omega)$-error estimate are proved under assumption that the solution of the dual problem (1.155) is in $H^2(\Omega)$. What does happend, if we have only $\psi \in H^1(\Omega)$?

6. Let us consider the DGM using the piecewise cubic polynomial approximations. Which order of convergence have the SIPG, NIPG and IIPG variants in the $|\!|\!|\cdot|\!|\!|$-norm and the L^2-norm if the exact solution u is
   (a) $u \in C^{\infty}(\overline{\Omega})$,
   (b) $u \in H^2(\Omega)$,
   (c) $u \in H^5(\Omega)$?

If you have any question, do not hesitate to contact me!