Numerical solution of IVP (ODE)

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Quiz # 3

Question $\#1$

Let us consider the **initial value problem**

- $y'(x) = f(x, y(x)), \qquad y(a) = \eta.$

Let y_k, $k = 0, 1, \ldots$ be the approximation of y at x_k, i.e. $y_k \approx y(x_k)$.

What is the global error G_k at step k?

$G_k = ?$

What is the local error L_k at step k?

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or error arrising in the one time step.

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Let y_k, $k = 0, 1, \ldots$ be the approximation of y at x_k, i.e. $y_k \approx y(x_k)$.

What is the relation between the global and local errors G_k and L_k, respectively?

(A) $G_k = \sum_{\ell=0}^{k} L_\ell$

(B) $G_k \leq \sum_{\ell=0}^{k} L_\ell$

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(D) none of the above ones

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Question #3

Let us consider the Euler method for the solution of ODEs:

- $y_{k+1} = y_k + h_k f(x_k, y_k)$,
- $y_0 = \eta$.

Let $\mathbb{J}_f \in \mathbb{R}^{m \times m}$ be the Jacobian matrix of f w..r.t. y.

The Euler method is stable if

(A) all eigenvalues of $\mathbb{J}_f$ have negative real parts.

(B) the spectral radius of the matrix $(\mathbb{I} + h_k \mathbb{J}_f)$ is < 1.

(C) h_k is smaller than the minimal eigenvalue of $\mathbb{J}_f$.

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Question #4

Let us consider the following the ODE: find $y : (a, b) \to \mathbb{R}^3$ such that

- $y'(x) = f(x, y(x))$.

Let the eigenvalues of the Jacobian $\{\frac{\partial f_i}{\partial y_j}\}_{i,j=1}^3$ are
$\lambda_1 = -1$, $\lambda_2 = -2000 + 2i$ and $\lambda_3 = -2000 - 2i$.

Find the maximal size of h_k, which guarantees the stability of the Euler method $y_{k+1} = y_k + h_k f(x_k, y_k)$?
(answer is a real positive number)

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Stability condition:
$|1 + h_k\lambda| < 1 \;\Leftrightarrow\; h_k \leq -2/\lambda$

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$$|1 + h_k \lambda| < 1 \;\Leftrightarrow\; h_k \leq -2/\lambda \quad \Rightarrow h_k \leq -2/\lambda \approx 0.001$$