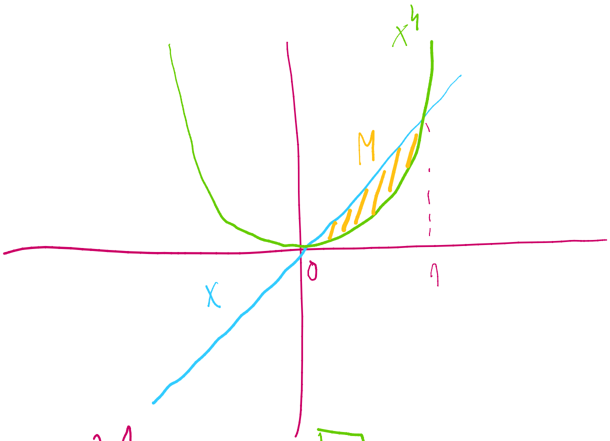
(1) $f(x) = x \quad g(x) = x^4$

$x = x^4 \Leftrightarrow x = 0, 1$

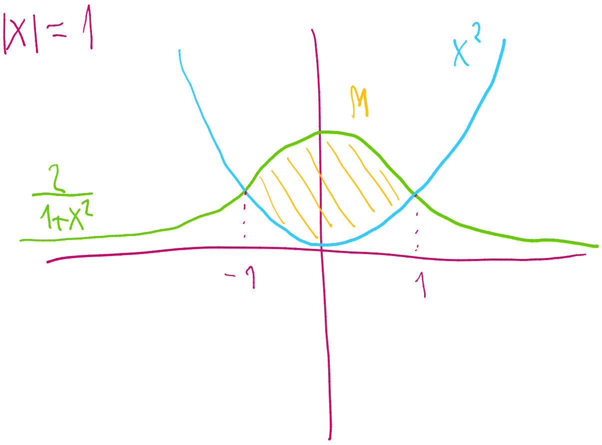


$M = \{(x,y) \in \mathbb{R}^2 : x \in [0,1],\ x^4 \leq y \leq x\}$

$$\text{plocha } M = \int_0^1 f - g = \int_0^1 x - x^4\,dx = \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2} - \frac{1}{5} = \boxed{\frac{3}{10}}$$

(2) $f(x) = \frac{2}{1+x^2} \quad g(x) = x^2$

$$\frac{2}{1+x^2} = x^2 \Leftrightarrow x^4 + x^2 - 2 = 0 \Leftrightarrow x^2 = \frac{-1 \pm \sqrt{9}}{2} \Leftrightarrow |x| = 1$$



$M = \{(x,y) \in \mathbb{R}^2 : x \in [-1,1],\ x^2 \leq y \leq \frac{2}{1+x^2}\}$

$$\text{plocha } M = \int_{-1}^{1} \frac{2}{1+x^2} - x^2\,dx = \left[2\operatorname{arctg} x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= 4\operatorname{arctg} 1 - \frac{2}{3} = \boxed{\pi - \frac{2}{3}}$$

(3) $f(x) = \arcsin x + \sqrt{1-x^2}, \quad x \in [-1,1]$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2} \cdot \frac{-2x}{\sqrt{1-x^2}} = \frac{1-x}{\sqrt{1-x^2}} = \sqrt{\frac{1-x}{1+x}} \Rightarrow [f'(x)]^2 = \frac{1-x}{1+x}$$

$$\text{délka grafu } f = \int_{-1}^{1} \sqrt{f'^2+1} = \int_{-1}^{1} \sqrt{\frac{1-x}{1+x} + 1}\,dx = \int_{-1}^{1} \sqrt{\frac{(1-x)+(1+x)}{1+x}}\,dx = \sqrt{2}\int_{-1}^{1} \frac{1}{\sqrt{1+x}}\,dx = \left| \begin{matrix} t = \sqrt{1+x} & dx = 2t\,dt \\ x = t^2 - 1 & -1 \to 0 \\ & 1 \to \sqrt{2} \end{matrix} \right|$$

$$= \sqrt{2}\int_0^{\sqrt{2}} \frac{1}{t} 2t\,dt = 2\sqrt{2}\,[t]_0^{\sqrt{2}} = \boxed{4}$$

(4) $f(x)=\log(\cos x) \quad x\in[0,\frac{\pi}{6}]$

$$f'(x)=\frac{-\sin x}{\cos x} \qquad (f'(x))^2=\frac{\sin^2 x}{\cos^2 x}$$

$$\text{délka grafu } f=\int_0^{\frac{\pi}{6}}\sqrt{\frac{\sin^2 x}{\cos^2 x}+1}\,dx \underset{\sin^2x+\cos^2x=1}{=} \int_0^{\frac{\pi}{6}}\sqrt{\frac{1}{\cos^2 x}}\,dx=\int_0^{\frac{\pi}{6}}\frac{1}{\cos x}\,dx=\int_0^{\frac{\pi}{6}}\frac{\cos x}{1-\sin^2 x}\,dx=\left|\begin{matrix} t=\sin x & 0\to 0 \\ dt=\cos x\,dx & \frac{\pi}{6}\to\frac{1}{2}\end{matrix}\right|$$

$$=\int_0^{\frac{1}{2}}\frac{1}{1-t^2}\,dt=\int_0^{\frac{1}{2}}\frac{-\frac{1}{2}}{t-1}+\frac{\frac{1}{2}}{t+1}\,dt=\frac{1}{2}\Big[-\log(1-t)+\log(1+t)\Big]_0^{\frac{1}{2}}=\frac{1}{2}\log\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right)=\boxed{\log\sqrt{3}}$$

(5) $\gamma(t)=(\cos t+t\sin t,\ \sin t-t\cos t) \quad t\in[0,2\pi]$

$$\gamma'(t)=(-\sin t+\sin t+t\cos t,\ \cos t-\cos t+t\sin t)=(t\cos t,\ t\sin t)$$

$$|\gamma'(t)|=\sqrt{(t\cos t)^2+(t\sin t)^2}=t$$

$$\text{délka } \gamma=\int_0^{2\pi} t\,dt=\left[\frac{t^2}{2}\right]_0^{2\pi}=\boxed{2\pi^2}$$

(6) $\gamma(t)=(e^{-t}\cos t,\ e^{-t}\sin t),\ t\in[0,\infty)$

$$\gamma'(t)=(-e^{-t}\cos t-e^{-t}\sin t,\ -e^{-t}\sin t+e^{-t}\cos t)$$

$$|\gamma'(t)|=\sqrt{e^{-2t}(\cos t-\sin t)^2+e^{-2t}(\cos t+\sin t)^2}=e^{-t}\sqrt{2\cos^2 t+2\sin^2 t}=\sqrt{2}\,e^{-t}$$

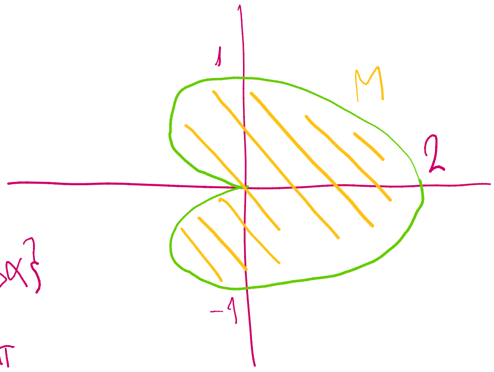
$$\text{délka } \gamma=\int_0^{\infty}|\gamma'(t)|=\int_0^{\infty}\sqrt{2}e^{-t}=\sqrt{2}\left[-e^{-t}\right]_0^{\infty}=\boxed{\sqrt{2}}$$

(7) $\gamma(t)=(\cos t,\sin t,\ t),\ t\in[0,T],\ T>0.$

$$\gamma'(t)=(-\sin t,\cos t,1),\quad |\gamma'(t)|=\sqrt{\sin^2 t+\cos^2 t+1}=\sqrt{2}$$

$$\text{délka } \gamma=\int_0^{T}\sqrt{2}\,dt=\boxed{\sqrt{2}\,T}$$

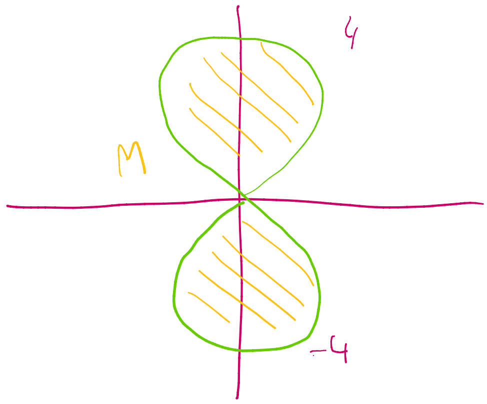
(8) $r(\alpha) = 1+\cos\alpha, \quad \alpha \in [0, 2\pi]$



$$M = \{(\rho\cos\alpha, \rho\sin\alpha) : 0 \le \alpha \le 2\pi,\ 0 \le \rho \le 1+\cos\alpha\}$$

$$\text{plocha } M = \int_0^{2\pi} \frac{1}{2} r^2(\alpha)\, d\alpha = \frac{1}{2}\int_0^{2\pi} (1+\cos\alpha)^2 d\alpha = \frac{1}{2}\int_0^{2\pi} 1 + 2\cos\alpha + \cos^2\alpha = \frac{1}{2}\cdot 2\pi + 0 + \frac{1}{4}2\pi = \boxed{\frac{3}{2}\pi}$$

(9) $r(\alpha) = 4\sin\alpha, \quad \alpha \in [0, 2\pi]$



$$M = \{(\rho\cos\alpha, \rho\sin\alpha) \in \mathbb{R}^2 : 0 \le \alpha \le 2\pi,\ 0 \le \rho \le 4\sin\alpha\}$$

$$\text{plocha } M = \int_0^{2\pi} \frac{1}{2} r^2(\alpha)\, d\alpha = \int_0^{2\pi} 2\sin^2\alpha\, d\alpha = \boxed{2\pi}$$

(10) $f(x) = \sqrt{R^2 - x^2}, \ x \in [-R, R]; \quad M = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le R^2\}$

$$= \{(x,y,z) \in \mathbb{R}^3 : \sqrt{y^2+y^2} \le f(x),\ x \in [-R,R]\}$$

rotační těleso vzniklé rotací grafu funkce f

$$\text{objem } M = \pi\int_{-R}^{R} f^2 = \pi\int_{-R}^{R} R^2 - x^2 dx = \pi\left[R^2x - \frac{x^3}{3}\right]_{-R}^{R}$$

$$= \pi\left(R^3 - (-R^3) - \frac{R^3}{3} - \left(-\frac{(-R)^3}{3}\right)\right) = 2\pi R^3\left(1 - \frac{1}{3}\right) = \boxed{\frac{4}{3}\pi R^3}$$

$$\text{povrch } M = 2\pi\int_{-R}^{R} |f|\sqrt{(f')^2+1} = 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\sqrt{\frac{x^2}{R^2-x^2}+1}\, dx = 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\cdot\sqrt{\frac{R^2}{R^2-x^2}}\, dx$$

$$f'(x) = \frac{-x}{\sqrt{R^2-x^2}}$$

$$= 2\pi\int_{-R}^{R} R\, dx = 2\pi\left[Rx\right]_{-R}^{R} = \boxed{4\pi R^2}$$

(11) $r(\alpha) = 1+\cos\alpha, \quad \alpha\in[0,\pi]$

$$M = \{(x,y,z)\in\mathbb{R}^3 : x = \rho\cos\alpha,\ \sqrt{y^2+z^2} = \rho\sin\alpha,\ \alpha\in[0,\pi],\ \rho\in[0,1+\cos\alpha]\}$$

objem $M = \frac{2}{3}\pi\int_0^\pi \rho^3 = \frac{2}{3}\pi\int_0^\pi (1+\cos\alpha)^3 d\alpha = \frac{2}{3}\pi\int_0^\pi 1+3\cos\alpha+3\cos^2\alpha+\cos^3\alpha\, d\alpha$

$$= \frac{2}{3}\pi\left(\pi+\frac{3}{2}\pi\right) = \boxed{\frac{5}{3}\pi^2}$$

(12) $M = \{(x,y)\in\mathbb{R}^2 : 0\le y\le\sqrt{r^2-x^2},\ x\in[0,r]\}$ (volíme $\rho = 1$)

$\rho=1$

$m \overset{\downarrow}{=}$ plocha $M = \frac{1}{4}\pi r^2$

$A = \int_0^r x\rho(x)\,dx, \quad B = \frac{1}{2}\int_0^r \rho^2(x)\,dx \rightarrow$ těžiště $M = \left(\frac{A}{m}, \frac{B}{m}\right)$

$$A = \int_0^r x\sqrt{r^2-x^2}\,dx = \left|\begin{matrix} t=r^2-x^2 & 0\to r^2 \\ dt=-2x\,dx & r\to 0\end{matrix}\right| = \int_{r^2}^0 -\frac{1}{2}\sqrt{t}\,dt = \int_0^{r^2}\frac{t^{\frac{1}{2}}}{2}dt = \left[\frac{t^{\frac{3}{2}}}{2\cdot\frac{3}{2}}\right]_0^{r^2} = \boxed{\frac{r^3}{3}}$$

$$B = \frac{1}{2}\int_0^r r^2-x^2\,dx = \frac{1}{2}\left[r^2x-\frac{x^3}{3}\right]_0^r = \frac{1}{2}\left(r^3-\frac{r^3}{3}\right) = \boxed{\frac{r^3}{3}}$$

$$\text{těžiště } M = \left(\frac{\frac{r^3}{3}}{\frac{1}{4}\pi r^2}, \frac{\frac{r^3}{3}}{\frac{1}{4}\pi r^2}\right) = \boxed{\left(\frac{4r}{3\pi}, \frac{4r}{3\pi}\right)}$$

(13) $M = \{(x,y,z)\in\mathbb{R}^3 : 0\le x\le r,\ 0\le y^2+z^2\le r^2-x^2\}$

těžiště $M = (X,0,0)$, kde $X = \frac{1}{m}\cdot\pi\int_0^r x\rho^2(x)\,dx$ $\left(m = \frac{2}{3}\pi r^3\right)$

$$X = \frac{3}{2\pi r^3}\cdot\pi\cdot\int_0^r x(r^2-x^2)\,dx = \frac{3}{2r^3}\left[\frac{r^2x^2}{2}-\frac{x^4}{4}\right]_0^r = \frac{3}{2\pi}\cdot\left(\frac{r^2}{2}-\frac{r^2}{4}\right) = \boxed{\frac{3r^2}{8\pi}}$$

$$\text{těžiště } M = \boxed{\left(\frac{3r^2}{8\pi}, 0, 0\right)}$$