Rungeovy-Kuttovy methody 4. řádu

\[ y_{k+1} = y_k + h \left( w_1 q_1 + w_2 q_2 + w_3 q_3 + w_4 q_4 \right) \]

- **Standardní formule:**
  \[ w_1 = w_4 = 1/6, \ w_2 = w_3 = 1/3, \]
  \[ q_1 = f(t, y), \]
  \[ q_2 = f(t + h/2, y + h/2 q_1), \]
  \[ q_3 = f(t + h/2, y + h/2 q_2), \]
  \[ q_4 = f(t + h, y + h q_3). \]

- **Tříosminová formule:**
  \[ w_1 = w_4 = 1/8, \ w_2 = w_3 = 3/8, \]
  \[ q_1 = f(t, y), \]
  \[ q_2 = f(t + h/3, y + h/3 q_1), \]
  \[ q_3 = f(t + 2h/3, y + h(-1/3 q_1 + q_2)),\]
  \[ q_4 = f(t + h, y + h(q_1 - q_2 + q_3)). \]
Asymptotický odhad chyby $e_n \leq Nh^p E_L(x_n - a)$

Eulerova metoda: $y_{n+1} = y - n + hf(x_n, y_n)$, $h = 1/2^{-6} = 0.015625$

<table>
<thead>
<tr>
<th>úloha</th>
<th>$y' = y$</th>
<th>$y(0) = 1$</th>
<th>$y' = -y$</th>
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chyba $\approx 4\%$ 40x větší chyba $\approx 4\%$ 10 000x větší
Asymptotický odhad chyby \( e_n \leq Nh^pE_L(x_n - a) \)

Eulerova metoda: \( y_{n+1} = y - n + hf(x_n, y_n) \), 
\( h = 1/2^{-6} = 0.015625 \)

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<th>odhad ( e_n )</th>
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chyba \( \approx 4\% \) 40x větší  
chyba \( \approx 4\% \) 10 000x větší
Úloha $y' = 1 - y^2$, $y(0) = 5$,
Runge-Kutta 4. řádu (standartní formule), $h = 0.04$

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Úloha $y' = 1 - y^2$, $y(0) = 5$, Runge-Kutta 4. řádu (standartní formule), $h = 0.04$

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Úloha \( y' = 1 - y, \ y(0) = 2, \) přesné řešení \( y = 1 + \exp(-x), \) výpočet metodou 2. řádu v jednoduché přesnosti

<table>
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<th>( y_n )</th>
<th>diskretizační ch.</th>
<th>zaokrouhlovací ch.</th>
<th>celková ch.</th>
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Vít Dolejší ODR ZNM 5 / 5
Vliv zaokrouhlovací chyby

Úloha $y' = 1 - y$, $y(0) = 2$, přesné řešení $y = 1 + \exp(-x)$, výpočet metodou 2. řádu v jednoduché přesnosti

<table>
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<tr>
<th>$x$</th>
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