Cvičení 14.4.
středa 14. dubna 2021 12:18

$$\sum (-1)^n a_n \quad a_n \geq 0 \quad \text{monotónní}$$

Nutná podmínka: $a_n \to 0$?

Absolutní konvergence: $\sum |a_n| < \infty$?

(1) $$\sum_{n=1}^{\infty} \frac{\sin(\frac{\pi}{3} n)}{n^2} \quad {\scriptstyle = a_n} \qquad |\sin(x)| \leq 1 \Rightarrow |a_n| \leq \frac{1}{n^2}$$

$$? \; \sum_{n=1}^{\infty} |a_n| < \infty \Leftarrow \sum \frac{1}{n^2} < \infty$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} a_n \text{ konverguje}$$

(2) $$\sum_{n=1}^{\infty} \frac{\sin(\frac{\pi}{3} n)}{n} \qquad \left|\frac{\sin(\frac{\pi}{3} n)}{n}\right| \leq \frac{1}{n} \to 0$$

omezené částečné součty

$$\sum a_n b_n \qquad b_n = \sin\left(\frac{\pi}{3} n\right) \quad \text{OČS} \checkmark$$

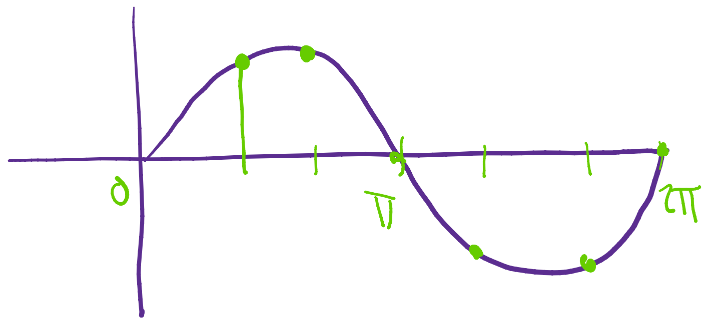
monotónní

$$a = \frac{1}{n} \quad \text{monotónní}, \; a_n \to 0 \quad \checkmark$$

- $\sum^{\infty} \sin(nx)$ má omezené částečné součty

- $\sum_{n=1}^{\infty} \sin(nx)$ má omezené částečné součty

- $\sin\left(\frac{\pi}{3}n\right) \to \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \ldots$



$\sum \frac{\sin\left(\frac{\pi}{3}n\right)}{n}$ konverguje podle Dirichletova kritéria

$\left|\sin \frac{\pi}{3}n\right| \to \left\{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, \ldots\right.$

$$\sum_{n=1}^{\infty} \left|\frac{\sin\left(\frac{\pi}{3}n\right)}{n}\right| \geq \sum_{n=1}^{\infty} \frac{\left|\sin\left(\frac{\pi}{3}(3n+1)\right)\right|}{3n+1} = \frac{\sqrt{3}}{2} \sum_{n=1}^{\infty} \frac{1}{3n+1} = \infty$$

(3) • $\sum_{n=1}^{\infty} \frac{z^n}{n}$ $\qquad z \in \mathbb{R}$

pro jaká z
- konverguje
- konverguje absolutně

$\frac{z^n}{n} \to |a_n| = \frac{|z|^n}{}$

$$a_n = \frac{z^n}{n} \quad \to \quad |a_n| = \frac{|z|^n}{n}$$

$$\sqrt[n]{|a_n|} = \frac{|z|}{\sqrt[n]{n}} \to |z| \quad , \text{ tedy } \begin{array}{l} |z|<1 \to K \\ |z|>1 \to \underline{\underline{D}} \end{array}$$

$$|z| = 1 \to \sum \frac{1}{n} = \infty$$

- $\sum \frac{z^n}{n}$ konverguje absolutně $\iff |z| < 1$

$|z| > 1 \to$ není splněna nutná podmínka $\to D$

$$|z| = 1 \begin{cases} z = 1 \to \sum \frac{1}{n} = \infty \\ z = -1 \to \sum \frac{(-1)^n}{n} \quad K \checkmark \end{cases}$$

$$\frac{1}{n} \searrow 0 \qquad (-1)^n \text{ OČS}$$

- $\sum \frac{z^n}{n}$ konverguje $\iff z \in [-1, 1)$

(4) kdy $\sum_{n=1}^{\infty} n^2 z^n$ konverguje / konverguje absolutně

$$a_n = n^2 z^n \to \sqrt[n]{|a_n|} = \overbrace{\sqrt[n]{n^2}}^{\nearrow 1} \cdot |z| \to |z| \begin{cases} <1 \to K \\ >1 \to D \end{cases}$$

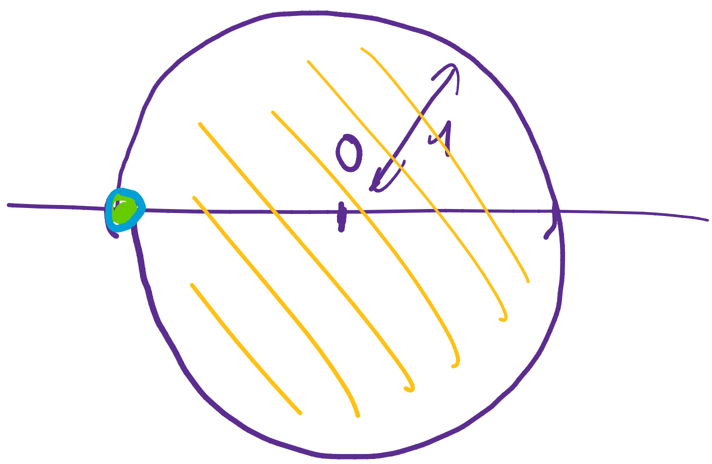
$a_n = n^2 z^n \rightarrow \sqrt[n]{|a_n|} = \sqrt[n]{n^2} \cdot |z| \rightarrow |z|$ $\searrow > 1 \rightarrow D$

$|z| = 1 \rightarrow \sum n^2 = \infty$

$\sum n^2 z^n$ AK $\Leftrightarrow |z| < 1$

$|z| \geq 1 \rightarrow$ není splněna nutná podmínka $\rightarrow D$

$\sum \underline{\underline{n^2}} z^n$ konv $\Leftrightarrow |z| < 1$



(5) kdy $\sum_{n=1}^{\infty} \frac{6^n}{n^2} z^n$ konverguje / konverguje absolutně ?

$a_n = \frac{1}{n^2} \cdot (6z)^n \rightarrow \sqrt[n]{|a_n|} = \frac{1}{\sqrt[n]{n^2}} \cdot 6|z| \rightarrow 6|z|$

$6|z| < 1 \rightarrow K \quad \left(|z| < \frac{1}{6} \right)$

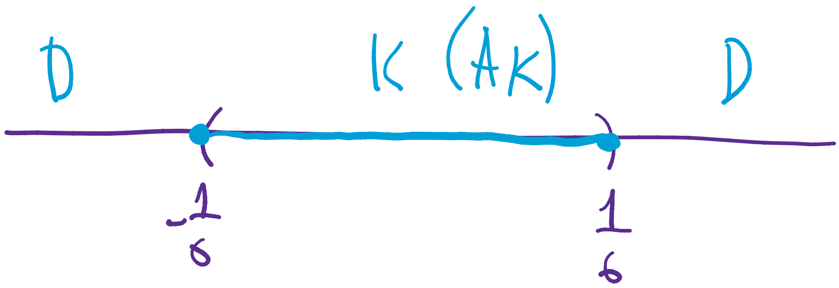
$> 1 \rightarrow D \quad \left(|z| > \frac{1}{6} \right)$

$> 1 \to D \qquad \left(|z| > \frac{1}{6}\right)$

$$|z| = \frac{1}{6} \to \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

$$\sum \frac{6^n}{n^2} z^n \quad AK \Leftrightarrow \quad |z| \leq \frac{1}{6}$$

pro $|z| > \frac{1}{6}$ není splněna nutná podmínka $\to D$



(6)
$$\sum_{n=1}^{\infty} \cos(\pi n) \cdot \left(\sqrt{n^3+3} - \sqrt{n^3+1}\right)$$

$$\cos(\pi n) = (-1)^{n+1} = a_n$$

$$b_n = \sqrt{n^3+3} - \sqrt{n^3+1} = \frac{n^3+3-n^3-1}{\sqrt{n^3+3}+\sqrt{n^3+1}}$$

$$= \frac{2}{\sqrt{n^3+1}+\sqrt{n^3+1}} \sim \frac{1}{n^\alpha} \qquad \alpha = \frac{3}{2} \to K$$

$$\sum_{n=1}^{\infty} \left| \cos(\pi n)\left(\sqrt{n^3+3} - \sqrt{n^3+1}\right) \right| < \infty$$

$n=1$

$\sum a_n b_n \quad AK \Rightarrow K$

(7) $$\sum_{n=1}^{\infty} \cos(\pi n) \cdot \frac{\sqrt{n+7}}{\sqrt{n}\sqrt{n+1}} = \sum (-1)^{n+1} \frac{\sqrt{n+7}}{\sqrt{n}\sqrt{n+1}}$$

$\sim \sqrt{n}$

AK ? $$\left| (-1)^{n+1} \frac{\sqrt{n+2}}{\sqrt{n}\sqrt{n+1}} \right| = \frac{\sqrt{n+7}}{\underset{\sqrt{n}}{\underset{\shortparallel}{\sqrt{n}}}\,\underset{\sqrt[3]{n}}{\sqrt{n+1}}} \sim \frac{1}{\sqrt{n}} = \frac{1}{n^{\frac{1}{2}}}$$

$\frac{1}{2} < 1 \Rightarrow$ nekonvergnuje absolutně

potřebujeme $\frac{\sqrt{n+7}}{\sqrt{n}\sqrt{n+1}} \searrow 0$

? $a_{n+1} \leq a_n$ $\quad$ $\frac{\sqrt{n+8}}{\cancel{\sqrt{n+1}}\sqrt{n+2}} \leq \frac{\sqrt{n+7}}{\sqrt{n}\cancel{\sqrt{n+1}}}$

$n \in \mathbb{N}$ $\quad\quad \Updownarrow$

$$n^2 + 8n = n(n+8) \leq (n+2)(n+7) = n^2 + 9n + 14$$

$$\sum_{n=1}^{\infty} \cos(\pi n) \frac{\sqrt{n+7}}{\sqrt{n}\sqrt{n+1}}$$ konvergnuje (ale ne absolutně)

(8) $\sum_{n=1}^{\infty} \underline{(-1)^n} \cdot \underline{\frac{n-1}{n+1}} \cdot \underline{\frac{1}{\sqrt[213]{n}}}$

$\sum (-1)^n \frac{1}{\sqrt[213]{n}}$ konverguje protože

K

$(-1)^n$ OČS

$\frac{1}{\sqrt[213]{n}} \searrow 0$

$\sum \underbrace{(-1)^n \frac{1}{\sqrt[213]{n}}}_{b_n} \cdot \underbrace{\frac{n-1}{n+1}}_{a_n}$ $a_n \nearrow 1$

(A) $\{a_n\}$ monotónní a omezená $\frac{n-1}{n+1} \to \underline{\underline{1}}$

$\sum b_n$ konverguje ✓

$a_{n+1} = \frac{n}{n+2}$ $a_n = \frac{n-1}{n+1}$ $< \underline{\underline{1}}$

$n^2+n = n(n+1) \geq (n+2)(n-1) = n^2+n-2$

• $a_{n+1} \geq a_n$

$\sum^{\infty} (-1)^n \frac{1}{\sqrt[213]{n}} \cdot \frac{n-1}{n+1}$ konverguje

$$\sum_{n=1} (-1)^n \frac{1}{\sqrt[2/3]{n}} \cdot \frac{n-1}{n+1} \quad \text{konverguje}$$

$$\frac{n-1}{n+1} = \frac{n+1-2}{n+1} = 1 - \frac{2}{n+1} \nearrow 1$$

(9) $$\sum_{n=1}^{\infty} \underline{\cos\frac{1}{n}} \cdot \underline{\cos n} \cdot \underline{\sqrt{\frac{n+2}{n(n+1)}}}$$

$\sum_{n=1}^{\infty} \cos n$ má omezené částečné součty

$\quad\quad\quad b_n$

$$a_n = \sqrt{\frac{n+2}{n(n+1)}} \qquad a_{n+1} \le a_n \; \checkmark$$

$$\frac{\sqrt{n+3}}{\cancel{\sqrt{n+1}}\sqrt{n+2}} \le \frac{\sqrt{n+2}}{\sqrt{n}\cancel{\sqrt{n+1}}}$$

$$n^2+\underline{3n} = n(n+3) \overset{\checkmark}{\le} (n+2)^2 = n^2 + \underline{4n} + \underline{4}$$

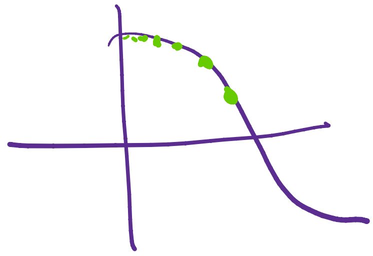
$$\sum \underbrace{\cos n \cdot \sqrt{\frac{n+2}{n(n+1)}}}_{} \quad \text{konverguje podle Dirichletova kritéria}$$

$$a_n = \cos\frac{1}{n} \quad b_n \qquad \sum_{n=1}^{\infty} b_n \;\text{konverguje}$$

$$\overset{\uparrow}{=} \qquad \sum_{n=1}$$

$\cos x$ klesající na $[0,\pi]$

tedy $\cos\frac{1}{n}$ je rostoucí



Tedy $\sum \cos\frac{1}{n} \cos n \sqrt{\frac{n+2}{n(n+1)}}$ konverguje
((D)+(A))

(10) $$\sum_{n=2}^{\infty} \frac{\sin(n+\frac{1}{n})}{\log(\log n)}$$

$$\sin\left(n+\tfrac{1}{n}\right) = \sin n\cdot\cos\tfrac{1}{n} + \cos n\cdot\sin\tfrac{1}{n}$$

$$S_2 = \sum_{n=2}^{\infty} \frac{\sin n}{\log(\log n)}\cdot\cos\frac{1}{n}\ , \quad \sum_{n=2}^{\infty} \frac{\cos n}{\log(\log n)}\sin\frac{1}{n} = S_1$$

$$\sin\frac{1}{n} \searrow 0 \qquad \log(\log n)^{-1} \searrow 0$$

$$\frac{\sin\frac{1}{n}}{\log(\log n)} \searrow 0\ , \quad \sum_{n=1}^{\infty} \cos n \text{ má OČS}$$

S_1 konverguje

$$\sum \underbrace{\frac{1}{\log(\log n)} \cdot \sin n}_{b_n}$$

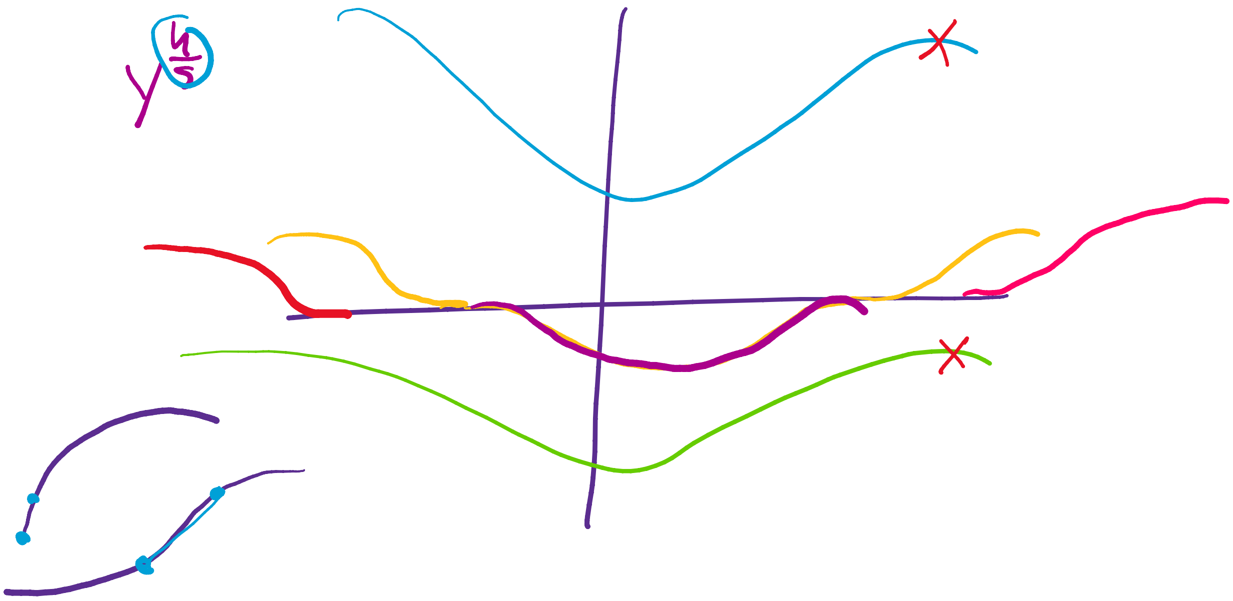
$\sum \sin n$ má OČS

$\frac{1}{\log(\log n)} \searrow 0$

konverguje podle (D)

$a_n = \cos\frac{1}{n}$ $\quad a_n \nearrow 1$

$\sum a_n b_n$ konverguje podle (A)



PS $\quad e^x\cos x \longrightarrow y_p = Ae^x\cos x + Be^x\underline{\sin x}$