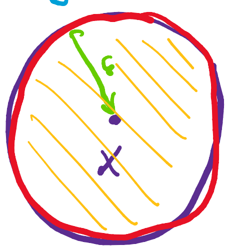
Cvičení 28.4

středa 28. dubna 2021 11:08

• $U(x,\varepsilon) = \{y \in \mathbb{R}^d : |x-y| < \varepsilon\}$ $\qquad B(x,\varepsilon) = \{y\in\mathbb{R}^d : |x-y| \le \varepsilon\}$

$|x-y| = \sqrt{(x_1-y_1)^2 + \dots + (x_d-y_d)^2}$

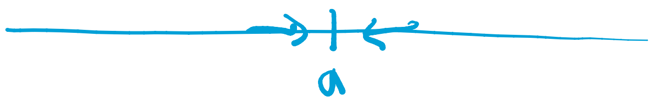
cvičení

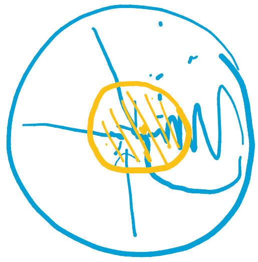


$P(a,\varepsilon) = \{y\in\mathbb{R}^d : 0<|y-a|<\varepsilon\}$

(L) $\lim\limits_{x\to a} f(x) = L \iff \forall \varepsilon>0\ \exists \delta>0\ \forall x\in P(a,\delta): f(x)\in U(L,\varepsilon)$

$x \in \mathbb{R}^d$, $a \in \mathbb{R}^d$, $L \in \mathbb{R}^*$, $U(L,\varepsilon)$ v $\mathbb{R}$





f je spojitá v bodě a, pokud $\lim\limits_{x\to a} f(x) = f(a)$

(S) $\forall \varepsilon>0\ \exists \delta>0\ \forall x\in U(a,\delta): f(x)\in U(f(a),\varepsilon)$

(1) x

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

spojitá na $\mathbb{R}^2$?

Pro $x \in \mathbb{R}^2 \setminus \{(0,0)\}$

• polynomy a elementární funkce jsou spojité (na D_f)

e^x, $\sin x$, $\log x$

- polynomy a elementární funkce jsou spojité (na D_f)

x^7+4xy^2

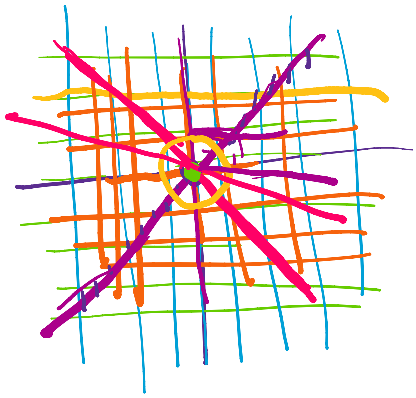
- součet, součin, podíl (pokud je definovaný), složení spojitých funkcí (zobrazení) jsou spojité

$$\frac{(e^x)^7+4e^x\sin y}{\operatorname{arctg}(x^2+\sin^7 y)}$$

← spojitá na D_f

f je spojitá na $\mathbb{R}^2 \setminus \{(0,0)\}$.

Spojitost v $(0,0)$. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0$



$$f(x,0) = \frac{x\cdot 0}{x^2+0} = 0$$

$$\lim_{x\to 0} f(x,0) = 0 = f(0,0)$$

$$\lim_{y\to 0} f(0,y) = 0 = f(0,0)$$

$$f(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2} \qquad f(x,-x) = \frac{-x^2}{x^2+x^2} = -\frac{1}{2}$$

$$f(r\cos\varphi, r\sin\varphi) = \frac{r\cos\varphi\, r\sin\varphi}{r^2\cos^2\varphi + r^2\sin^2\varphi} = \boxed{\cos\varphi\cdot\sin\varphi}$$

(2)

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & (x,y)\neq(0,0) \\ & (x,y)=(0,0) \end{cases}$$

3

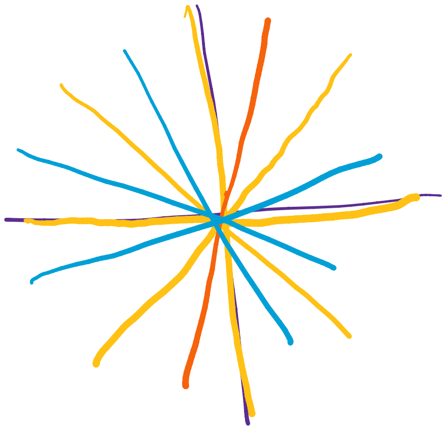
2

$$f(x,y) = \begin{cases} \dfrac{\cdots}{x^2+y^2} & \\ 0 & (x,y)=(0,0) \end{cases}$$

$x \in \mathbb{R}^2 \setminus \{(0,0)\} \to$ spojitá (podíl polynomů, jmenovatel $\neq 0$)

$(x,y)=(0,0)$

$$f(x,0)=0 \qquad f(0,y)=0$$



$$f(x,x)=\frac{x^3}{x^2+x^2}=\frac{x}{2}$$

$$\lim_{x\to 0} f(x,x)=0$$

$$f(x,-x)=\frac{x^3}{x^2+x^2}=\frac{x}{2} \to 0$$

$$f(x,Lx)=\frac{x\cdot L^2x^2}{x^2+L^2x^2}=\frac{L^2x}{1+L^2} \to 0$$

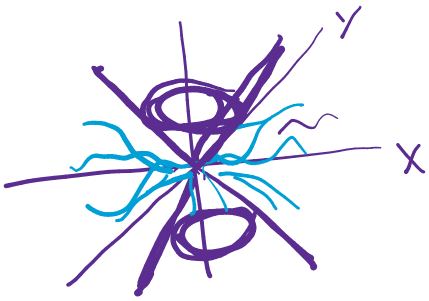
$$f(r\cos\varphi, r\sin\varphi)=\frac{r\cos\varphi\cdot r^2\sin^2\varphi}{r^2\cos^2\varphi+r^2\sin^2\varphi}=r\cos\varphi\sin^2\varphi$$

$\lim\limits_{r\to 0+} g(r)=0\,?$

$$\left|f(r\cos\varphi, r\sin\varphi)-\underbrace{f(0,0)}_{0}\right| \le r\,\underbrace{|\cos\varphi|}_{\le 1}\cdot\underbrace{\sin^2\varphi}_{\le 1} \le r = g(r)$$

$r = \sqrt{x^2+y^2} < \delta$

$$\lim_{r\to 0+} r = 0$$



$$\forall \varepsilon>0\ \exists \delta>0\ \forall (x,y)\in P((0,0),\delta) : |f(x,y)-f(0,0)|<\varepsilon$$

$\delta=\varepsilon$ $\qquad \sqrt{x^2+y^2}<\delta \to |f(x,y)-f(0,0)|\le\sqrt{x^2+y^2}$

$|x|\le\sqrt{x^2+y^2}$

$-\sqrt{x^2+y^2} \le f(x,y)-f(0,0) \le \sqrt{x^2+y^2}$

$$-\sqrt{x^2+y^2} \le f(x,y) - f(0,0) \le \sqrt{x^2+y^2}$$

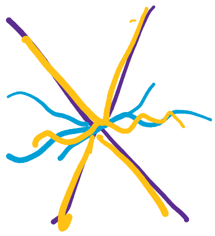
$|x| \le \sqrt{x^2+y^2}$

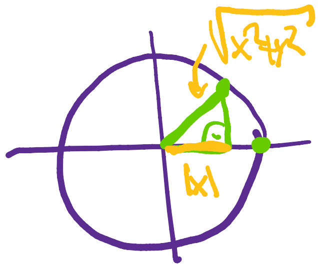
$|x| = \sqrt{x^2}$

$$\left|\frac{xy^2}{x^2+y^2}\right| \le \frac{|x|\cdot y^2}{x^2+y^2} \le \frac{\sqrt{x^2+y^2}\cdot \cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} = \sqrt{x^2+y^2}$$

$\left(\sqrt{x^2+y^2}\right)^2$

$y^2 \le x^2+y^2$



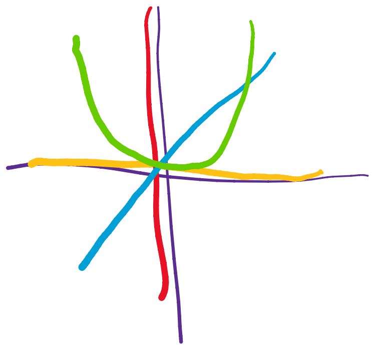


(3)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} \quad , f(x,y)$$

$y = Lx$

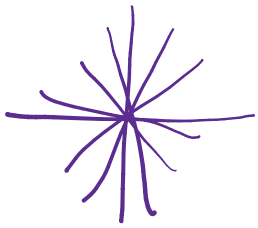
$y = x^\alpha$



$f(x,0) = 0 = L\,?$

$f(0,y) = 0$

$$f(x,x) = \frac{x^3}{x^4+x^2} = \frac{x}{x^2+1} \to 0$$



$$f(x,-x) = \frac{-x^3}{x^4+x^2} = \frac{-x}{x^2+1} \to 0$$

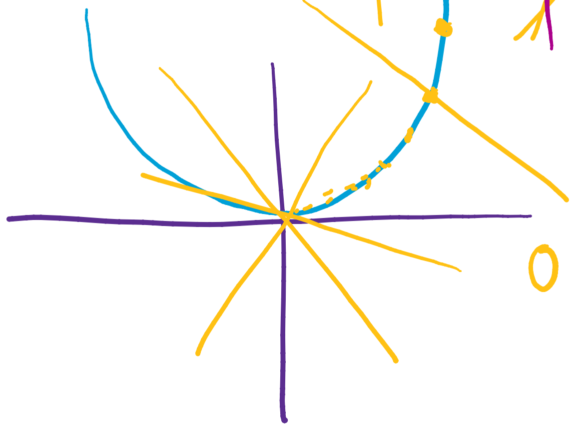
$$\left|f(r\cos\varphi, r\sin\varphi)\right| = \left|\frac{r^2\cos^2\varphi\cdot r\sin\varphi}{r^4\cos^4\varphi + r^2\sin^2\varphi}\right| = \left|\frac{r\cos^2\varphi\sin\varphi}{r^2\cos^4\varphi + \sin^2\varphi}\right| \le g(r)$$

$$f(x,Lx) = \frac{x^3\cdot L}{x^4+L^2x^2} = \frac{xL}{x^2+L^2} \to 0$$

$$f(x,x^2) = \frac{x^2\cdot x^2}{x^4+x^4} = \frac{1}{2}$$

$$f(x,-x^2) = \frac{x^2\cdot(-x^2)}{\ldots} = -\frac{1}{\ldots}$$

$$f(x,-x^2)=\frac{x^2\cdot(-x^2)}{x^4+x^4}=-\frac{1}{2}$$

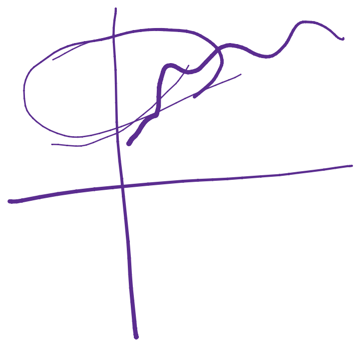


(4) $$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

spojitost v $(0,0)$?

$$\left|f(r\cos\varphi, r\sin\varphi)\right|=\left|\frac{\cancel{r}\cos\varphi\, r\sin\varphi}{\sqrt{\cancel{r^2}\cos^2\varphi+\cancel{r^2}\sin^2\varphi}}\right|=\left|r\cos\varphi\sin\varphi\right|\leq r=g(r)$$

(5) $$\lim_{(x,y)\to(0,0)}\overbrace{\log(\underbrace{x^2+y^2}_{r^2})\sqrt{\underbrace{x^2+y^2}_{r^2}}}^{f(x,y)}=0$$



$$\left|f(r\cos\varphi, r\sin\varphi)\right|=\left|\log r^2\cdot r\right|=g(r)$$

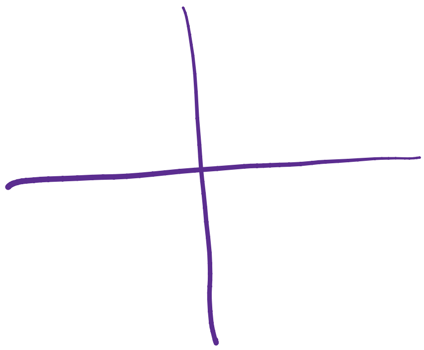
$$\lim_{r\to0+}\log(r^2)\cdot r=0$$

$\lim_{r\to 0+}$ ...

(7) $(x^2+y^2)^{x^2y^2} \to e^{\boxed{x^2y^2\log(x^2+y^2)}}$

(8) $\dfrac{\sqrt{x^2+y^2+1}-1}{x^2+y^2} \to$ $\boxed{a^2-b^2}$

(10)



$$\left.\begin{array}{l}\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y)\right) = L \\ \lim\limits_{(x,y)\to(0,0)} f(x,y) = M\end{array}\right\} L = M$$

- vyzkoušet $f(x,0),\ f(0,x),\ f(x,x),\ f(x,x^\alpha),\ f(x,Lx)$
- polární souřadnice $\to g(r)$

Parciální derivace

$$f(x,y,z) = xyz - x^3y + 3z^4$$

$$\frac{\partial f}{\partial x}(x,y,z) = (Cx - Dx^3 + E)' = C - 3Dx^2 = yz - 3yx^2$$

$$= yz - 3x^2y + 0$$

$$\frac{\partial f}{\partial y}(x,y,z) = xz - x^3 \qquad \frac{\partial f}{\partial z} = xy + 12z^3$$

$$f(x,y) = \sin(xy) \to \frac{\partial f}{\partial x}(x,y),\ \frac{\partial f}{\partial y}(x,y)$$

$f(x,y) = \sin(xy) \to \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)$

$(x^y)^z$

$f(x,y,z) = x^{(y^z)}$ $\quad \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

$\frac{\partial f^2}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ | $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ ✓

$f(x,y) = x^2 y - e^{y^3 x}$

$f(x,y) = \sin(xy) \Rightarrow \frac{\partial f}{\partial x} = y\cos(xy) \quad \frac{\partial f}{\partial y} = x\cos(xy)$

$(x^\alpha)' = \alpha x^{\alpha-1}$

$f(x,y,z) = x^{(y^z)}$ $\quad \frac{\partial f}{\partial x} = y^z x^{y^z - 1}$

$(a^x)' = \log a \cdot a^x$ $\quad (x^\alpha)' = \alpha x^{\alpha - 1}$, $\alpha = y^z$

$\frac{\partial f}{\partial y} = z \cdot y^{z-1} \cdot \log x \cdot x^{(y^z)}$

$\frac{\partial f}{\partial z} = \log y \cdot y^z \cdot \log x \cdot x^{(y^z)}$