Cvičení 23.4.

pátek 23. dubna 2021 11:26

- $\int \sqrt{e^{2t} + e^{4t}} = \int e^t \sqrt{1 + e^{2t}} \quad \Big| \; u = e^t$

- $\left(\frac{x^7 + C}{7x^4} \right)^7 \quad > 0 \qquad z = \sqrt{y} \qquad y = 0$

---

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi}{3} n\right)}{n^2} =: a_n \qquad |a_n| \leq \frac{1}{n^2} \to 0$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \implies \sum_{n=1}^{\infty} |a_n| < \infty \quad \text{(srovnávací krit)}$$

$$\implies \sum_{n=1}^{\infty} a_n \text{ konverguje}$$

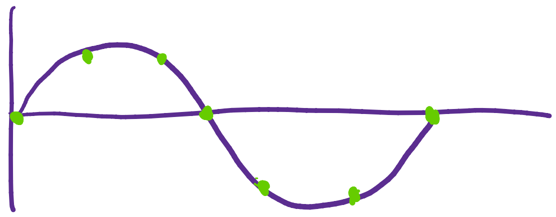
(2) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi}{3} n\right)}{n} =: a_n \qquad |a_n| \leq \frac{1}{n} \to 0$

$\displaystyle\sum_{n=1}^{\infty} \sin(nx)$ má omezené částečné součty pro $x \in \mathbb{R}$

∞

$$\left(\sum_{n=1}^{\infty} \cos(nx) \quad \text{———} \;||\; \text{———} \quad x \neq 2k\pi,\ k\in\mathbb{Z} \right)$$

$\pi \cdot \frac{n}{3}$



- $\left\{ \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \dots \right\}$

$\sum a_n b_n$ $\quad a_n$ monotónní $\quad a_n \to 0$

b_n má OČS

potom $\sum a_n b_n$ konverguje

$a_n = \frac{1}{n} \searrow 0 \quad b_n = \sin\left(\frac{\pi}{3} n\right)$ OČS

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi}{3}n\right)}{n} \quad \text{konverguje}$$

$|b_n| \to \left\{ \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, \dots \right\}$

$$\sum_{n=1}^{\infty} \left| \frac{\sin\left(\frac{\pi}{3}n\right)}{n} \right| \geq \sum_{n=1}^{\infty} \frac{\left|\sin\left(\frac{\pi}{3}(3n+1)\right)\right|}{3n+1} = \sum_{n=1}^{\infty} \frac{\sqrt{3}}{2} \cdot \frac{1}{3n+1} = \infty$$

(4) $\sum_{n=1}^{\infty} n^7 z^n \qquad z \in \mathbb{R}$

pro které $z \in \mathbb{R}$ řada konverguje (absolutně)

$a_n = n^7 z^n \to |a_n| = n^7 |z|^n$

$\sqrt[n]{n^7 |z|^n} = |z| \sqrt[n]{n^7} \to |z| \begin{cases} < 1 & \to AK \\ > 1 & \to D \text{ neplatí } a_n \to 0 \end{cases}$

$|z| = 1 \to |a_n| = n^7 \to \infty$ neplatí $a_n \to 0 \Rightarrow D$

- $a_n \to 0 \iff |a_n| \to \infty$

$\sum_{n=1}^{\infty} n^7 z^n$ konverguje (i absolutně) právě tehdy, když $|z| < 1$

(5) $\sum_{n=1}^{\infty} \frac{6^n}{n^2} z^n \; {}^{=a_n} \qquad z \in \mathbb{R}$

$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{(6|z|)^n}{n^2}} = \frac{6|z|}{\sqrt[n]{n^2}} \to 6|z| \begin{cases} |z| < \frac{1}{6} & AK \\ |z| > \frac{1}{6} & D \quad \cancel{a_n \to 0} \end{cases}$

$|z| = \frac{1}{6} \Rightarrow |a_n| = \frac{1}{n^2}$ a $\sum \frac{1}{n^2} < \infty$

[illegible] $\sum 6^n z^n$ [illegible] lutně pro $|z| \le \frac{1}{6}$

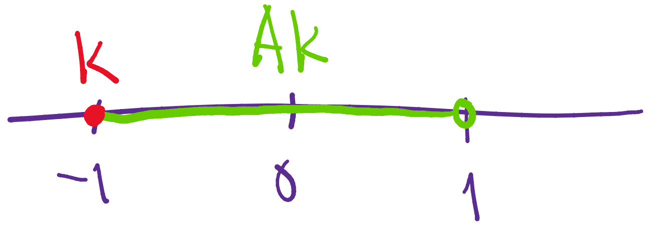
Tedy řada $\sum \frac{6^n z^n}{n^2}$ konverguje absolutně pro $|z| \leq \frac{1}{6}$
diverguje jinak

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$ $=a_n$ $\qquad z \in \mathbb{R}$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{|z|^n}{n}} = \frac{|z|}{\sqrt[n]{n}} \to |z| \begin{cases} <1 & \text{AK} \\ >1 & \text{D (nepl. } a_n \to 0) \end{cases}$$

$$|z| \begin{cases} z=1 \to \displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = \infty \\ z=-1 \to \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \end{cases}$$

$\left.\begin{array}{l} (-1)^n \text{ má OČS} \\ \frac{1}{n} \searrow 0 \end{array}\right\} \to$ K (Dirichlet)



(6) $$\sum_{n=0}^{\infty} \overbrace{\underbrace{\cos(\pi n)}_{(-1)^n} \underbrace{\left(\sqrt{n^3+3} - \sqrt{n^3+1}\right)}_{a^2-b^2 = (a+b)(a-b)}}^{a_n} = \qquad a_n \overset{?}{\to} 0$$

$$\sum^{\infty} \left(\cos(\pi n)\right) \frac{n^3+3-(n^3+1)}{\ldots}$$

$$= \sum_{n=0}^{\infty} \left(\cos(\pi n)\right) \frac{n^3+3-(n^3+1)}{\sqrt{n^3+3}+\sqrt{n^3+1}}$$

$$= \underbrace{\sum_{n=0}^{\infty} \overbrace{|\cos(\pi n)|}^{\leq 1} \frac{2}{\sqrt{n^3+3}+\sqrt{n^3+1}}}_{\alpha_n \to 0} \to \sim \frac{1}{n^\alpha} \quad \alpha = \frac{3}{2}$$

$$\sum \frac{1}{n^{\frac{3}{2}}} < \infty \quad \left(\frac{3}{2} > 1\right)$$

$$|\alpha_n| \leq \frac{2}{\sqrt{n^3+3}+\sqrt{n^3+1}} \sim \frac{1}{n^{\frac{3}{2}}}$$

Tedy $\sum_{n=0}^{\infty} |\alpha_n| < \infty$ a $\sum \alpha_n$ konverguje

(7)

$$\underbrace{\sum_{n=1}^{\infty} \cos(\pi n) \cdot \frac{\sqrt{n+7}}{\sqrt{n}\sqrt{n+1}}}_{\alpha_n}$$

$$\sim \frac{1}{n^\alpha} \quad \alpha = \frac{1}{2}$$

$\alpha_n \to 0 \quad |\cos(\pi n)| \leq 1 \quad \sqrt{\frac{n+7}{n^2+n}} \to 0$

$$\sim \sqrt{\frac{1}{n}}$$

$$|\cos \pi n| = 1 \quad |\alpha_n| = \sqrt{\frac{n+7}{n^2+n}} \sim \sqrt{\frac{1}{n}}$$

$\sum \frac{1}{\sqrt{n}} = \infty \to \sum \alpha_n$ nekonverguje absolutně

$\sum^{\infty}$ ... $-(-1)^n \to$ OČS ✓

$$\sum_{n=1}^{\infty} a_n b_n \qquad b_n = (-1)^n \rightarrow \text{OČS} \checkmark$$

$$a_n = \underline{\sqrt{\frac{n+7}{n^2+n}}} \searrow 0$$

$$a_{n+1}^2 \leq a_n^2, \quad \frac{n+8}{\underbrace{(n+1)^2+n+1}_{(n+1)(n+2)}} \leq \frac{n+7}{\underbrace{n^2+n}_{(n+1)n}}$$

$$\frac{n+8}{n+2} \leq \frac{n+7}{n}$$

$$\cancel{n^2} + 8n \leq \cancel{n^2} + 16n + 14 \qquad n \in \mathbb{N}$$

Tedy $\sum (-1)^n \sqrt{\frac{n+7}{n(n+1)}}$ konverguje (neabsolutně) podle Dirichletova krit.

(8) $$\sum_{n=1}^{\infty} \overbrace{(-1)^n \cdot \frac{n-1}{n+1} \cdot \frac{1}{\sqrt[213]{n}}}^{a_n} = \frac{1}{n^{\frac{1}{213}}}$$

$$\frac{n-1}{n+1} \rightarrow 1$$

$$|a_n| \sim \underline{\underline{\frac{1}{n^{\alpha}}}} \qquad \alpha = \frac{1}{213} < 1$$

řada nekonverguje absolutně

$$\sum (-1)^n \cdot \frac{n-1}{n+1} \cdot \frac{1}{\sqrt[213]{n}}$$

$$\sum \underbrace{(-1)^n}_{b_n} \cdot \underbrace{\frac{n-1}{n+1}}_{\tilde{a}_n} \cdot \underbrace{\frac{1}{\sqrt[2013]{n}}}_{a_n}$$

$$\left.\begin{array}{l} b_n \text{ má OČS} \\ a_n \searrow 0 \end{array}\right\} \Rightarrow \sum (-1)^n \frac{1}{\sqrt[2013]{n}} \text{ konv.}$$

$$\sum a_n b_n \quad \left.\begin{array}{l} \bullet\ a_n \text{ monotónní a omezená} \\ \bullet\ b_n \text{ konverguje} \end{array}\right\} \Rightarrow \sum a_n b_n \text{ konv}$$

$$\tilde{a}_n = \frac{n \overset{\pm 1}{-} 1}{n+1} \nearrow 1 \qquad \tilde{b}_n = a_n b_n$$

$$\frac{n-1}{n+1} = 1 - \frac{2}{n+1} \qquad \sum \hat{b}_n \text{ konverguje}$$

Tedy $\sum_{n=1}^{\infty} \hat{a}_n \hat{b}_n = \sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n+1} \frac{1}{\sqrt[2013]{n}}$ konverguje

(9) $$\sum_{n=1}^{\infty} \underbrace{\cos\left(\frac{1}{n}\right)}_{c_n} \underbrace{\cos n}_{a_n} \underbrace{\sqrt{\frac{n+2}{n(n+1)}}}_{b_n}$$

(10) $$\sum_{n=2}^{\infty} \frac{\sin\left(n+\frac{1}{n}\right)}{\log(\log n)}$$

$$\boxed{\sum \sin(n) \text{ OČS}}$$

$$\cancel{\Downarrow}$$

$$\sum \sin\left(n+\frac{1}{n}\right) \text{ OČS}$$

$\sum_{n=1}^{\infty} a_n$ má omezené částečné součty

✓ $\sum_{n=1} a_n$ má omezené částečné součty

$$\sum b_n \to \sqrt{\frac{n+2}{n(n+1)}} \searrow 0$$

$$b_{n+1}^2 \leq b_n^2$$

$$\frac{n+3}{\cancel{(n+1)}(n+2)} \leq \frac{n+2}{n\cancel{(n+1)}}$$

$$n^2+3n \leq n^2+4n+4 \quad \checkmark$$

Tedy $\sum \cos n \sqrt{\frac{n+2}{n(n+1)}}$ konverguje podle Dirichleta

$C_n = \cos\left(\frac{1}{n}\right)$ $\quad \frac{1}{n} \searrow 0$ cos klesající na pravém okolí 0

$C_n \nearrow 1$

(Abel) $\to$ $\sum C_n a_n b_n$ konverguje.

$$\sin\left(n+\frac{1}{n}\right) = \sin n \cos\frac{1}{n} + \cos n \sin\frac{1}{n}$$

klesající

$$\sum_{n=2}^{\infty} \sin n \cdot \cos\frac{1}{n} \frac{1}{\log(\log n)} + \sum_{n=2}^{\infty} \cos n \sin\frac{1}{n} \cdot \frac{1}{\log(\log n)}$$

$\frac{1}{\log(\log n)} \searrow 0$ $\quad$ log rostoucí $\to$ log(log) rostoucí

$\frac{1}{\log(\log n)} \searrow 0$ log rostoucí $\to$ log (log) rostoucí

$\sum \sin n$ $\sum \cos n$ OČS Tedy

$\sum \sin n \frac{1}{\log(\log n)}$, $\sum \cos n \frac{1}{\log(\log n)}$ konvergují podle Dir.

$\cos \frac{1}{n}$ rostoucí omezený $\cos \frac{1}{n} \nearrow 1$

Tedy obě řady konvergují

$\sum a_n < \infty \sim \sum 2^n a_{2^n}$