# Numerical quadratures

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# Newton - Cotes quadratures: weights and nodes

| NC | $w_j$ | $x_j$ |
|---|---|---|
| $n=1$ | $\frac{1}{2}\quad\frac{1}{2}$ | $0\quad 1$ |
| $n=2$ | $\frac{1}{6}\quad\frac{4}{6}\quad\frac{1}{6}$ | $0\quad\frac{1}{2}\quad 1$ |
| $n=3$ | $\frac{1}{8}\quad\frac{3}{8}\quad\frac{3}{8}\quad\frac{1}{8}$ | $0\quad\frac{1}{3}\quad\frac{2}{3}\quad 1$ |
| $n=4$ | $\frac{7}{90}\quad\frac{32}{90}\quad\frac{12}{90}\quad\frac{32}{90}\quad\frac{7}{90}$ | $0\quad\frac{1}{4}\quad\frac{2}{4}\quad\frac{3}{4}\quad 1$ |
| $n=5$ | $\frac{19}{288}\quad\frac{75}{288}\quad\frac{50}{288}\quad\frac{50}{288}\quad\frac{75}{288}\quad\frac{19}{288}$ | $0\quad\frac{1}{5}\quad\frac{2}{5}\quad\frac{3}{5}\quad\frac{4}{5}\quad 1$ |
| $n=6$ | $\frac{41}{840}\quad\frac{216}{840}\quad\frac{27}{840}\quad\frac{272}{840}\quad\frac{27}{840}\quad\frac{216}{840}\quad\frac{41}{840}$ | $0\quad\frac{1}{6}\quad\frac{2}{6}\quad\frac{3}{6}\quad\frac{4}{6}\quad\frac{5}{6}\quad 1$ |

# Gauss quadratures: weights and nodes

| $G_k$ | $j$ | $w_j$ | $x_j$ |
|---|---|---|---|
| $n = 0$ | 0 | 1.00000000000000 | 0.50000000000000 |
| $n = 1$ | 0 | 0.50000000000000 | 0.21132486540519 |
| | 1 | 0.50000000000000 | 0.78867513459481 |
| $n = 2$ | 0 | 0.27777777777778 | 0.11270166537926 |
| | 1 | 0.44444444444444 | 0.50000000000000 |
| | 2 | 0.27777777777778 | 0.88729833462074 |
| $n = 3$ | 0 | 0.17392742256873 | 0.06943184420297 |
| | 1 | 0.32607257743127 | 0.33000947820757 |
| | 2 | 0.32607257743127 | 0.66999052179243 |
| | 3 | 0.17392742256873 | 0.93056815579703 |
| $n = 4$ | 0 | 0.11846344252809 | 0.04691007703067 |
| | 1 | 0.23931433524968 | 0.23076534494716 |
| | 2 | 0.28444444444444 | 0.50000000000000 |
| | 3 | 0.23931433524968 | 0.76923465505284 |
| | 4 | 0.11846344252809 | 0.95308992296933 |

# Gauss quandatures: weights and nodes(2)

| | | | |
|---|---|---|---|
| $n = 5$ | 0 | 0.08566224618959 | 0.03376524289842 |
| | 1 | 0.18038078652407 | 0.16939530676687 |
| | 2 | 0.23395696728635 | 0.38069040695840 |
| | 3 | 0.23395696728635 | 0.61930959304160 |
| | 4 | 0.18038078652407 | 0.83060469323313 |
| | 5 | 0.08566224618959 | 0.96623475710158 |
| $n = 6$ | 0 | 0.06474248308443 | 0.02544604382862 |
| | 1 | 0.13985269574464 | 0.12923440720030 |
| | 2 | 0.19091502525256 | 0.29707742431130 |
| | 3 | 0.20897959183673 | 0.50000000000000 |
| | 4 | 0.19091502525256 | 0.70292257568870 |
| | 5 | 0.13985269574464 | 0.87076559279970 |
| | 6 | 0.06474248308443 | 0.97455395617138 |
| $n = 7$ | 0 | 0.05061426814519 | 0.01985507175123 |
| | 1 | 0.11119051722669 | 0.10166676129319 |
| | 2 | 0.15685332293894 | 0.23723379504184 |
| | 3 | 0.18134189168918 | 0.40828267875218 |
| | 4 | 0.18134189168918 | 0.59171732124782 |
| | 5 | 0.15685332293894 | 0.76276620495816 |
| | 6 | 0.11119051722669 | 0.89833323870681 |
| | 7 | 0.05061426814519 | 0.98014492824877 |

# Numerical quadratures – composite rules

$$I(f) = \int_0^1 e^x \,\mathrm{d}x = e^x - 1 \approx 1.718281828459045$$

- trapezoid rule ($n = 1$)
- Simpson rule ($n = 2$)
- Gauss rule $n = 1$
- Gauss rule $n = 6$

- $h = 1, \quad 1/2, \quad 1/4, \quad 1/8, \; \ldots, \; 1/1024$
- $R_h = I(f) - Q_h(f)$
- $q = R_{2h}/R_h$
- order $p$: $\quad \frac{R_{2h}}{R_h} = 2^{p+1} \qquad \Longrightarrow \quad p + 1 = \log_2(R_{2h}/R_h)$

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- $R_h = I(f) - Q_h(f)$
- $q = R_{2h}/R_h$
- order $p$: $\frac{R_{2h}}{R_h} = 2^{p+1} \qquad \Longrightarrow \quad p + 1 = \log_2(R_{2h}/R_h)$

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- $h = 1, \quad 1/2, \quad 1/4, \quad 1/8, \ \ldots, \ 1/1024$
- $R_h = I(f) - Q_h(f)$
- $q = R_{2h}/R_h$
- order $p$: $\quad \frac{R_{2h}}{R_h} = 2^{p+1} \qquad \Longrightarrow \quad p + 1 = \log_2(R_{2h}/R_h)$

# $\int_0^1 \exp(x)\,\mathrm{d}x$ – trapezoid rule

| $m$ | $h$ | $N$ | $T_h(f)$ | $R_h(f)$ | $R_{2h}/R_h$ | $p+1$ |
|---|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 1.859140914229523E+00 | 1.408591E-01 | — | — |
| 1 | $2^{-1}$ | 2 | 1.753931092464825E+00 | 3.564926E-02 | 3.9512 | 1.9823 |
| 2 | $2^{-2}$ | 4 | 1.727221904557517E+00 | 8.940076E-03 | 3.9876 | 1.9955 |
| 3 | $2^{-3}$ | 8 | 1.720518592164302E+00 | 2.236764E-03 | 3.9969 | 1.9989 |
| 4 | $2^{-4}$ | 16 | 1.718841128579994E+00 | 5.593001E-04 | 3.9992 | 1.9997 |
| 5 | $2^{-5}$ | 32 | 1.718421660316327E+00 | 1.398319E-04 | 3.9998 | 1.9999 |
| 6 | $2^{-6}$ | 64 | 1.718316786850093E+00 | 3.495839E-05 | 4.0000 | 2.0000 |
| 7 | $2^{-7}$ | 128 | 1.718290568083479E+00 | 8.739624E-06 | 4.0000 | 2.0000 |
| 8 | $2^{-8}$ | 256 | 1.718284013366820E+00 | 2.184908E-06 | 4.0000 | 2.0000 |
| 9 | $2^{-9}$ | 512 | 1.718282374686094E+00 | 5.462270E-07 | 4.0000 | 2.0000 |
| 10 | $2^{-10}$ | 1024 | 1.718281965015814E+00 | 1.365568E-07 | 4.0000 | 2.0000 |

exact value $I(f) = e - 1 \approx 1.718281828459045\text{E}{+}00$

# $\int_0^1 \exp(x)\,\mathrm{d}x$ – trapezoid rule

| $m$ | $h$ | $N$ | $T_h(f)$ | $R_h(f)$ | $R_{2h}/R_h$ | $p+1$ |
|---|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 1.859140914229523E+00 | 1.408591E-01 | — | — |
| 1 | $2^{-1}$ | 2 | 1.753931092464825E+00 | 3.564926E-02 | 3.9512 | 1.9823 |
| 2 | $2^{-2}$ | 4 | 1.727221904557517E+00 | 8.940076E-03 | 3.9876 | 1.9955 |
| 3 | $2^{-3}$ | 8 | 1.720518592164302E+00 | 2.236764E-03 | 3.9969 | 1.9989 |
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| 8 | $2^{-8}$ | 256 | 1.718284013366820E+00 | 2.184908E-06 | 4.0000 | 2.0000 |
| 9 | $2^{-9}$ | 512 | 1.718282374686094E+00 | 5.462270E-07 | 4.0000 | 2.0000 |
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# $\int_0^1 \exp(x)\,\mathrm{d}x$ – Simpson rule

| $m$ | $h$ | $N$ | $S_h(f)$ | $R_h(f)$ | $R_{2h}/R_h$ | $p+1$ |
|---|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 1.718861151876593E+00 | 5.793234E-04 | — | — |
| 1 | $2^{-1}$ | 2 | 1.718318841921747E+00 | 3.701346E-05 | 15.6517 | 3.9682 |
| 2 | $2^{-2}$ | 4 | 1.718284154699897E+00 | 2.326241E-06 | 15.9113 | 3.9920 |
| 3 | $2^{-3}$ | 8 | 1.718281974051891E+00 | 1.455928E-07 | 15.9777 | 3.9980 |
| 4 | $2^{-4}$ | 16 | 1.718281837561772E+00 | 9.102727E-09 | 15.9944 | 3.9995 |
| 5 | $2^{-5}$ | 32 | 1.718281829028015E+00 | 5.689702E-10 | 15.9986 | 3.9999 |
| 6 | $2^{-6}$ | 64 | 1.718281828494606E+00 | 3.556089E-11 | 15.9999 | 4.0000 |
| 7 | $2^{-7}$ | 128 | 1.718281828461268E+00 | 2.223111E-12 | 15.9960 | 3.9996 |
| 8 | $2^{-8}$ | 256 | 1.718281828459185E+00 | 1.394440E-13 | 15.9427 | 3.9948 |
| 9 | $2^{-9}$ | 512 | 1.718281828459054E+00 | 8.881784E-15 | 15.7000 | 3.9727 |
| 10 | $2^{-10}$ | 1024 | 1.718281828459047E+00 | 1.776357E-15 | 5.0000 | 2.3219 |

exact value $I(f) = e - 1 \approx 1.718281828459045\text{E}{+}00$

| $m$ | $h$ | $N$ | $S_h(f)$ | $R_h(f)$ | $R_{2h}/R_h$ | $p+1$ |
|---|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 1.718861151876593E+00 | 5.793234E-04 | — | — |
| 1 | $2^{-1}$ | 2 | 1.718318841921747E+00 | 3.701346E-05 | 15.6517 | 3.9682 |
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| 4 | $2^{-4}$ | 16 | 1.718281837561772E+00 | 9.102727E-09 | 15.9944 | 3.9995 |
| 5 | $2^{-5}$ | 32 | 1.718281829028015E+00 | 5.689702E-10 | 15.9986 | 3.9999 |
| 6 | $2^{-6}$ | 64 | 1.718281828494606E+00 | 3.556089E-11 | 15.9999 | 4.0000 |
| 7 | $2^{-7}$ | 128 | 1.718281828461268E+00 | 2.223111E-12 | 15.9960 | 3.9996 |
| 8 | $2^{-8}$ | 256 | 1.718281828459185E+00 | 1.394440E-13 | 15.9427 | 3.9948 |
| 9 | $2^{-9}$ | 512 | 1.718281828459054E+00 | 8.881784E-15 | 15.7000 | 3.9727 |
| 10 | $2^{-10}$ | 1024 | 1.718281828459047E+00 | 1.776357E-15 | 5.0000 | 2.3219 |

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# $\int_0^1 \exp(x)\,\mathrm{d}x$ – Gauss rule $n = 1$

| $m$ | $h$ | $N$ | $G_h(f)$ | $R_h(f)$ | $R_{2h}/R_h$ | $p+1$ |
|---|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 1.717896378007504E+00 | 3.854505E-04 | — | — |
| 1 | $2^{-1}$ | 2 | 1.718257165052592E+00 | 2.466341E-05 | 15.6284 | 3.9661 |
| 2 | $2^{-2}$ | 4 | 1.718280277824108E+00 | 1.550635E-06 | 15.9054 | 3.9914 |
| 3 | $2^{-3}$ | 8 | 1.718281731400156E+00 | 9.705889E-08 | 15.9762 | 3.9979 |
| 4 | $2^{-4}$ | 16 | 1.718281822390608E+00 | 6.068437E-09 | 15.9940 | 3.9995 |
| 5 | $2^{-5}$ | 32 | 1.718281828079732E+00 | 3.793128E-10 | 15.9985 | 3.9999 |
| 6 | $2^{-6}$ | 64 | 1.718281828435338E+00 | 2.370726E-11 | 15.9999 | 4.0000 |
| 7 | $2^{-7}$ | 128 | 1.718281828457563E+00 | 1.481926E-12 | 15.9976 | 3.9998 |
| 8 | $2^{-8}$ | 256 | 1.718281828458953E+00 | 9.237056E-14 | 16.0433 | 4.0039 |
| 9 | $2^{-9}$ | 512 | 1.718281828459038E+00 | 7.327472E-15 | 12.6061 | 3.6560 |
| 10 | $2^{-10}$ | 1024 | 1.718281828459046E+00 | 1.332268E-15 | 5.5000 | 2.4594 |

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## $\int_0^1 \exp(x)\,\mathrm{d}x$ – Gauss rule $n = 1$

| $m$ | $h$ | $N$ | $G_h(f)$ | $R_h(f)$ | $R_{2h}/R_h$ | $p+1$ |
|---|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 1.717896378007504E+00 | 3.854505E-04 | — | — |
| 1 | $2^{-1}$ | 2 | 1.718257165052592E+00 | 2.466341E-05 | 15.6284 | 3.9661 |
| 2 | $2^{-2}$ | 4 | 1.718280277824108E+00 | 1.550635E-06 | 15.9054 | 3.9914 |
| 3 | $2^{-3}$ | 8 | 1.718281731400156E+00 | 9.705889E-08 | 15.9762 | 3.9979 |
| 4 | $2^{-4}$ | 16 | 1.718281822390608E+00 | 6.068437E-09 | 15.9940 | 3.9995 |
| 5 | $2^{-5}$ | 32 | 1.718281828079732E+00 | 3.793128E-10 | 15.9985 | 3.9999 |
| 6 | $2^{-6}$ | 64 | 1.718281828435338E+00 | 2.370726E-11 | 15.9999 | 4.0000 |
| 7 | $2^{-7}$ | 128 | 1.718281828457563E+00 | 1.481926E-12 | 15.9976 | 3.9998 |
| 8 | $2^{-8}$ | 256 | 1.718281828458953E+00 | 9.237056E-14 | 16.0433 | 4.0039 |
| 9 | $2^{-9}$ | 512 | 1.718281828459038E+00 | 7.327472E-15 | 12.6061 | 3.6560 |
| 10 | $2^{-10}$ | 1024 | 1.718281828459046E+00 | 1.332268E-15 | 5.5000 | 2.4594 |

exact value $I(f) = e - 1 \approx 1.718281828459045\mathrm{E}{+}00$

# $\int_0^1 \exp(x)\,\mathrm{d}x$ – Gauss rule $n = 6$

| $m$ | $h$ | $N$ | $G_h(f)$ | $R_h(f)$ | $R_{2h}/R_h$ | $p+1$ |
|---|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 1.718281828459045E+00 | 0.000000E+00 | — | — |
| 1 | $2^{-1}$ | 2 | 1.718281828459045E+00 | 0.000000E+00 | 0.0000 | 0.0000 |
| 2 | $2^{-2}$ | 4 | 1.718281828459045E+00 | 0.000000E+00 | 0.0000 | 0.0000 |
| 3 | $2^{-3}$ | 8 | 1.718281828459046E+00 | 4.440892E-16 | 0.0000 | 0.0000 |
| 4 | $2^{-4}$ | 16 | 1.718281828459045E+00 | 0.000000E+00 | 0.0000 | 0.0000 |
| 5 | $2^{-5}$ | 32 | 1.718281828459045E+00 | 0.000000E+00 | 0.0000 | 0.0000 |
| 6 | $2^{-6}$ | 64 | 1.718281828459045E+00 | 0.000000E+00 | 0.0000 | 0.0000 |
| 7 | $2^{-7}$ | 128 | 1.718281828459046E+00 | 4.440892E-16 | 0.0000 | 0.0000 |
| 8 | $2^{-8}$ | 256 | 1.718281828459045E+00 | 2.220446E-16 | 0.0000 | 0.0000 |
| 9 | $2^{-9}$ | 512 | 1.718281828459046E+00 | 6.661338E-16 | 0.0000 | 0.0000 |
| 10 | $2^{-10}$ | 1024 | 1.718281828459047E+00 | 1.554312E-15 | 0.0000 | 0.0000 |

exact value $I(f) = e - 1 \approx 1.718281828459045\text{E}{+}00$

# $\int_0^1 \exp(x)\,\mathrm{d}x$ – Gauss rule $n = 6$

| $m$ | $h$ | $N$ | $G_h(f)$ | $R_h(f)$ | $R_{2h}/R_h$ | $p+1$ |
|---|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 1.718281828459045E+00 | 0.000000E+00 | — | — |
| 1 | $2^{-1}$ | 2 | 1.718281828459045E+00 | 0.000000E+00 | 0.0000 | 0.0000 |
| 2 | $2^{-2}$ | 4 | 1.718281828459045E+00 | 0.000000E+00 | 0.0000 | 0.0000 |
| 3 | $2^{-3}$ | 8 | 1.718281828459046E+00 | 4.440892E-16 | 0.0000 | 0.0000 |
| 4 | $2^{-4}$ | 16 | 1.718281828459045E+00 | 0.000000E+00 | 0.0000 | 0.0000 |
| 5 | $2^{-5}$ | 32 | 1.718281828459045E+00 | 0.000000E+00 | 0.0000 | 0.0000 |
| 6 | $2^{-6}$ | 64 | 1.718281828459045E+00 | 0.000000E+00 | 0.0000 | 0.0000 |
| 7 | $2^{-7}$ | 128 | 1.718281828459046E+00 | 4.440892E-16 | 0.0000 | 0.0000 |
| 8 | $2^{-8}$ | 256 | 1.718281828459045E+00 | 2.220446E-16 | 0.0000 | 0.0000 |
| 9 | $2^{-9}$ | 512 | 1.718281828459046E+00 | 6.661338E-16 | 0.0000 | 0.0000 |
| 10 | $2^{-10}$ | 1024 | 1.718281828459047E+00 | 1.554312E-15 | 0.0000 | 0.0000 |

exact value $I(f) = e - 1 \approx 1.718281828459045\text{E}{+}00$

# Numerical quadratures – composite rules

$$I(f) = \int_0^1 \sqrt{x}\,\mathrm{d}x = \frac{2}{3} \approx 0.66666666666666666667$$

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# $\int_0^1 \sqrt{x}\,\mathrm{d}x$ – trapezoid rule

| $n$ | $h$ | $N$ | $T_h(f)$ | $R_h(f)$ | $R_{h/2}/R_h$ | order |
|---|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 5.000000000000000E-01 | 1.666667E-01 | — | — |
| 1 | $2^{-1}$ | 2 | 6.035533905932737E-01 | 6.311328E-02 | 2.6408 | 1.4010 |
| 2 | $2^{-2}$ | 4 | 6.432830462427466E-01 | 2.338362E-02 | 2.6990 | 1.4324 |
| 3 | $2^{-3}$ | 8 | 6.581302216244542E-01 | 8.536445E-03 | 2.7393 | 1.4538 |
| 4 | $2^{-4}$ | 16 | 6.635811968772282E-01 | 3.085470E-03 | 2.7667 | 1.4681 |
| 5 | $2^{-5}$ | 32 | 6.655589362789417E-01 | 1.107730E-03 | 2.7854 | 1.4779 |
| 6 | $2^{-6}$ | 64 | 6.662708113785069E-01 | 3.958553E-04 | 2.7983 | 1.4846 |
| 7 | $2^{-7}$ | 128 | 6.665256572968257E-01 | 1.410094E-04 | 2.8073 | 1.4892 |
| 8 | $2^{-8}$ | 256 | 6.666165489765280E-01 | 5.011769E-05 | 2.8136 | 1.4924 |
| 9 | $2^{-9}$ | 512 | 6.666488815499515E-01 | 1.778512E-05 | 2.8180 | 1.4946 |
| 10 | $2^{-10}$ | 1024 | 6.666603622189838E-01 | 6.304448E-06 | 2.8210 | 1.4962 |

exact value $I(f) = \frac{2}{3} \approx 0.6666666666666667$

# $\int_0^1 \sqrt{x}\,\mathrm{d}x$ – Simpson rule

| $n$ | $h$ | $N$ | $S_h(f)$ | $R_h(f)$ | $R_{h/2}/R_h$ | order |
|---|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 6.380711874576983E-01 | 2.859548E-02 | — | — |
| 1 | $2^{-1}$ | 2 | 6.565262647925707E-01 | 1.014040E-02 | 2.8200 | 1.4957 |
| 2 | $2^{-2}$ | 4 | 6.630792800850236E-01 | 3.587387E-03 | 2.8267 | 1.4991 |
| 3 | $2^{-3}$ | 8 | 6.653981886281528E-01 | 1.268478E-03 | 2.8281 | 1.4998 |
| 4 | $2^{-4}$ | 16 | 6.662181827461796E-01 | 4.484839E-04 | 2.8284 | 1.5000 |
| 5 | $2^{-5}$ | 32 | 6.665081030783619E-01 | 1.585636E-04 | 2.8284 | 1.5000 |
| 6 | $2^{-6}$ | 64 | 6.666106059362655E-01 | 5.606073E-05 | 2.8284 | 1.5000 |
| 7 | $2^{-7}$ | 128 | 6.666468462030957E-01 | 1.982046E-05 | 2.8284 | 1.5000 |
| 8 | $2^{-8}$ | 256 | 6.666596590744270E-01 | 7.007592E-06 | 2.8284 | 1.5000 |
| 9 | $2^{-9}$ | 512 | 6.666641891086617E-01 | 2.477558E-06 | 2.8284 | 1.5000 |
| 10 | $2^{-10}$ | 1024 | 6.666657907176324E-01 | 8.759490E-07 | 2.8284 | 1.5000 |

exact value $I(f) = \frac{2}{3} \approx 0.6666666666666667$

# $\int_0^1 \sqrt{x}\,\mathrm{d}x$ – Gauss rule, $n = 1$

| $n$ | $h$ | $N$ | $G_h(f)$ | $R_h(f)$ | $R_{h/2}/R_h$ | order |
|---|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 6.738873386790492E-01 | 7.220672E-03 | — | — |
| 1 | $2^{-1}$ | 2 | 6.692395023997495E-01 | 2.572836E-03 | 2.8065 | 1.4888 |
| 2 | $2^{-2}$ | 4 | 6.675777015359070E-01 | 9.111035E-04 | 2.8239 | 1.4977 |
| 3 | $2^{-3}$ | 8 | 6.669888717455580E-01 | 3.222205E-04 | 2.8276 | 1.4996 |
| 4 | $2^{-4}$ | 16 | 6.667805949572163E-01 | 1.139283E-04 | 2.8283 | 1.4999 |
| 5 | $2^{-5}$ | 32 | 6.667069467851046E-01 | 4.028012E-05 | 2.8284 | 1.5000 |
| 6 | $2^{-6}$ | 64 | 6.666809078632009E-01 | 1.424120E-05 | 2.8284 | 1.5000 |
| 7 | $2^{-7}$ | 128 | 6.666717016914930E-01 | 5.035025E-06 | 2.8284 | 1.5000 |
| 8 | $2^{-8}$ | 256 | 6.666684468168600E-01 | 1.780150E-06 | 2.8284 | 1.5000 |
| 9 | $2^{-9}$ | 512 | 6.666672960448092E-01 | 6.293781E-07 | 2.8284 | 1.5000 |
| 10 | $2^{-10}$ | 1024 | 6.666668891854427E-01 | 2.225188E-07 | 2.8284 | 1.5000 |

exact value $I(f) = \frac{2}{3} \approx 0.666666666666666667$

# $\int_0^1 \sqrt{x}\,\mathrm{d}x$ – Gauss rule, $n = 6$

| $n$ | $h$ | $N$ | $G_h(f)$ | $R_h(f)$ | $R_{h/2}/R_h$ | order |
|---|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 6.669130850887391E-01 | 2.464184E-04 | — | — |
| 1 | $2^{-1}$ | 2 | 6.667537887353612E-01 | 8.712207E-05 | 2.8284 | 1.5000 |
| 2 | $2^{-2}$ | 4 | 6.666974689694490E-01 | 3.080230E-05 | 2.8284 | 1.5000 |
| 3 | $2^{-3}$ | 8 | 6.666775569252534E-01 | 1.089026E-05 | 2.8284 | 1.5000 |
| 4 | $2^{-4}$ | 16 | 6.666705169545143E-01 | 3.850288E-06 | 2.8284 | 1.5000 |
| 5 | $2^{-5}$ | 32 | 6.666680279489899E-01 | 1.361282E-06 | 2.8284 | 1.5000 |
| 6 | $2^{-6}$ | 64 | 6.666671479526478E-01 | 4.812860E-07 | 2.8284 | 1.5000 |
| 7 | $2^{-7}$ | 128 | 6.666668368269568E-01 | 1.701603E-07 | 2.8284 | 1.5000 |
| 8 | $2^{-8}$ | 256 | 6.666667268274141E-01 | 6.016075E-08 | 2.8284 | 1.5000 |
| 9 | $2^{-9}$ | 512 | 6.666666879367035E-01 | 2.127004E-08 | 2.8284 | 1.5000 |
| 10 | $2^{-10}$ | 1024 | 6.666666741867594E-01 | 7.520093E-09 | 2.8284 | 1.5000 |

exact value $I(f) = \frac{2}{3} \approx 0.666666666666666667$

# $\int_0^1 \exp(x)\,\mathrm{d}x$ – half-step error estimate (Simpson)

| $n$ | $h$ | $N$ | $S_h(f)$ | $R_h(f)$ | estim |
|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 1.718861151876593E+00 | 5.793234E-04 | — |
| 1 | $2^{-1}$ | 2 | 1.718318841921747E+00 | 3.701346E-05 | 3.615400E-05 |
| 2 | $2^{-2}$ | 4 | 1.718284154699897E+00 | 2.326241E-06 | 2.312481E-06 |
| 3 | $2^{-3}$ | 8 | 1.718281974051891E+00 | 1.455928E-07 | 1.453765E-07 |
| 4 | $2^{-4}$ | 16 | 1.718281837561772E+00 | 9.102727E-09 | 9.099341E-09 |
| 5 | $2^{-5}$ | 32 | 1.718281829028015E+00 | 5.689702E-10 | 5.689171E-10 |
| 6 | $2^{-6}$ | 64 | 1.718281828494606E+00 | 3.556089E-11 | 3.556062E-11 |
| 7 | $2^{-7}$ | 128 | 1.718281828461268E+00 | 2.223111E-12 | 2.222518E-12 |
| 8 | $2^{-8}$ | 256 | 1.718281828459185E+00 | 1.394440E-13 | 1.389111E-13 |
| 9 | $2^{-9}$ | 512 | 1.718281828459054E+00 | 8.881784E-15 | 8.704149E-15 |
| 10 | $2^{-10}$ | 1024 | 1.718281828459047E+00 | 1.776357E-15 | 4.736952E-16 |

$$R_h(f) = |I(f) - S_h(f)|, \qquad \text{estim} = \left| \frac{S_{2h}(f) - S_h(f)}{2^{p+1} - 1} \right|$$

# $\int_0^1 \exp(x)\,\mathrm{d}x$ – half-step error estimate (Simpson)

| $n$ | $h$ | $N$ | $S_h(f)$ | $R_h(f)$ | estim |
|---|---|---|---|---|---|
| 0 | $2^{0}$ | 1 | 1.718861151876593E+00 | 5.793234E-04 | — |
| 1 | $2^{-1}$ | 2 | 1.718318841921747E+00 | 3.701346E-05 | 3.615400E-05 |
| 2 | $2^{-2}$ | 4 | 1.718284154699897E+00 | 2.326241E-06 | 2.312481E-06 |
| 3 | $2^{-3}$ | 8 | 1.718281974051891E+00 | 1.455928E-07 | 1.453765E-07 |
| 4 | $2^{-4}$ | 16 | 1.718281837561772E+00 | 9.102727E-09 | 9.099341E-09 |
| 5 | $2^{-5}$ | 32 | 1.718281829028015E+00 | 5.689702E-10 | 5.689171E-10 |
| 6 | $2^{-6}$ | 64 | 1.718281828494606E+00 | 3.556089E-11 | 3.556062E-11 |
| 7 | $2^{-7}$ | 128 | 1.718281828461268E+00 | 2.223111E-12 | 2.222518E-12 |
| 8 | $2^{-8}$ | 256 | 1.718281828459185E+00 | 1.394440E-13 | 1.389111E-13 |
| 9 | $2^{-9}$ | 512 | 1.718281828459054E+00 | 8.881784E-15 | 8.704149E-15 |
| 10 | $2^{-10}$ | 1024 | 1.718281828459047E+00 | 1.776357E-15 | 4.736952E-16 |

$$R_h(f) = |I(f) - S_h(f)|, \qquad \text{estim} = \left| \frac{S_{2h}(f) - S_h(f)}{2^{p+1} - 1} \right|$$