Ordinary differential equations 1

Write a simple code for the numerical solution of

$$y' = -2\alpha(x-1)y, \qquad x \in (0,10), \qquad y(0) = \exp(-\alpha), \quad \alpha = 2. \tag{0.1}$$

The exact solution is $y(x) = c\exp[-\alpha(x-1)^2]$ and thus $y''(x) \approx x^2\exp[-\alpha x^2] \quad \text{for } x \gg 1$.

Use the explicit Euler method with the fixed and adaptively chosen time step h_k according to the formula

$$\frac{1}{2}y''(x_k+\tau_k h_k)h_k^2 = \mathrm{TOL} \qquad \Leftrightarrow \qquad h_k = \sqrt{2\mathrm{TOL}/y''(\cdot)}. \tag{0.2}$$

The second order derivative can be approximated by the difference

$$y'' \approx \frac{y'_k - y'_{k-1}}{x_k - x_{k-1}} = \frac{f(x_k,y_k) - f(x_{k-1},y_{k-1})}{x_k - x_{k-1}}. \tag{0.3}$$

Use the code `adapt_time.f90` from the archive:
`http://msekce.karlin.mff.cuni.cz/~dolejsi/Vyuka/NS_source/ODE/methods.tgz` (archive)

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1. Study the code and perform preliminarly numerical tests.
2. Modify the code in order to compute with the fixed time step. Which value of h_k does guarantee the stability of the scheme? (approximately)
3. Demonstrate that the choice (0.2) – (0.3) practically guarantees the stability condition.

More advanced tasks

1. Include to the code the evaluation of the error of the computations.
2. Investigate the dependence of the error on the size of fixed time step and the tolerance for the adaptive time step.
3. Compare the computations with adaptive and fixed time steps from the point of view of efficiecny (number of time steps).
4. Compare the computed errors with the corresponding error estimates. Do not forget to distinguish between the global and local errors!

Ordinary differential equations 2

Let us consider the system

$$u' = 998u + 1998v, \qquad (0.4)$$
$$v' = -999u - 1999v$$

with the Jacobi matrix reads

$$J_f = \begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix},$$

and its eigenvalues -1 and -1000 (stiff system). Exact solution $u(x) = 2\mathrm{e}^{-x} - \mathrm{e}^{-1000\,x}$, $v(x) = -\mathrm{e}^{-x} + \mathrm{e}^{-1000\,x}$.

Write a simple code for solving (0.4) by the explicit and implicit Euler methods and demonstrate the stability of the corresponding numerical method.

Use the code `stiff.f90` from the archive (either the interactive code `stiff` or C-shell script `Stiff.sh`):
`http://msekce.karlin.mff.cuni.cz/~dolejsi/Vyuka/NS_source/ODE/methods.tgz` (archive)

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1. Study the code and perform preliminarly numerical tests.
2. Find experimentaly the limit value of h_k which guarantees the stability of the explicit method
3. Compare the accuracy of the explicit and implicit method for different h_k.

More advanced tasks

1. Modify the code in such a way that the time step is chosen adaptively (use the technique (0.2)–(0.3)).
2. Include to the code the evaluation of the error of the computations (use previous task).
3. Investigate the dependence of the error on h_k and TOL for both schemes.
4. Compare the computed errors with the corresponding error estimates.
5. Modify the code and employ the Crank-Nicolson method.